On the convergence of the V-type hyperplane constrained method for singular value decomposition

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Abstract

In this paper, we investigate the convergence of the V-type hyperplane constrained method for singular value decomposition. The V-type method involves employing the Newton type iteration to solve the nonlinear systems with the searching range of right singular vectors constrained on a hyperplane. First, we discuss the nonsingularity of the Jacobian matrix appearing in the Newton type iteration. Next, we clarify the convergence of the Newton type iteration. Finally, we prove that singular value decomposition is computable by the V-type method.

Keywords: singular value decomposition, nonlinear system, Newton’s iterative method.

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1. Introduction

For a rectangular matrix $A \in \mathbb{R}^{m \times n}$ with $m \leq n$, there exist suitable orthogonal matrices $U = (u_1 \cdots u_m) \in \mathbb{R}^{m \times m}$ and $V = (v_1 \cdots v_n) \in \mathbb{R}^{n \times n}$ such that

$$A = U \Sigma V^T, \quad \Sigma = (\sigma_{ij})_{i,j=1,\ldots,m}$$

where $V^T$, $O_{m,n-m}$ and $\delta_{ij}$ denote the transpose of $V$, the $m \times (n-m)$ zero matrix and the Kronecker delta, respectively. The nonnegative $\sigma_k$ are called singular values. The columns of $u_k$ in $U$ and $v_k$ in $V$ are called left and right singular vectors, respectively. Eq. (1) is then called the singular value decomposition (SVD) of $A$. In this paper, for simplicity, let us call the pairs $(\sigma_k, u_k, v_k)$ for $k = 1, \ldots, m$ the singular pairs. The SVD of $A$ with $m > n$ has the same form as that of $A^T$ with $m \leq n$. So, it is enough to discuss the case where $m \leq n$.

In [1], some of the authors proposed six types of SVD methods, collectively named as SN-SVD. The six types of SN-SVD are based on solving the nonlinear systems related to SVD. Of them, three types employ the Newton type iteration for solving the nonlinear systems. The rest make use of the inverse iteration instead of the Newton type iteration. The methods using the Newton type iteration are named in [1] as the PJU-type, the PJV-type and the SJ-type methods. In particular, the solution of every nonlinear system appearing in the PJU-type and the PJV-type methods is strictly constrained on a hyperplane. Hence, the PJU-type method is also called the hyperplane constrained method [2,3]. The searching range of the left singular vector $u_k$ is directly constrained in the PJU-type. The PJV-type, however, differs from the PJU-type in that the searching range of the right singular vector $v_k$ is constrained. Although the PJU-type and the PJV-type methods should be precisely called the U-type and the V-type hyperplane constrained method, respectively, in this paper, we simply call them the U-type method and the V-type method.

Several numerical experiments in [1] demonstrate that the U-type and the V-type are superior to the other types with respect to numerical accuracy. Some convergence theorems of the U-type method are proved in [2]. It is shown in [3] that the U-type method generates more accurate SVD than some standard methods. A hybrid method of the U-type method is also proposed. The V-type method, however, has not been investigated, except for the elementary numerical results. The aim of this paper is to theoretically discuss some of the basic properties related to the convergence of the V-type method.

2. The V-type Newton type iteration

In this section, we first explain an iterative method employed in the V-type method for a singular pair, and next we investigate some of its properties.

Let us begin our analysis by considering the nonlinear system,

$$A u = \sigma u, \quad A^T v = \sigma v, \quad \|u\|_2 = 1, \quad \|v\|_2 = 1, \quad (2)$$

$$A v_k = 0, \quad k = m + 1, \ldots, n. \quad (3)$$

Note here that the solutions of (2) become the singular pairs $(\sigma_k, u_k, v_k)$ for $k = 1, \ldots, m$. Let $V_k = \text{span}(u_1, \ldots, u_k)$. Moreover, let $V_k^\perp$ denotes the orthogonal complement of $V_k$. Then all of $v_k$ in (3) are given by the orthonormal basis of $V_k^\perp$. For example, the singular
vectors $v_{m+1}, \ldots, v_n$ are computable by employing the Gram-Schmidt method if $v_1, \ldots, v_m$ are as given in (2). Therefore, in this paper, we mainly discuss an iterative method for computing the solutions of (2).

