H^2 gradient method for the coefficient identification problem in a partial differential equation

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Abstract

A numerical method is considered herein for solving the problem of coefficient identification in a linear partial differential equation used to detect damages in steel-concrete composite beams. To identify the numerically unknown coefficient, an iterating algorithm based on the adjoint method is presented. The H^1 gradient method is applied to set the search direction in our algorithm. The search direction is selected based on the solution to the weak equation with a bilinear form in the H^2 Sobolev space. The effectiveness of our method is confirmed via numerical experiments.

Keywords coefficient identification problem in PDE, H^1 gradient method, adjoint method

Research Activity Group Mathematical Design

1. Introduction

A new numerical method is considered herein for solving the problem of coefficient identification in a partial differential equation by applying the topological optimization technique [1]. This is applicable for detecting damages in steel-concrete composite beams, as proposed by Morassi et al. [2]. A steel-concrete composite beam is obtained by connecting two beams, one metallic and the other reinforced concrete, by means of small metallic elements (connectors) that are welded on the top flange of the metallic beams and immersed in the concrete to hinder sliding at the concrete-steel interface (Fig. 1). A

Fig. 1. Steel-concrete composite beam [2].

one-dimensional mechanical model of the small vibrations of such a composite beam, as proposed by Morassi et al. [3], is the focus of this study. Hereafter, the quantities relative to the concrete beam (upper beam in Fig. 1) and to the steel beam (lower beam in Fig. 1) are defined by indices $i = 1, 2$, respectively. Let us suppose that the system vibrates freely in the plane, with small oscillations around a straight and not stressed configuration of equilibrium. Consequently, in accordance with the traditional kinematic hypotheses of the beam theory under small axial and flexural deformations, the actual

configuration is thoroughly defined by assigning axial displacements $u_i(x, t)$, $i = 1, 2$, and transversal displacements, $v_i(x, t)$, $i = 1, 2$, of y abscissa's transversal section at a moment of time t [2]. In the simple model [3], we assume that the transversal displacements are the same, namely $v = v_1 = v_2$. The length of a beam is denoted by L . $a_i \in C^1[0, L]$ and $j_i \in C^2[0, L]$ are the axial stiffness and flexural stiffness of the cross-section, respectively. $\rho_i \in C^0[0, L]$ is the linear mass density of the i -th beam. The coefficient $k \in C^1[0, L]$ represents the shearing stiffness per unit length of the connection. Then, the shearing coefficient is satisfied according to the following condition:

$$0 \leq k(x) \leq \bar{k}, \quad \forall x \in [0, L], \quad (1)$$

where $\bar{k}$ is a given positive constant. The mechanical model for infinitesimal free vibrations of this steel-concrete composite beam is described by the following system of partial differential equations:

$$\begin{cases} Cw_{,tt} - A_k w = 0 & \text{in } (0, L) \times (0, T), \\ w(\cdot, 0) = 0, \quad w_{,t}(\cdot, 0) = 0 & \text{in } (0, L), \\ Dw(0, \cdot) = \bar{U}, \quad Dw(L, \cdot) = 0 & \text{on } (0, T), \end{cases} \quad (2)$$

where $w = (u_1(x, t), u_2(x, t), v(x, t))^T$ and $C = \text{diag}(\rho_1, \rho_2, \rho)$. The quantity ρ is defined as $\rho = \rho_1 + \rho_2$ and T is the length of observation time. $\bar{U} = (\bar{U}_1, \bar{U}_2, \bar{v}, \bar{\varphi})^T$ is the vector of Dirichlet data at $x = 0$. A_k is the partial differential operator defined as follows:

$$A_k w = \begin{bmatrix} (a_1 u_{1,x})_{,x} + k(u_2 - u_1 + v_{,x} e_s) \\ (a_2 u_{2,x})_{,x} - k(u_2 - u_1 + v_{,x} e_s) \\ -(j v_{,xx})_{,xx} + k(u_2 - u_1 + v_{,x} e_s) e_s \\ + (k e_c^2 v_{,x})_{,x} \end{bmatrix},$$

where $j = j_1 + j_2$ and D is the trace operator defined by $Dw = (u_1, u_2, v, v_x)^T$. Finally, e_s is the half-height of the steel beam and $e_c = e - e_s$, where e is the distance between the axes of the two beams forming the system.

