On the uniqueness of stable anisotropic capillary hypersurfaces that contact the edge of a wedge

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Abstract

Anisotropic energies serve to model the boundaries of some types of crystals and liquid crystals. We study the stability of capillary hypersurfaces with anisotropic energy in a wedge, which contact with the edge of the wedge. We prove that stable capillary hypersurfaces in a wedge are, up to rescaling, only parts of the Wulff shape under some boundary conditions. Here the Wulff shape is the model of a single crystal introduced by G. Wulff.

Keywords anisotropic energy, capillary surface, crystal, Wulff shape

Research Activity Group Geometric Shape Generation

1. Introduction

We study equilibrium hypersurfaces in the $(n+1)$ -dimensional Euclidean space $\mathbb{R}^{n+1}$ for anisotropic surface energies. At first, we introduce anisotropic energies and the Wulff shape. Let γ be a positive continuous function on the n -dimensional unit sphere S^n and let $X : M \rightarrow \mathbb{R}^{n+1}$ be an immersion from an n -dimensional oriented connected compact C^∞ manifold M with $\partial M \neq \emptyset$ into $\mathbb{R}^{n+1}$. Let ν be the Gauss map of X , that is, ν is the unit normal vector field along X . The anisotropic energy of X is defined as $\mathcal{F}_\gamma(X) := \int_M \gamma(\nu) dA$, where dA is the n -dimensional volume form of M induced by X . This energy was introduced by J. W. Gibbs in order to model the shape of small crystals. It is known that, for any positive number $V > 0$, among all closed hypersurfaces as above enclosing the same $(n+1)$ -dimensional volume V , there exists a unique (up to translation in $\mathbb{R}^{n+1}$) minimizer $W_\gamma(V)$ of $\mathcal{F}_\gamma$ [1]. The minimizer $W_\gamma = W_\gamma(V_0)$ for the specific value $V_0 := (n+1)^{-1} \int_{S^n} \gamma(\nu) dS^n$ is called the Wulff shape for γ . The Wulff shape is a model of small crystals introduced by G. Wulff [2]. When $\gamma \equiv 1$, $\mathcal{F}_\gamma$ is the area of X and W_γ is the unit sphere. In general, the Wulff shape is not smooth, but if γ is of class C^2 and has a certain convexity (that is, $n \times n$ matrix $D^2\gamma + \gamma I_n$ is positive definite at each point on S^n , denoted by $D^2\gamma + \gamma I_n > 0$, where $D^2\gamma$ denotes the Hessian of γ on S^n and I_n is the identity map on $T_x S^n$ ($\forall x \in S^n$)), then W_γ is a smooth strictly convex closed hypersurface.

Recently, variational problems for hypersurfaces with anisotropic energy have been studied by many authors. Concerning the uniqueness of stable solutions, B. Palmer showed that stable, connected, oriented, closed equilibrium hypersurfaces are (up to translations and homothety) the Wulff shape [3] (stable closed hypersurface is defined similarly to the definition of stable anisotropic hypersurface which will be given in Definition 3). In this

