On the size of determinants in the class number formulae of cyclotomic function fields

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Abstract

We discuss the size of the determinants, which appear in the determinant formulae of the relative class numbers of cyclotomic function fields. These are the determinants of integer symmetric matrices, whose entries are 0 or 1. We show that, for a smaller characteristic, the determinants are significantly large (in the absolute value) compared to the determinants of randomly generated such matrices, while for a larger characteristic, it is not the case. We explain why this happens by comparing some upper bounds.

Keywords cyclotomic function field, class number, determinant formula

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1. Introduction

In this article, we discuss the size of the determinants of specific 0/1 (or ± 1) matrices, which appear in the determinant formula of the relative class numbers of cyclotomic function fields. This determinant formula was proved by Jung-Ahn [1, 2].

We show that, for a small characteristic, the determinant formula for cyclotomic function fields produces 0/1 (or ± 1) matrices, which have extraordinary huge determinants compared to the random 0/1 (or ± 1) matrices. On the contrary, for a larger characteristic, the matrices in the determinant formula are not ones of significant determinant in the random matrices.

We can explain these phenomena by upper bounds of the relative class numbers of cyclotomic function fields, which are a consequence of some algebraic class number formulae and the rationality of the zeta functions proved by A. Weil.

2. Background and notation

For a cyclotomic field $\mathbb{Q}(\mu_n)$, the determinant formulae of the relative class numbers are well known. One of them is due to Hazama [3]. His formula expresses the relative class numbers of the p -th cyclotomic fields as the determinants of ± 1 matrices. T. Taniguchi (Kanazawa Institute of Technology) found that the determinants of ± 1 matrices, which appear in the determinant formula, are extraordinarily huge in the family of random ± 1 matrices of the same size.

We first expected that we were able to show that the same holds for the cyclotomic function fields case. But our hope comes true only for function fields of small characteristics, not for those of larger characteristics. We provide the reason why the determinant of the 0/1 ma-

trices, which appear in the determinant formulae, varies.

First, we recall some notion, especially on the arithmetic of cyclotomic function fields. The interested readers are referred to Rosen [4, Chapter 12].

For a prime number p , let $\mathbb{F}$ be the finite field of q -elements (q is a power of p). We denote $A = \mathbb{F}[T]$ the ring of polynomials in an indeterminate T over $\mathbb{F}$, $k = \mathbb{F}(T)$ the field of fractions of A . Let ρ be a Carlitz module, which is an $\mathbb{F}$ -algebra homomorphism defined by the following:

$$\rho: A \ni T \mapsto \phi + \mu \in \text{End}_{\mathbb{F}}(\mathbb{G}_{a,k}),$$

where the right-hand side is the endomorphism ring of the additive group $\mathbb{G}_{a,k}$ over k , $\phi(x) = x^q$ is the Frobenius endomorphism, and μ is the multiplication by T endomorphism. It is well known that for each $m = m(T) \in A$, $\rho(m)(x)$ is a separable polynomial with A -coefficients. We fix, once and for all, a separable closure k^s of k . Let Λ_m be the m -th division points of ρ , namely,

$$\Lambda_m = \{z \in k^s \mid \rho(m)(z) = 0\}.$$

We denote by $K_m = k(\Lambda_m)$ the field generated by Λ_m over k , and call it *the m -th cyclotomic function field*.

For example, $K_T = k(\Lambda_T)$, where $\Lambda_T = \{z \in k^s \mid (\phi + \mu)(z) = 0\}$. Since $(\phi + \mu)(z) = z^q + Tz$, $K_T = k(\sqrt[q]{-T})$.

An analogy between cyclotomic fields and cyclotomic function fields is obvious. The ring $\mathbb{Z}$ of rational integers is considered as the subring of the endomorphism ring of the multiplicative group $\mathbb{G}_{m,\mathbb{Q}}$ over the field $\mathbb{Q}$ of rationals via

$$\psi: \mathbb{Z} \ni n \rightarrow (x \mapsto x^n) \in \text{End}(\mathbb{G}_{m,\mathbb{Q}}),$$

we find that the group μ_n of the n -th roots of unity is

$$\mu_n = \{z \in \overline{\mathbb{Q}}^\times \mid \psi(n)(z) = z^n = 1\}.$$

Then, the n -th cyclotomic field is $\mathbb{Q}(\mu_n)$.

The m -th cyclotomic function field K_m satisfies many properties similar to the n -th cyclotomic field $\mathbb{Q}(\mu_n)$. For example, K_m/k is a finite Abelian extension with the Galois group $\text{Gal}(K_m/k) \cong (A/mA)^\times$. This extension is unramified outside m . There is also an analog of the maximal real subfield, namely, the subfield corresponds to the subgroup $(\mathbb{F}/mA)^\times$ of $\text{Gal}(K_m/k)$. We denote it by K_m^+ , and call it the maximal real subfield of K_m .