Let us define function $\Phi_U : [0, L] \times [0, T] \rightarrow \mathbb{R}^3$ such that

$$\Phi_U(x, t) = \begin{bmatrix} \bar{U}_1(t)\zeta(x) \\ \bar{U}_2(t)\zeta(x) \\ \bar{\varphi}(t)\psi(x) + \bar{v}\zeta(x) \end{bmatrix}, \quad (3)$$

where

$$\zeta(x) = 2(x/L)^3 - 3(x/L)^2 + 1, \quad \psi(x) = x(1 - x/L)^2.$$

Then, our direct problem (2) can be represented as follows:

$$\begin{cases} C\hat{w}_{,tt} - A_k\hat{w} = \hat{f} & \text{in } (0, L) \times (0, T), \\ \hat{w}(\cdot, 0) = -\Phi(\cdot, 0), \quad \hat{w}_{,t}(\cdot, 0) = -\Phi_{,t}(\cdot, 0) & \text{in } (0, L), \\ D\hat{w}(0, \cdot) = D\hat{w}(L, \cdot) = 0 & \text{on } (0, T), \end{cases} \quad (4)$$

where $\hat{w} = w - \Phi$ and $f = A_k\Phi - C\Phi_{,tt}$. By integration by parts, the weak form of (4) can be represented as follows:

$$\int_0^T ((C\hat{w}_{,tt}, m) + a_k(\hat{w}, m)) dt = \int_0^T (\hat{f}, m) dt \quad (5)$$

for any $m = (g_1, g_2, h) \in C_0^\infty([0, L]; (H_0^1(0, L))^2 \times H_0^2(0, L))$. Here, $(\cdot, \cdot)$ means the inner product on $(L^2(0, L))^3$. $a_k(\cdot, \cdot)$ is the bilinear form on $(H^1(0, L))^2 \times H^2(0, L)$ defined by $a_k(\hat{w}, m) = \alpha^{(0)}(\hat{w}, m) + \alpha_k^{(1)}(\hat{w}, m)$, where

$$\begin{aligned} \alpha^{(0)}(\hat{w}, m) &= \int_0^L (a_1 u_{1,x} g_1 + a_2 u_{2,x} g_2 + j v_{,xx} h_{,xx}) dx, \\ \alpha_k^{(1)}(\hat{w}, m) &= \int_0^L k(u_2 - u_1 + v_{,x} e_s)(g_2 - g_1 + h_{,x} e_s) dx \\ &\quad + \int_0^L k e_c^2 v_{,x} h_{,x} dx. \end{aligned}$$

We can prove the well-posedness of the direct problem (2) by applying Theorem 30.1 from [4] to the weak problem (5).

Theorem 1 Assume that $\bar{U} \in (C^3[0, T])^4$ with $d^i \bar{U}/dt^i|_{t=0} = \mathbf{0}$ ($i = 0, 1, 2, 3$). Then, there is a unique solution to the direct problem (2) with $w \in (H^2((0, T); H^1(0, L)))^2 \times H^2((0, T); H^2(0, L))$ and $w_{,ttt} \in (L^2((0, L); L^2(0, L)))^3$.

Typical damage occurring in real steel-concrete systems includes the deterioration of the connection, which causes a decrease in the stiffness coefficient k . Since the inaccessibility of the connection from the exterior makes direct inspection difficult, an inverse problem applicable to such applications comprises estimating the coefficient k based on suitable non destructive techniques. For this problem, Morassi et al. [3] proved that the shearing stiffness k could be uniquely determined from the measurement of the frequency response function of the composite system taken at one end of the beam. Moreover, Belishev et al. [5] proposed the boundary control method to solve this inverse problem based on measurements taken at

both the ends of the beam. For a full system in which the transversal displacements are not the same, our research group proposed a numerical method based on the projected gradient method [6]. Using this method, we could obtain good numerical reconstruction if the measured data have no errors. However, it failed to obtain a stable numerical solution for our inverse problem when the measurement data contained errors.

