Generalizations of Ho-Lee's binomial interest rate model

II: randomization

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Received March 10, 2020, Accepted July 29, 2020

Abstract

In the present paper, we introduce a generalization of Ho-Lee's binary tree interest rate model, which can be regarded as a Bayesian type extension, by combining the technique used by Akahori-Aoki-Nagata with a randomization by a Polya urn type reinforcement. Moreover, a multinomial extension of the generalization of the Ho-Lee model is also discussed.

Keywords term structure of interest rates, discrete model, Bayesian interpretation

Research Activity Group Mathematical Finance

1. Introduction

Modelling of term structure of interest rates has been a central issue in mathematical finance. The spot rate approach by Vasicek [1] and Cox, Ingersol and Ross [2] has been extended to multi-factor affine class approach (c.f. Duffie and Kan [3], Duffie, Filipović and Schachermayer [4], and Filipovic and Teichmann [5]), on one hand, the potential or state-price density approach discussed by Flesaker and Hughston [6], Rogers [7], and by more recent Akahori, Hishida, Teichmann and Tsuchiya [8], on the other. The yield curve approach by Heath, Jarrow and Morton [9], HJM for short, which is inspired by Ho-Lee's binomial tree interest rate model [10], has been also a major way to model term structure of interest rates, Brace-Gatrek-Musiela's LIBOR model [11] as a one of the most successful successor.

Since the term structure of interest rates in continuous time is naturally of infinite dimensional, its discrete analogue as well as its discretization becomes quite problematic.

In the spot rate approach or its generalization, one can work on discretization of a continuous-time Markov process by a Markov chain, but in many cases, the state space of the Markov chain cannot be a proper lattice. For the yield curve approach, Ho-Lee model is the right discrete-time analogue, but in its original form the Ho-Lee model can express only a constant (term structure of) volatility. Akahori, Aoki and Nagata [12] proposed a generalization, combining state-price density approach. The paper [12] was titled "PART I", and the present paper is the PART II after a long silence, introducing a Bayesian type generalization of Ho-Lee model, employing "randomization" by a reinforcement scheme of the Polya urn type. We may regard our scheme as *Bayesian* approach.

In the approach, the term structure of interest rates is still described by a Markov chain on a lattice or so-called multi-nomial tree, and it can be considered as a

parameter-randomization of a classical model.

The present paper is organized as follows. In Section 2 we recall the mathematical structure of the reinforcement scheme of Polya urn type by looking at a simple situation. In Section 3, after recalling the generalization of Ho-Lee model by Akahori, Aoki and Nagata [12], we introduce its randomization. In Section 4, we introduce the multi-nomial randomization of Ho-Lee model. In Section 5, we conclude the paper with comments on our on-going studies.

2. Pólya urn process

Let $(R_t, W_t)_{t \in \mathbb{Z}_+}$ be a Markov chain on $\mathbb{N}^2$ defined by

$$\begin{aligned} P\left((R_{t+1}, W_{t+1}) = (R_t + 1, W_t) \mid R_t, W_t\right) \\ = 1 - P\left((R_{t+1}, W_{t+1}) = (R_t, W_t + 1) \mid R_t, W_t\right) \\ = \frac{R_t}{R_t + W_t}, \quad t \in \mathbb{Z}_+. \end{aligned} \quad (1)$$

It is normally called *Pólya urn process*: if one picks up a red ball at t -th experiment, then she put two balls into the urn, and the probability to pick a red ball is its ratio in the urn, as usual.

One can easily see from the urn picture that the increments $(\Delta R_t := R_{t+1} - R_t)_{t \in \mathbb{Z}_+}$, taking values in $\{0, 1\}$, satisfy

$$(\Delta R_1, \dots, \Delta R_n) \stackrel{d}{=} (\Delta R_{\sigma(1)}, \dots, \Delta R_{\sigma(n)}), \quad (2)$$

for any permutation $\sigma \in \mathfrak{S}_n$. In other words, they form

an *exchangeable* family. Since for any $k \in \{0, \dots, n\}$,

$$\begin{aligned} & P(\Delta R_{\sigma(1)} = 1, \dots, \Delta R_{\sigma(k)} = 1, \\ & \quad \Delta R_{\sigma(k+1)} = 0, \dots, \Delta R_{\sigma(n)} = 0 \mid R_0, W_0) \\ &= P(\Delta R_1 = 1, \dots, \Delta R_k = 1, \\ & \quad \Delta R_{k+1} = 0, \dots, \Delta R_n = 0 \mid R_0, W_0), \end{aligned}$$

for any $\sigma \in \mathfrak{S}_n$, we have

$$\begin{aligned} & P(R_t = k \mid R_0, W_0) \\ &= \binom{n}{k} P(\Delta R_1 = 1, \dots, \Delta R_k = 1, \\ & \quad \Delta R_{k+1} = 0, \dots, \Delta R_n = 0 \mid R_0, W_0), \end{aligned}$$

which is, by a repeated use of (1),

$$\begin{aligned} &= \binom{n}{k} \frac{R_0}{R_0 + W_0} \cdots \frac{R_0 + k - 1}{R_0 + W_0 + k - 1} \\ & \quad \times \frac{W_0}{R_0 + W_0 + k} \cdots \frac{W_0 + n - k - 1}{R_0 + W_0 + n - 1}. \end{aligned} \quad (3)$$