Fig. 1. A surface $X(M)$ in a wedge type domain Ω

paper, we study a free boundary problem for hypersurfaces with anisotropic energy whose supporting surfaces form a wedge shape. The hypersurfaces we study are as follows (Fig 1). Let $\Pi_1, \Pi_2 \subset \mathbb{R}^{n+1}$ be n -dimensional closed half hyperplanes which are not parallel to each other and $\partial\Pi_1 = \partial\Pi_2$, and we consider the wedge-like domain $\Omega \subset \mathbb{R}^{n+1}$ whose boundary is $\Pi_1 \cup \Pi_2$. Let N_i be the unit normal vector field on Π_i outward from Ω ($i = 1, 2$). Note that the pair N_1, N_2 is linearly independent. We call $\Pi_0 := \Pi_1 \cap \Pi_2$ the edge of the wedge. Let $X : M \rightarrow \mathbb{R}^{n+1}$ be an immersion from an n -dimensional oriented connected compact C^∞ manifold M with $\partial M \neq \emptyset$ into Ω whose supporting surface is $\Pi_1 \cup \Pi_2$ and satisfying $\partial X(M) \cap \Pi_i \neq \emptyset$ ($i = 1, 2$). We assume that ∂M is homeomorphic to S^{n-1} and $X(M) \cap \Pi_0 \neq \emptyset$. Furthermore, we assume that the restriction of X to ∂M is an embedding. Let $C_i := \partial X(M) \cap \Pi_i$ ($i = 1, 2$) and $\sigma_i := X^{-1}(C_i)$ ($i = 1, 2$). And set $\sigma_0 := \sigma_1 \cap \sigma_2$ and $C_0 \subset \Pi_0$ to be the $(n-1)$ -dimensional domain bounded by $X(\sigma_0)$. Let $D_i \subset \Pi_i$ be the n -dimensional domain bounded by C_i and C_0 ($i = 1, 2$). We denote by $V(X)$ the volume bounded by $X(M)$ and $\Pi_1 \cup \Pi_2$. For these

 Fig. 3. Construction of the specific admissible variation Y_t

- (ii) The k_i^γ are not real in general. However, H_r^γ are always real.
- (iii) The anisotropic mean curvatures satisfy $\Lambda = 1/n(-\operatorname{div}_M D\gamma + nH)$ where H is the mean curvature of X .

Next, we introduce a useful formula, which is called the Steiner-type formula. This is a generalization of Weyl's tube formula [7].

Fact 8 (Steiner-type formula cf. [6]). Assume that $\gamma : S^n \rightarrow \mathbb{R}_{>0}$ is of class C^2 . Let $X : M \rightarrow \mathbb{R}^{n+1}$ be an immersion from an n -dimensional oriented connected compact manifold M with or without boundary into $\mathbb{R}^{n+1}$. We define the anisotropic parallel hypersurface X_t as

$$X_t := X + t\xi_\gamma : M \rightarrow \mathbb{R}^{n+1} \quad (t \in \mathbb{R}). \quad (8)$$

Then, the anisotropic energy $\mathcal{F}_\gamma(X_t)$ of X_t has the following representation:

$$\mathcal{F}_\gamma(X_t) = \int_M \gamma(\nu) \sum_{r=0}^n (-1)^r t^r ({}_nC_r) H_r^\gamma dA. \quad (9)$$

3. Proof of Theorem 4

3.1 Euler-Lagrange equations

Let $X : M \rightarrow \bar{\Omega}$ be an anisotropic capillary hypersurface in the wedge-like closed domain $\bar{\Omega}$ which does contact with the edge of the wedge defined in the introduction. First, we can prove that the Euler-Lagrange equations of the variational problem for the energy E with free boundary and volume constraint by the same way as the proof of Lemma 1 in [4].

Proposition 9 (Euler-Lagrange equations cf. [4]). X is an anisotropic capillary hypersurface if and only if the following conditions (i) and (ii) hold:

- (i) The anisotropic mean curvature Λ of X is constant on M .
- (ii) $\langle \tilde{\xi}, N_i \rangle = \omega_i$ on C_i ($i = 1, 2$).

These conditions imply that an anisotropic capillary hypersurface is a generalization of the classical capillary hypersurface in the absence of gravity.

3.2 An admissible variation constructed by using parallel variation

In order to prove Theorem 4, as [4], we will compute the second variation of the total energy for a specific ad-

missible variation as follows (Fig 3). First, we consider the anisotropic parallel hypersurface $X_t = X + t\tilde{\xi}$ ($t \in \mathbb{R}$, $|t| \ll 1$). If $\omega_1, \omega_2 > 0$, then X_t does not necessarily satisfy the boundary condition. But, since the condition (ii) in Proposition 9 and the pair N_1, N_2 is linearly independent, we can take a suitable translation $Z_t := X_t + tb$ ($\exists b \in \mathbb{R}^{n+1}$) so that Z_t satisfies the boundary condition. But Z_t is not in general, volume preserving. Hence we take a suitable homothety

$$Y_t := \mu(t)Z_t = \mu(t)(X + t(\tilde{\xi} + b)), \quad (10)$$

where $\mu(t)$ satisfies $\mu(t) > 0$, $\mu(0) = 1$ and $V(Y_t) = V(Y_0)$ ($\forall t$). Therefore Y_t is an admissible variation of X . We will compute the second variation of the total energy for the variation Y_t .