In this paper, a numerical method is considered to obtain a stable solution for the problem of coefficient identification in (2). Our inverse problem is used to reconstruct the stiffness coefficient k from dynamical measurements taken at one end of the boundary and at certain interior locations of the system. Let $\bar{Q}(t) = (\bar{N}_1(t), \bar{N}_2(t), \bar{S}(t), \bar{M}(t))^T$ be the vector of Neumann data for the system (2) as evaluated at $x = 0$, $t \in [0, T]$, defined by

$$\begin{aligned} a_1 u_{1,x}|_{x=0} &= \bar{N}_1, \quad a_2 u_{2,x}|_{x=0} = \bar{N}_2, \\ -(j v_{,xx})_{,x} - k(u_2 - u_1 + v_{,x} e_s) e_s - k e_c^2 v_{,x}|_{x=0} &= \bar{S}, \\ -j v_{,xx}|_{x=0} &= \bar{M}. \end{aligned}$$

Moreover, $\bar{u}_i(x, t)$ denotes the longitudinal displacements on $I \times [0, T]$, where I is an open interval in $[0, L]$. Then, our inverse problem is formulated as follows:

Coefficient identification problem. Determine the unknown shearing stiffness k from the knowledge of the measurements $\bar{Q}$ and $\bar{u}_i$ ($i = 1, 2$).

To numerically solve our inverse problem, an iteration method based on the technique of topology optimization of density type is proposed. The direct and inverse problems of density type are introduced using a density type shearing coefficient which is defined as the composition of given sigmoid and unknown control functions. The unknown shearing stiffness function is approximately identified by finding the control function. A cost functional with a control function is introduced using the measured data, then, an unconstrained minimizing problem is produced. The H^1 gradient method [7] is applied to set the search direction in our method. We show the effectiveness of our method using numerical experiments.

2. H^2 gradient method

First, a direct problem for density type is introduced. Let $\phi \in C^1(\mathbb{R})$ be a sigmoid function satisfying $0 \leq \phi(\xi) \leq 1$, $\forall \xi \in \mathbb{R}$. Then, the shearing function of density type is defined as $\tilde{k}(\theta) = \bar{k}\phi(\theta(x))$. Here, θ is a control function that belongs to $C^1[0, L]$. We note that the function $\tilde{k}(\theta)$ satisfies the condition (1). Then, the direct problem of density type can be introduced as follows:

$$\begin{cases} Cw_{,tt} - A_{\tilde{k}(\theta)} w = 0 & \text{in } (0, L) \times (0, T), \\ w(\cdot, 0) = 0, \quad w_{,t}(\cdot, 0) = 0 & \text{in } (0, L), \\ Dw(0, \cdot) = \bar{U}, \quad Dw(L, \cdot) = 0 & \text{on } (0, T). \end{cases} \quad (6)$$

This problem is well-posed because the shearing function of density type belongs to $C^1[0, L]$ and satisfies the condition (1). Using the solution to (6), our inverse problem of density type can be defined.

Coefficient identification problem of density

type. Determine the control function $\theta \in C^1[0, L]$ from the knowledge of the measurements $\bar{Q}$ and $\bar{u}_i$ ($i = 1, 2$).