We then notice that the product in (3) can be rewritten by factorials, then, Gamma functions, as

$$\begin{aligned} &= \binom{n}{k} \frac{\frac{(R_0+k-1)!}{(R_0-1)!} \frac{(W_0+n-k-1)!}{(W_0-1)!}}{\frac{(R_0+W_0+n-1)!}{(R_0+W_0-1)!}} \\ &= \binom{n}{k} \frac{\frac{\Gamma(R_0+k)}{\Gamma(R_0)} \frac{\Gamma(W_0+n-k)}{\Gamma(W_0)}}{\frac{\Gamma(R_0+W_0+n)}{\Gamma(R_0+W_0)}}, \end{aligned}$$

and by Beta functions as

$$= \binom{n}{k} \frac{B(R_0 + k, W_0 + n - k)}{B(R_0, W_0)}. \quad (4)$$

The story continues if we write the Beta function in numerator in (4) by the integral form. We now arrive at the following formula:

$$\begin{aligned} & P(R_t = k \mid R_0, W_0) \\ &= \frac{1}{B(R_0, W_0)} \int_0^1 \binom{t}{k} x^{R_0+k-1} (1-x)^{W_0+t-k-1} dx. \end{aligned} \quad (5)$$

The formula (5) has an interesting, or some may say, *magical*, interpretation: *Pólya urn process is equivalent to a simple random walk whose probability to move 1 (otherwise doesn't move) at each step is an initially chosen Beta(R_0, W_0) distributed random variable*. Just separate the integrand as

$$\binom{t}{k} x^k (1-x)^{t-k} \times x^{R_0-1} (1-x)^{W_0-1}. \quad (6)$$

The interpretation can be something more in the context of Bayesian statistics, where unknown parameters are set to be random variables. In Bayesian inference, one starts with a belief on the distribution of the parameter in question, called *prior* distribution. In our case, it is Beta(R_0, W_0). After the experiments, one will change the belief to a new one called *posterior*, according as the

Bayesian formula:

$$\text{posterior} \propto \text{prior} \times \text{likelihood function}. \quad (7)$$

In this case, the likelihood function is the probability mass at k of Binary(t, x). Therefore, the right-hand-side of (7) is nothing but (6), meaning that the probability mass at k of Pólya urn process is equal to that of the posterior distribution under the observation that k red balls are picked out of n trials, of Bernoulli(x).

The mathematical secret behind the above story is *de Finetti's theorem*, which asserts that an exchangeable family of random variables can be made conditionally i.i.d.. Pólya urn process is an easiest example to which the *magical* randomization argument applies.

3. Ho-Lee model and its randomization

As discussed in Introduction, the Ho-Lee model [10] is recognized as a first *yield curve* model, where the curve is described by recombining binary tree. In Akahori, Aoki, and Nagata [12], it is proven that the model can be realized by so-called *state price density approach*, by choosing state price density D as

$$D_t = \exp(a_t X_t), \quad t \in \mathbb{Z}_+,$$

where $a : \mathbb{Z}_+ \rightarrow \mathbb{R}$ is a parameter that corresponds to *term structure of volatility*, and X is a random walk such that

$$\begin{aligned} & P(X_{t+1} - X_t = 1 \mid X_t) \\ &= 1 - P(X_{t+1} - X_t = 0 \mid X_t) =: q, \end{aligned}$$

which is also a parameter. The price at t of zero-coupon bond of maturity T , denoted by $P(t, T)$, under Ho-Lee model is given by

$$\begin{aligned} & P(t, T) \\ &= D_t^{-1} E[D_T \mid X_t] \\ &= e^{(a_T - a_t)X_t} \times E[e^{a_T(X_T - X_t)}] \\ &= e^{(a_T - a_t)X_t} \sum_{k=0}^{T-t} \binom{T-t}{k} e^{a_T k} q^k (1-q)^{T-t-k} \\ &= e^{(a_T - a_t)X_t} (e^{a_T} q + 1 - q)^{T-t}. \end{aligned} \quad (8)$$

Our first result states that the randomization of q to Beta(R, W) in the final expression in (8) still defines an arbitrage-free interest rate model. We keep the notations and the definitions in the previous section.