Let O_m and O_m^+ be the rings of integers of K_m and K_m^+ , respectively, which are the integral closures of A in K_m and K_m^+ . These are Dedekind domains.

We denote the class numbers (the order of the groups of divisor classes of degree zero) of K_m and K_m^+ by h_m and h_m^+ . It is well known that $h_m^+ \mid h_m$. We denote by h_m^- the quotient and call it *the relative class number* of K_m . We also denote the class numbers (the order of the Picard groups) of O_m and O_m^+ by $\tilde{h}_m$ and $\tilde{h}_m^+$ respectively. In this case, it is also known that $\tilde{h}_m^+ \mid \tilde{h}_m$. We denote by $\tilde{h}_m^-$ the quotient $\tilde{h}_m/\tilde{h}_m^+$, and call it *the relative class number* of O_m .

3. Determinant formulae for the relative class numbers of cyclotomic function fields

Jung and Ahn [1, 2] showed that the determinant formula of h_m^- and $\tilde{h}_m^-$. We recall their results. Let $p(T) \in A$ be a monic irreducible polynomial of degree d . For $n \geq 1$, let $m = p(T)^n$. We set

$$\mathbb{M}_m = \{a \in A \mid a \neq 0, \deg(a) < \deg(m), (a, m) = 1\},$$

$$\mathbb{M}_m^+ = \{a \in \mathbb{M}_m \mid a : \text{monic}\},$$

$$\mathbb{M}_m^- = \mathbb{M}_m \setminus \mathbb{M}_m^+.$$

For $a \in A$, let $\bar{a} \in \mathbb{M}_m$ be the unique element that satisfies $a \equiv \bar{a} \pmod{m}$. Let $\text{sgn}(a)$ be the leading coefficient of a , and $\text{sgn}_m(a) := \text{sgn}(\bar{a})$. For $a \in \mathbb{M}_m$, set $\bar{a} = a/\text{sgn}_m(a) \in \mathbb{M}_m^+$.

For $a, b \in \mathbb{M}_m$, we set $\langle ab \rangle$ by

$$\langle ab \rangle := \begin{cases} 1 & (\text{sgn}_m(ab) = 1), \\ 0 & (\text{sgn}_m(ab) \neq 1). \end{cases}$$

Then Jung-Ahn's class number formula states that [2, Propositions 2.5, 2.6]:

$$h_m^- = |\det(\langle ab \rangle - \langle a\bar{b} \rangle)_{a,b \in \mathbb{M}_m^-}|, \quad (1)$$

$$\tilde{h}_m^- = |\det(\langle ab \rangle)_{a,b \in \mathbb{M}_m^-}|. \quad (2)$$

We remark that the size of these matrices are given by the following:

$$\#\mathbb{M}_m^- = \varphi(m) \left(1 - \frac{1}{q-1}\right),$$

where $\varphi(m)$ is Euler's totient function for polynomials. In this case $m = p(T)^n$, $\varphi(m) = q^{(n-1)d}(q^d - 1)$. The matrix in (1) is a 0, ± 1 matrix, while the matrix in (2) is

a 0/1 symmetric matrix. It is known, by Yin [5, Lemma 3] that the relation between h_m^- and $\tilde{h}_m^-$:

$$h_m^- = (q-1)^s \tilde{h}_m^-, \quad (3)$$

where $s = q^{(n-1)d}(q^d - 1)/(q - 1) - 1$.

4. Note on 0/1 matrices and ± 1 matrices

As we have just remarked, $\tilde{h}_m^-$ is given by the determinant of a 0/1 symmetric matrix. We can rewrite it into a ± 1 matrix by simple transformation: Set $r = \#\mathbb{M}_m^- = \varphi(m)[1 - 1/(q-1)]$, and $H = (h_{a,b})_{a,b \in \mathbb{M}_m^-}$ as follows:

$$h_{a,b} = \begin{cases} -1 & (\text{sgn}_m(ab) = 1), \\ 1 & (\text{sgn}_m(ab) \neq 1). \end{cases}$$

Then we have

$$2^r \tilde{h}_m^- = \pm \det \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & H & \vdots \\ 1 & \cdots & 1 \end{pmatrix}. \quad (4)$$

We consider mainly 0/1 matrices below.

5. Inequality for relative class numbers

It is well-known that the absolute value of the determinant of a complex matrix is bounded by the product of the norms of all the column vectors. For any complex square matrix $M = (m_{ij})$ of size r , let $\mathbf{m}_j$ be the j -th column vector. Let $|\mathbf{m}_j|$ be its Euclidean norm. Then we have Hadamard's inequality:

$$|\det(M)| \leq \prod_{j=1}^r |\mathbf{m}_j|. \quad (5)$$

If M has bounded elements $|m_{ij}| \leq 1$ ($1 \leq i, j \leq r$), then we have

$$|\det(M)| \leq r^{\frac{r}{2}}.$$

Applying this to (4) with $m = p(T)$, we have:

Proposition 1.

$$\tilde{h}_m^- \leq \frac{\left[\frac{(q-2)(q^d-1)}{q-1} + 1 \right]^{\frac{(q-2)(q^d-1)+q-1}{2(q-1)}}}{2^{\frac{(q-2)(q^d-1)}{q-1}}}. \quad (6)$$

6. Numerical observation

We conducted some numerical observations to measure how close the determinant expression in (2) to the right-hand side of (6). Unless otherwise stated, we assume that m is an irreducible monic polynomial ($n = 1$) hereafter.