To identify the unknown control function θ , the adjoint method is adopted. The unknown θ is determined by minimizing the cost functional $J : C^1[0, L] \rightarrow \mathbb{R}^+ := [0, \infty)$ which is defined as

$$\begin{aligned} J(\theta) &= \tilde{J}(\tilde{k}(\theta)) \\ &= \int_0^T \frac{|Q[\tilde{k}(\theta)] - \bar{Q}|^2}{\|\bar{Q}\|_{L^2}^2} dt \\ &\quad + \int_0^T \int_I \sum_{i=1}^2 \frac{|u_i[\tilde{k}(\theta)] - \bar{u}_i|^2}{\|\bar{u}\|_{L^2}^2} dx dt, \end{aligned}$$

where $Q[\tilde{k}(\theta)]$ is the Neumann boundary value at $x = 0$, which is obtained using the solution to (6). We introduce two types of boundary value problems to find the derivative of J .

$$\begin{cases} C\tilde{V}_{,tt} - A_{\tilde{k}}\tilde{V} = \frac{-2[(u_1 - \bar{u}_1)\chi_I, (u_2 - \bar{u}_2)\chi_I, 0]^T}{\|u\|_{L^2}^2} & \text{in } (0, L) \times (0, T), \\ \tilde{V}(\cdot, T) = \frac{\Phi_Q(\cdot, T)}{\|Q\|_{L^2}^2}, \quad \tilde{V}_{,t}(\cdot, T) = \frac{\partial_t \Phi_Q(\cdot, T)}{\|Q\|_{L^2}^2} & \text{in } (0, L), \\ D\tilde{V}(0, \cdot) = \frac{2(Q[\tilde{k}] - \bar{Q})}{\|Q\|_{L^2}^2}, \quad D\tilde{V}(L, \cdot) = 0 & \text{on } (0, T), \end{cases}$$

where χ_I is the characteristic function of I , i.e. $\chi_I(x) = 1$ for $x \in I$ and $\chi_I(x) = 0$ for $x \in [0, L] \setminus I$ and Φ_Q is the function (3) by using $2(Q[\tilde{k}] - \bar{Q})$ instead of $\bar{U}$.

$$\begin{cases} CU_{\varphi,tt}^{(\tilde{k})} - A_{\tilde{k}}U_{\varphi}^{(\tilde{k})} = F(\varphi)w[\tilde{k}] & \text{in } (0, L) \times (0, T), \\ U_{\varphi}^{(\tilde{k})}(\cdot, 0) = U_{\varphi,t}^{(\tilde{k})}(\cdot, 0) = 0 & \text{in } (0, L), \\ DU_{\varphi}^{(\tilde{k})}(0, \cdot) = DU_{\varphi}^{(\tilde{k})}(L, \cdot) = 0 & \text{on } (0, T), \end{cases}$$

where

$$F(\varphi)w = \begin{bmatrix} \varphi(u_2 - u_1 + v_{,x}e_s) \\ -\varphi(u_2 - u_1 + v_{,x}e_s) \\ (\varphi(u_2 - u_1) + v_{,x}e_s)e_s, x + (\varphi e_c^2 v_{,x}), x \end{bmatrix}.$$

Using [4, Theorem 30.1], we can prove that these problems are well-posed. Moreover, we guarantee $\tilde{V} \in (C^0([0, T]; H^1(0, L)))^2 \times C^0([0, T]; H^2(0, L))$ with $V_{,t} \in (C^0([0, T]; L^2(0, L)))^2 \times C^0([0, T]; L^2(0, L))$ and $U_{\varphi}^{(\tilde{k})} \in (C^0([0, T]; H_0^1(0, L)))^2 \times C^0([0, T]; H_0^2(0, L))$ with $U_{\varphi,t}^{(\tilde{k})} \in (C^0([0, T]; L^2(0, L)))^2 \times C^2([0, T]; L^2(0, L))$.

Similar to [6, Theorem 3.1], the derivative of J can be obtained.