3.3 Anisotropic energies for the boundary of a hypersurface

In order to compute the second variation, we will define a new anisotropic energy for the boundary of considered hypersurfaces. First, we define the anisotropic energy for C_i . Since $\omega_i < \gamma(N_i)$ ($i = 1, 2$), we can define the n -dimensional cross section domain of W_γ by the plane P_i defined as follows

$$P_i := \{x \in \mathbb{R}^{n+1} \mid \langle x, N_i \rangle = \omega_i\} \quad (i = 1, 2). \quad (11)$$

Then, set $\hat{o}_i = \omega_i N_i$. It is clear that P_i is parallel to Π_i . We denote by $\hat{W}_i$ the boundary of the cross section domain (the left side of Fig. 4). We can regard $\hat{o}_i$ as the origin of P_i . Since $\hat{W}_i$ is convex and since $\hat{o}_i$ is in the inside of $\hat{W}_i$, there is a positive support function $\hat{\gamma}_i : S^{n-1} \rightarrow \mathbb{R}_{>0}$ such that its Wulff shape is $\hat{W}_i$. For later use, let ξ_i be the Cahn-Hoffman map for $\hat{\gamma}_i$. Now, we define the anisotropic energy of $\chi_i := X|_{\sigma_i}$ by

$$\hat{\mathcal{F}}(\chi_i) := \int_{\sigma_i} \hat{\gamma}_i(\rho) ds, \quad (12)$$

where ds is the volume element of χ_i and ρ is the outward unit normal of σ_i in the plane Π_i . By using this anisotropic energy, we can show the balancing formula (13) below. Set $\hat{\mathcal{F}}_i(D_i) := \hat{\mathcal{F}}(\chi_i) + h_i \mathcal{H}^{n-1}(C_0)$, where let $h_i := \langle \tilde{\xi}_i(x), \rho_0 \rangle$, ($x \in \sigma_0$) where ρ_0 is the unit outward normal vector on C_0 from D_i . Then, we can prove that h_i is constant in x .

Lemma 10.

$$\hat{\mathcal{F}}_i(D_i) = -n\Lambda \mathcal{H}^n(D_i) \quad (i = 1, 2). \quad (13)$$

Secondly, we define an anisotropic energy for C_0 . Since $\tilde{\xi}(\sigma_0) \subset P_1 \cap P_2$, we can consider the convex hull of $\tilde{\xi}(\sigma_0)$ in $P_1 \cap P_2$, denoted by W_0 (the right side of Fig. 4). Since W_0 is convex, there is a support function $\hat{\gamma}_0 : S^{n-2} \rightarrow \mathbb{R}_{>0}$ such that its Wulff shape is $\hat{W}_0 := \partial W_0$. Let ξ_0 be the Cahn-Hoffman map of $\hat{\gamma}_0$. We define the anisotropic energy of $\chi_0 := X|_{\sigma_0}$ by

$$\hat{\mathcal{F}}_0(\chi_0) := \int_{\sigma_0} \hat{\gamma}_0(\eta) d\tilde{s} \quad (14)$$

where η is the outward normal vector field along ∂C_0 and $d\tilde{s}$ is the volume element of ∂C_0 .

Remark 11. If $n = 2$, then $\hat{\mathcal{F}}_0(\chi_0) = \mathcal{H}^1(W_0)$ is satis-

Fig. 4. $\hat{W}_i$ (left, the green domain) and W_0 (right, the blue domain (line))

fied.