We observed that

- when $q = 3$, $\tilde{h}_m^-$ is close to Hadamard's bound (this is a similar phenomenon to the cyclotomic field case). Cf. Table 1,
- when $q > 3$, $\tilde{h}_m^-$ is no longer close to Hadamard's bound. Cf. Table 2.

We tabulate parts of these numerical observations: When $q = 3$, $\tilde{h}_m^-$ tend to huge (Table 1).

On the other hand, when $q > 3$, $\tilde{h}_m^-$ is no longer huge (Table 2).

Table 1. Comparison of decimal digits of $\tilde{h}_m^-$ and the right-hand side of (6) bound ($q = 3$).

m	the digits of $\tilde{h}_m^-$	digits of RHS of (6)
$T^4 + T + 2$	17	22
$T^5 + 2T + 1$	80	91
$T^6 + T^3 + T + 2$	325	359

 Table 2. Comparison of decimal digits of $\tilde{h}_m^-$ and the right-hand side of (6) ($q = 5, 7, 29$).

q	m	digits of $\tilde{h}_m^-$	digits of RHS of (6)
5	$T^4 + 2$	398	486
7	$T^3 + 3T + 5$	198	266
29	$T^2 + T + 20$	551	936

 Fig. 1. The size of matrices: 40. The dot is the digits of $\tilde{h}_m^-$, $q = 3$, $m = T^4 + T^3 + 2T + 1$.

 Fig. 2. The size of matrices: 40. The dot is the digits of $\tilde{h}_m^-$, $q = 7$, $m = T^2 + 6T + 3$.

We calculate 100,000 samples of randomly generated symmetric 0/1 matrices of the same size as in the determinant formula (2). This computation had been performed by our program written in C using Pari library [6]. We observed that

- when $q = 3$, $\tilde{h}_m^-$ is quite huge than the determinants of randomly generated 0/1 symmetric matrices (cf. Fig. 1),
- when $q > 3$, $\tilde{h}_m^-$ is no longer larger than the determinants of randomly generated 0/1 symmetric matrices or is smaller than them (cf. Figs. 2 and 3).

 Fig. 3. The size of matrices: 108. The dot is the digits of $\tilde{h}_m^-$, $q = 11$, $m = T^2 + 7T + 2$.

7. Investigation of numerical observation

Rosen [7, Theorem 3] showed the following bound of the relative class number h_m^- of K_m :

$$h_m^- \leq \left(\frac{q^d - 1}{q - 1} \right)^{\frac{(q-2)(q^d-1)}{2(q-1)}}.$$

This bound is derived from the explicit forms of the zeta functions of K_m , more precisely, it is derived from the rationality of the zeta functions and the fact that the numerators of zeta functions are polynomials of degree $2g_m$, where g_m is the genus of K_m .

We have, by (3):

Proposition 2.

$$\tilde{h}_m^- \leq \frac{\left(\frac{q^d - 1}{q - 1} \right)^{\frac{(q-2)(q^d-1)}{2(q-1)}}}{(q - 1)^{\frac{q^d-1}{q-1} - 1}}. \quad (7)$$

The relationship between these two upper bounds (6) and (7) is given as follows by elementary calculation:

Theorem.

$$\tilde{h}_m^- \leq \frac{\left(\frac{q^d - 1}{q - 1} \right)^{\frac{(q-2)(q^d-1)}{2(q-1)}}}{(q - 1)^{\frac{q^d-1}{q-1} - 1}} \quad (8)$$

$$\leq \frac{\left[\frac{(q-2)(q^d-1)}{q-1} + 1 \right]^{\frac{(q-2)(q^d-1)+q-1}{2(q-1)}}}{2^{\frac{(q-2)(q^d-1)}{q-1}}}. \quad (9)$$

We remark that, for $q = 3$ in (8), (9), and putting $t = \frac{q^d-1}{q-1}$, we see that

$$\tilde{h}_m^- \leq \frac{2t^{\frac{t}{2}}}{2^t} \leq \frac{(t+1)^{\frac{t+1}{2}}}{2^t},$$

which shows that these two upper bounds are close in this case, and explains Table 1 and Fig. 1.

On the other hand, for $q > 3$, the expressions in (8) and (9) are very different. This explains that $\tilde{h}_m^-$ is far less than the upper bound (9), which is given by Hadamard's bound.

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