Theorem 2 *The derivative of the functional J can be given as*

$$\frac{dJ}{d\theta}(\theta) = \frac{d\tilde{J}}{d\tilde{k}} \circ \frac{d\tilde{k}}{d\theta} \in H^{-2}(0, L).$$

Here, $\frac{d\tilde{J}}{d\tilde{k}}$ is the derivative obtained as

$$\begin{aligned} \left\langle \frac{d\tilde{J}}{d\tilde{k}}(\tilde{k}), \varphi \right\rangle &= \int_0^T \alpha_{\varphi}^{(1)}(w[\tilde{k}], \tilde{V}) dt \\ &\quad + \left\langle C\partial_t U_{\varphi}^{(\tilde{k})}(T), \Phi_Q(\cdot, T) \right\rangle - \left\langle CU_{\varphi}^{(\tilde{k})}(T), \partial_t \Phi_Q(\cdot, T) \right\rangle \end{aligned}$$

Table 1. Physical constants [2].

Constants	Value
L	3.50 m
A_1	$3.00 \times 10^{-2} \text{ m}^2$
J_1	$9.00 \times 10^{-6} \text{ m}^4$
ρ_1	73.19 kgm^{-1}
E_1	$4.2863 \times 10^{10} \text{ Nm}^{-2}$
A_2	$1.64 \times 10^{-3} \text{ m}^2$
J_2	$5.41 \times 10^{-6} \text{ m}^4$
ρ_2	12.90 kgm^{-1}
E_2	$2.1 \times 10^{11} \text{ Nm}^{-2}$
e_s	0.07 m
e_c	0.03 m
T	$4.0 \times 10^{-3} \text{ s}$
n_p	16

for $\forall \varphi \in H^2(0, L)$.

The unknown design variable θ is identified numerically by the following process:

$$\theta_{\ell+1} = \theta_{\ell} + \epsilon_{\ell} \frac{s_{\theta}^{(\ell)}}{\|s_{\theta}^{(\ell)}\|_{H^2}} \quad (\ell = 0, 1, 2, \dots),$$

where $\epsilon_{\ell} > 0$ is a suitable step size. We adopt the idea of H^1 gradient method to set the search direction s_{θ} . The following weak equation for $H^2(0, L)$ is defined to identify the search direction:

$$a_{\theta}(s_{\theta}, \varphi) = - \left\langle \frac{dJ}{d\theta}(\theta), \varphi \right\rangle, \quad \forall \varphi \in H^2(0, L), \quad (7)$$

where $a_{\theta}(\cdot, \cdot)$ is a bounded and coercive bilinear form of $H^2(0, L)$ defined by

$$a_{\theta}(\varphi, \psi) = \alpha_{\theta}(\varphi'', \psi'')_{L^2} + \beta_{\theta}(\varphi', \psi')_{L^2} + \gamma_{\theta}(\varphi, \psi)_{L^2}.$$

Here, $\alpha_{\theta} > 0$, $\beta_{\theta} > 0$, and $\gamma_{\theta} > 0$ are given positive constants. From the Lax-Milgram theorem, the search direction s_{θ} is uniquely determined, and belongs to $H^2(0, L)$. We refer to this numerical process as the H^2 gradient method.

3. Numerical experiment

Some numerical results by applying our method to the identification of the coefficient k are presented herein. The physical constants of a composite beam are summarized in Table 1. These values are used in our numerical experiments after performing nondimensionalization.

We assume that our target coefficients k for the damaged configuration of the composite beam are defined by

$$k(x) = \begin{cases} 11.85 + 3.95 \cos 14\pi \left(x - \frac{3}{7}\right) & \left(\frac{3}{7} < x < \frac{4}{7}\right), \\ 15.80 & (\text{otherwise}). \end{cases}$$

The Dirichlet boundary value should be given by $\bar{U}(t) = (C_u B_5(t), C_u B_5(t), 0, 0)^T$, where $C_u = 1.50 \times 10^{-2}$ and B_5 is a 5th order normalized B-spline function. The Neumann boundary data and the interior measured data are obtained by numerically solving the problem (2) for the damaged beam with $\bar{U}$. To numerically solve the

direct problem, we use the HHT method [8] for time integration and introduce linear spline functions and cubic Hermite functions for approximating u_i ($i = 1, 2$) and v , respectively. Here, the intervals $[0, L]$ and $[0, T]$ are divided into 400 and 14400 equally spaced subintervals, respectively. To numerically solve the initial-boundary value problems in our inverse analysis, the HHT method is used for time integration with the linear elements approximating u_i and the cubic Hermite elements for approximating v . The intervals $[0, L]$ and $[0, T]$ are coarsely divided into 200 and 7200 equally spaced subintervals, respectively. The weak equation (7) for the search direction is solved numerically using FEM with cubic Hermite elements.