3.4 Second variation of $E(Y_t)$

First of all, the following proposition can be proved.

Lemma 12.

$$\frac{d}{dt}V(X_t) = E(X_t) = \mathcal{F}_\gamma(X_t) + \sum_{i=1}^2 \omega_i \mathcal{H}^n(D_i^t), \quad (15)$$

where D_i^t is the n -dimensional domain in $\Pi_i + t\tilde{\xi}$ bounded by $C_i + t\tilde{\xi}$ and the $(n-1)$ -dimensional domain enclosed by $X_t(\sigma_0)$ ($i = 1, 2$).

We can rewrite the two terms of the right hand side of (15) as a power series in t explicitly. By using the Steiner-type formula (9), we have

$$\begin{aligned} \mathcal{F}_\gamma(X_t) &= \mathcal{F}_\gamma(X) - n\Lambda\mathcal{F}_\gamma(X)t \\ &\quad + nC_2 \int_M \gamma(\nu) H_2^\gamma dA \cdot t^2 + O(t^3). \end{aligned} \quad (16)$$

Next, we will compute $\mathcal{H}^n(D_i^t)$. By using the anisotropic energy (12) and (14) of C_i and C_0 , respectively, we can prove,

$$\begin{aligned} \mathcal{H}^n(D_i^t) &= \mathcal{H}^n(D_i) + \hat{\mathcal{F}}_i(D_i)t \\ &\quad - \frac{n-1}{2} \left(\int_{\sigma_i} \hat{\gamma}_i(\rho) \hat{\Lambda} ds - \frac{1}{n-1} \hat{\mathcal{F}}_0(\chi_0) h_i \right) t^2 \\ &\quad + O(t^3), \end{aligned} \quad (17)$$

where $\hat{\Lambda}$ is the anisotropic mean curvature of C_i .

Substituting (16) and (17) into (15), the total energy $E(X_t)$ can be represented as a power series in t . By long computation, we can get the second variation of $E(Y_t)$ as follows:

Lemma 13.

$$\begin{aligned} \frac{d^2}{dt^2}E(Y_t)|_{t=0} &= -(n-1) \sum_{i=1}^2 \omega_i B_i \\ &\quad - \frac{1}{n} \int_M \gamma(\nu) \sum_{1 \leq i < j \leq n} (k_i^\gamma - k_j^\gamma)^2 dA \end{aligned} \quad (18)$$

where

$$B_i := \int_{\sigma_i} \hat{\gamma}_i \hat{\Lambda} ds + \frac{(\mathcal{F}_i(D_i))^2}{n\mathcal{H}^n(D_i)} - \frac{1}{n-1} \hat{\mathcal{F}}_0(\chi_0) h_i. \quad (19)$$

3.5 Stability

Firstly, we can get the following inequalities.

Lemma 14. (1) If $n = 2$, then $B_i \geq 0$.

(2) If $n \geq 3$ and D_i is convex, then $B_i \geq 0$.

We must assume D_i are convex for $n \geq 3$ in order to prove $B_i \geq 0$ since we use the Alexandrov-Fenchel inequality ([8, Theorem 6.3.1]), which is a result in convex analysis. If $n \geq 3$, then we assume that D_i are convex. If X is stable, then the second variation of the total energy of X for any admissible variations is non-negative. But by Lemmas 13 and 14, we have $\frac{d^2}{dt^2}E(Y_t)|_{t=0} \leq 0$. Therefore, $\frac{d^2}{dt^2}E(Y_t)|_{t=0} = 0$ and hence $k_1^\gamma = \dots = k_n^\gamma$ is satisfied. It is known that any hypersurface that satisfies $k_1^\gamma = \dots = k_n^\gamma$ is (up to translation and homothety) a part of the Wulff shape [9]. Conversely, if $X(M)$ is a part of a homothety of the Wulff shape W_γ , then we can prove that X is stable by similar way to the proof of [10, Theorem 4.1]. Therefore, we have proved Theorem 4.

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