Proposition 1 *The bond market given by*

$$\begin{aligned} & P(t, T) = \frac{e^{(a_T - a_t)R_t}}{B(R_t, W_t)} \int_0^1 (e^{a_T} x + 1 - x)^{T-t} \\ & \quad \times x^{R_t-1} (1-x)^{W_t-1} dx, \quad t \leq T \in \mathbb{N} \end{aligned} \quad (9)$$

is arbitrage-free under the natural filtration of (R, W) .

Proof It will be proven as a special case of Theorem 2 below.

(QED)

Note that even when a_T is independent of T , the randomized Ho-Lee model can describe a non-trivial term

structure, which is in sharp contrast to the classical Ho-Lee model.

The forward rates from T to $T + 1$ for $T = t, t + 1, \dots$ are given by

$$f(t, T) := \log \frac{P(t, T)}{P(t, T + 1)}. \quad (10)$$

In the classical Ho-Lee model (8), the forward rate f is always a simple random walk for each T . In fact, by substituting (8) into (10), we see that

$$\begin{aligned} f(t, T) &= (a_T - a_{T+1})X_t + t \log \frac{e^{a_{T+1}q + 1 - q}}{e^{a_T q + 1 - q}} \\ &\quad + \log \frac{(e^{a_T q + 1 - q})^T}{(e^{a_{T+1}q + 1 - q})^{T+1}}, \end{aligned}$$

and thus

$$\begin{aligned} f(t, T) &= f(0, T) + t \log \frac{e^{a_{T+1}q + 1 - q}}{e^{a_T q + 1 - q}} \\ &\quad + (a_T - a_{T+1})(X_t - X_0). \end{aligned} \quad (11)$$

On the other hand, the “randomized” zero-coupon bond (9) allows a richer shapes. To simulate, we use the following approximation:

$$\begin{aligned} f(t, T) &\simeq (a_T - a_{T+1})R_t \\ &\quad + \log \frac{B(R_t, W_t) + (T - t)a_TB(R_t + 1, W_t)}{B(R_t, W_t) + (T + 1 - t)a_{T+1}B(R_t + 1, W_t)}, \end{aligned} \quad (12)$$

where $(\tilde{R}, \tilde{W})$ is a copy of the Polya Urn process defined by (1). The approximation is based on the following observation:

$$\begin{aligned} E \left[e^{a_T \tilde{R}_{T-t}} \mid \tilde{R}_0 = R_t, \tilde{W}_0 = W_t \right] \\ \simeq 1 + a_T E \left[\tilde{R}_{T-t} \mid \tilde{R}_0 = R_t, \tilde{W}_0 = W_t \right] \\ = 1 + a_T (T - t) \frac{B(R_t + 1, W_t)}{B(R_t, W_t)}. \end{aligned}$$

Here we give some illustrative examples to show the richness, but only through 3D graphs of Figs. 1 and 2. Though the parameter a is the same and the expectation of the randomized q is equal to the non-random one, the shapes of the sample paths of the term structure are quite different.

Note that while the classical Ho-Lee model can express only such a “regular” sample path in the space of the curves, the randomized one can potentially express more variety of curves, including the “irregular” path in Fig. 2 expressing a “bump”, often observed in reality. Moreover, the randomized Ho-Lee model can give a Bayesian interpretation of the appearance of such a bump as a reflection of singular beliefs resulted from singular events.

4. Multinomial extension

The Bayesian interpretation of Pólya urn process, which we have described in Section 2, can be imported

Fig. 1. A sample path of the forward rate (11) of classical Ho-Lee model with $a_t = -0.03 \log(t + 0.3) \cos 1.21(t + 2.3)$ and $q = 13/14$.

Fig. 2. A sample path of the forward rate (12) of “randomized” Ho-Lee model with $a_t = -0.03 \log(t + 0.3) \cos 1.21(t + 2.3)$ and $R_0/(R_0 + W_0) = 13/14$.

to the randomized Ho-Lee model given in Proposition 1 as follows: the traders in the bond market agree with,

- the bond price is given by the formula (8),
- the value of a as a deterministic function,
- the value of $\{(R_s, W_s)\}_{s \leq t}$ at each time t ,

but they do not have a solid agreement with the value of q , which they treat as a random variable. The market equilibrium is given by Bayesian estimation of q , with prior distribution at t is $B(R_t, W_t)$.

Proposition 1 now reads: the bond price formed by the above agreements is arbitrage-free.