The function ϕ in the shearing stiffness of the density type is given as follows:

$$\phi(\theta) = \frac{1 + \tanh(10\theta)}{2}.$$

The initial control function is $\theta_0 \equiv 1.0$ and the internal observation range is $I = (L/5, L/4)$. The stop criterion of our algorithm is $J(\theta_\ell) < 1.0 \times 10^{-3}$.

In our inverse analysis, the Neumann data and internal measurements with errors are used as follows:

$$Q^\delta(t) = Q(t - \delta \cdot T) + \delta Q(t),$$

$$u_i(x, t) = u_i(x, t - \delta \cdot T) + \delta u_i(t).$$

Here, the j -th elements of $\delta Q(t)$ and $\delta u_i(t)$ are given errors according to normal distributions with variances of $(\delta \cdot \|Q\|_\infty)^2$ and $(\delta \cdot \|u_i(t)\|_\infty)^2$, respectively, where $\|\cdot\|_\infty$ is the maximum norm. We suppose the mean values of the random errors are set to zero and $\delta = 1.0 \times 10^{-2}$. In these numerical experiments, we check the influence the choice of the parameter $(\alpha_\theta, \beta_\theta, \gamma_\theta)$ in the bilinear form $a_\theta(s_\theta, \phi)$ on the identified coefficient $k(\theta)$. Fig. 2 shows the identified coefficient k with $\alpha_\theta = 1.0 \times 10^{-6}$, $\beta_\theta = 2.0 \times 10^{-3}$, and $\gamma_\theta = 1.0$. The identified k is in good agreement with the exact value for the undamaged area. The accuracy of the identified result for the damaged area is less accurate, however it is acceptable and stable for the measured data with random errors. The approximated coefficient with $\alpha_\theta = 1.0 \times 10^{-6}$, $\beta_\theta = 2.0 \times 10^{-3}$, and $\gamma_\theta = 1.0 \times 10^{-2}$ is obtained as shown in Fig. 3. The reconstruction of the coefficient k is less accurate, therefore, the identified result depends on the choice of parameters in (7). Fig. 4 shows the identified result for $\alpha_\theta = 1.0 \times 10^{-8}$, $\beta_\theta = 2.0 \times 10^{-5}$, and $\gamma_\theta = 1.0 \times 10^{-2}$. The accuracy of this result is similar to that shown in Fig. 2. Therefore, it is evident that the ratios of the parameters are more important than the choice of size.

From these results, the effectiveness and potential of our method is determined. Our future study will present the strategy used for the choice of the parameters in (7) to obtain stable and highly accurate result. Moreover the mathematical background of our method will be considered.

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Fig. 2. Identified coefficient $k(\theta)$ ($\alpha_\theta = 1.0 \times 10^{-6}$, $\beta_\theta = 2.0 \times 10^{-3}$, $\gamma_\theta = 1.0$).

Fig. 3. Identified coefficient $k(\theta)$ ($\alpha_\theta = 1.0 \times 10^{-6}$, $\beta_\theta = 2.0 \times 10^{-3}$, $\gamma_\theta = 1.0 \times 10^{-2}$).

Fig. 4. Identified coefficient $k(\theta)$ ($\alpha_\theta = 1.0 \times 10^{-8}$, $\beta_\theta = 2.0 \times 10^{-5}$, $\gamma_\theta = 1.0 \times 10^{-2}$).

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