Let us replace (2) with the nonlinear system

$$Av = \sigma u, \quad A^T u = \sigma v, \quad (z, v) = C,$$

with an arbitrary vector $z \in \mathbb{R}^n$ and a constant $C \in \mathbb{R}\backslash\{0\}$, where $(z, v) = z^T v$. Then we immediately have the following theorem.

**Theorem 1** If $(z, v_k) \neq 0$ for $k = 1, \ldots, m$, the solutions of (4) are $(\sigma, u, v) = (\sigma_k, \alpha_k u_k, \alpha_k v_k)$ for $k = 1, \ldots, m$, where $\alpha_k = C/(z, v_k)$.

From the 2nd and the 3rd equations in (4), it is obvious that $\sigma$ is a function of $u$, namely, $\sigma = \sigma(u) = (w, u) C^{-1}$, $w = Az$. Hence, (4) yields the nonlinear system,

$$H(x) = 0,$$

$$H(x) := \begin{pmatrix} Av - \sigma(u)u \\ A^T u - \sigma(v)u \end{pmatrix}, \quad x := \begin{pmatrix} u \\ v \end{pmatrix}.$$  

Applying Newton’s iterative method to (5), we obtain the iteration,

$$x^{(\ell+1)} = \Phi(x^{(\ell)}), \quad \ell = 0, 1, \ldots,$$

$$x^{(0)} := \begin{pmatrix} u^{(0)} \\ v^{(0)} \end{pmatrix},$$

$$\Phi(x) := x - (J(x))^{-1}H(x),$$

$$J(x) := \begin{pmatrix} -\sigma(u)I_m - wv C^{-1}A & A \\ A^T - vw C^{-1} & -\sigma(v)I_n \end{pmatrix}.$$  

Let $\Gamma$ be the hyperplane $\Gamma = \{v \mid (z, v) = C\}$. Then a theorem for the existing range of $x$ is as follows.

**Theorem 2** If $v^{(0)} \in \Gamma$, namely, $(z, v^{(0)}) = C$, then $v^{(\ell)} \in \Gamma$ for $\ell = 1, 2, \ldots \text{ in (7)}$.

**Proof** From (7) and (9), it is obvious that $J(x^{(\ell)}) (x^{(\ell+1)} - x^{(\ell)}) = -H(x^{(\ell)})$. Combining it with (6), (8) and (10), we derive

$$(-\sigma(u^{(\ell)})I_m - \frac{w^{(\ell)}v^{(\ell)}}{C}) u^{(\ell+1)} + Av^{(\ell+1)} = -\sigma(u^{(\ell)})u^{(\ell)},$$

$$(A^T - \frac{v^{(\ell)}w^{(\ell)}}{C}) u^{(\ell+1)} - \sigma(v^{(\ell)})v^{(\ell+1)} = -\sigma(u^{(\ell)})v^{(\ell)}.$$  

The elimination of $u^{(\ell+1)}$ from the above two leads to $u^{(\ell+1)} = -\sigma(u^{(\ell)})(B + pq^T)^{-1}p$, where $B = AA^T - \sigma(u^{(\ell)})^2 I_m, p = A v^{(\ell)} + \sigma(u^{(\ell)})u^{(\ell)}$ and $q = -w C^{-1}$. With the help of the Sherman-Morrison formula (cf. [4]),

$$(B + pq^T)^{-1} = \left(I_m - \frac{B^{-1}pq^T}{1 + (q, B^{-1}p)}\right) B^{-1},$$

we derive

$$u^{(\ell+1)} = -\sigma(u^{(\ell)}) \frac{u^{(\ell)}}{1 - (w, u^{(\ell)}) C^{-1}}, \quad u^{(0)} = B^{-1} p.$$  

Substituting (12) into (11), we have

$$v^{(\ell+1)} = \frac{v^{(\ell)} - A^T u^{(\ell)}}{1 - (w, u^{(\ell)}) C^{-1}}.$$  

Suppose that $v^{(\ell)} \in \Gamma$ for some $\ell$, namely, $(z, v^{(\ell)}) = C$,

then from (13) and $w = Az$, it follows that $(z, v^{(\ell+1)}) = (C - (w, u^{(\ell)})) / (1 - (w, u^{(\ell)}) C^{-1}) = C$, which implies that $v^{(\ell+1)} \in \Gamma$. By induction, it is concluded that $v^{(\ell+1)} \in \Gamma$ for $\ell = 1, 2, \ldots$ if $v^{(0)} \in \Gamma$.