This argument, however, is not fully consistent in that observations of the term structure $T \mapsto P(\cdot, T)$ could give a better estimation of q . In fact, in practice, assuming the Ho-Lee model (8), all the parameters can be calibrated from the observations of the term structure, and no Bayesian estimate is needed. Of course this procedure will naturally result in over-fitting. To be consistent with these two points of view, we will introduce, in this section, a multi-nomial extension of the one given in Section 3. Let $\mathfrak{R}_t := (R_t^1, \dots, R_t^N)$, $t = 0, 1, \dots$, be a

Markov chain such that

$$P(R_{t+1}^i = R_t^i + 1 \mid \mathfrak{R}_t) = \frac{R_t^i}{\sum_{j=1}^N R_t^j},$$

$$i = 1, \dots, N.$$

Then, by a similar argument as we did for Polya urn in Section 2, for $(k_1, \dots, k_N)$ such that $k_1 + \dots + k_N = t$,

$$\begin{aligned} P(R_t^1 = k_1, \dots, R_t^N = k_N \mid \mathfrak{R}_0 = \vec{r}) \\ = \mathbf{B}(\vec{r})^{-1} \\ \times \int_{T^N} \frac{t!}{k_1! \dots k_N!} q_1^{r_1-1+k_1} \dots q_N^{r_N-1+k_N} dq_1 \dots dq_N, \end{aligned}$$

where

$$T^N := \{(x^1, \dots, x^N) \in [0, 1]^N : x^1 + \dots + x^N = 1\},$$

$$\vec{r} = (r_1, \dots, r_N) \in \mathbb{Z}^N,$$

and $\mathbf{B}$ is the multivariate beta function, namely

$$\begin{aligned} \mathbf{B}(\alpha_1, \dots, \alpha_N) &:= \int_{T^N} \prod_{j=1}^N x_j^{\alpha_j-1} dx_j \\ &= \frac{\prod_{j=1}^N \Gamma(\alpha_j)}{\Gamma(\sum_{j=1}^N \alpha_j)}. \end{aligned}$$

Theorem 2 *The bond market given by*

$$\begin{aligned} P(t, T) &= e^{\sum_{i=1}^N (a_T^i - a_t^i) R_t^i} \\ &\times \int_{T^N} (e^{a_T^1} q_1 + \dots + e^{a_T^N} q_N)^{T-t} \mu_{\mathfrak{R}_t}(dq_1 \dots dq_N) \end{aligned} \quad (13)$$

is arbitrage-free, where

$$\mu_{(r_1, \dots, r_N)}(dq_1 \dots dq_N) := \prod_{j=1}^N q_j^{r_j-1} dq_j.$$

Proof We will show that the right-hand-side of (13) is equal to $D_t^{-1} E[D_T \mid \mathfrak{R}_t]$ with $D_t = e^{\sum_{i=1}^N a_t^i R_t^i}$. With this choice, we have

$$\begin{aligned} D_t^{-1} E[D_T \mid \mathfrak{R}_t] \\ &= e^{-\sum_{i=1}^N a_t^i R_t^i} \cdot E \left[e^{\sum_{i=1}^N a_T^i R_T^i} \mid \mathfrak{R}_t \right] \\ &= e^{\sum_{i=1}^N (a_T^i - a_t^i) R_t^i} \cdot E \left[e^{\sum_{i=1}^N a_T^i (R_T^i - R_t^i)} \mid \mathfrak{R}_t \right], \end{aligned}$$

and

$$\begin{aligned} E \left[e^{\sum_{i=1}^N a_T^i (R_T^i - R_t^i)} \mid \mathfrak{R}_t \right] \\ &= \sum_{\substack{k_1 + \dots + k_N = T-t \\ k_1 \geq 0, \dots, k_N \geq 0}} e^{\sum_{i=1}^N a_T^i k_i} \\ &\quad \times P(R_T^i - R_t^i = k_i, i = 1, \dots, N \mid \mathfrak{R}_t) \\ &= \sum_{\substack{k_1 + \dots + k_N = T-t \\ k_1 \geq 0, \dots, k_N \geq 0}} e^{\sum_{i=1}^N a_T^i k_i} \mathbf{B}(\mathfrak{R}_t)^{-1} \frac{(T-t)!}{k_1! \dots k_N!} \\ &\quad \times \int_{T^N} q_1^{R_t^1-1+k_1} \dots q_N^{R_t^N-1+k_N} dq_1 \dots dq_N \\ &= \int_{T^N} (e^{a_T^1} q_1 + \dots + e^{a_T^N} q_N)^{T-t} \mu_{\mathfrak{R}_t}(dq_1 \dots dq_N). \end{aligned} \quad (\text{QED})$$

5. Concluding remark

This paper introduced a randomization of multinomial extension of Ho-Lee model by Akahori, Aoki and Nagata. Though we have commented that the extension is necessary for the Bayesian interpretation to work, we did not give a rigorous mathematical formulation of the Bayesian interpretation. Actually, we need also the randomization of $T \mapsto a$, the term structure of volatility to make it rigorous. This will be done in the forthcoming paper by J. Akahori, Y. Chiba, Y. Semba and Y. Ryu.

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