(QED)

In [1], some of the authors propose the iteration

$$x^{(\ell+1)} = \Psi(\Phi(x^{(\ell)})), \quad \ell = 0, 1, \ldots,$$  

$$\Psi(x) := \frac{C}{(z, v) x}$$

in place of (7). We call the iteration (14) the V-type Newton type iteration. The function $\Psi$ is employed for scaling the singular vectors in order to keep $v^{(\ell)} \in \Gamma$ for $\ell = 1, 2, \ldots$. However, it turns out from Theorem 2 that $\Psi$ acts as an identity mapping. In other words, the Newton type iteration (14) is theoretically reduced to the iteration (7).

**Theorem 3** If $v^{(0)} \in \Gamma$, then the Newton type iteration (14) is equivalent to Newton’s iteration (7) for $\ell = 0, 1, \ldots$.

In the case of Newton type iteration (14), it is necessary to compute the inverse of the Jacobian matrix $J(x)$. Accordingly, it is important to guarantee the nonsingularity and the boundedness of $J(x)$. Some properties of $J(x)$ are shown in the following theorems.

**Theorem 4** The Jacobian matrix $J(x)$ is nonsingular if and only if

$$\sigma(u)^{n-m} \left\{ \sum_{i=1}^{m} \left[ \sigma_i(v_i, z)(\sigma_i(v, v) + \sigma(u)(u_i, u)) \prod_{j \neq i} (\sigma_j^2 - \sigma(u)^2) \right] C^{-1} + \prod_{i=1}^{m} (\sigma_i^2 - \sigma(u)^2) \right\} \neq 0,$$

where $\prod_{i \neq j}$ denotes the product over $j = 1, \ldots, i-1, i+1, \ldots, m$.

**Proof** As a formula for the determinant of a block matrix, it is shown in [4] that

$$\det(B_1 B_2 B_3) = \det(B_4) \det(B_1 - B_2 B_3^{-1} B_5).$$  

Let $B_1 = -\sigma(u)I_m - vw C^{-1}, B_2 = A, B_3 = A^T - vw C^{-1}$ and $B_4 = -\sigma(u)I_n$ in (15), then the left-hand side is just equal to the determinant of $J(x)$ in (10). Hence it follows that

$$\det(J(x)) = (-1)^n \sigma(u)^{n-m} D,$$  

$$D := \det(AA^T - \sigma(u)^2 I_m - (Av + \sigma(u)u)w C^{-1}).$$

Let $U^T u = \tilde{u}, V^T v = \tilde{v}$ and $\tilde{z} = \tilde{x}$. Note here that $AA^T = US^2 U^T, Av = US \tilde{v}$ and $w = Az = U \tilde{z}$. By taking into account that $\det(UUT) = 1$, we derive

$$D = \det(S^2 - \sigma(u)^2 I_m - (\Sigma \tilde{v} + \sigma(u)\tilde{u})\tilde{z}^T S \tilde{C}^{-1}).$$

Let $\tilde{B} = S^2 - \sigma(u)^2 I_m, \tilde{p} = \Sigma \tilde{v} + \sigma(u)\tilde{u}$ and $\tilde{q} = -\Sigma \tilde{z}$. Then $D$ becomes $\det(\tilde{B} + \tilde{pq}^T)$. From the formula for rank one update (cf. [4]), namely, $\det(\tilde{B} + \tilde{pq}^T) = \det(\tilde{B}(1 + (\tilde{q}, \tilde{B}^{-1}\tilde{p}))$, we have

$$D = \det(\tilde{B} + (\tilde{q}, \adj(\tilde{B})\tilde{p})$$,

where $\adj(\tilde{B})$ is the adjugate matrix of $\tilde{B}$. Since it is ob-
Theorem 5 If the singular values of A are distinct and not zero, then $J(\alpha_kx_k)$ for $k = 1, \ldots, m$ are nonsingular.

Proof Let $x = \alpha_kx_k$ in (16), namely, $u = \alpha_ku_k$ and $v = \alpha_kv_k$ where $\alpha_k = C/(z, v_k)$. By taking into account that $\sigma(\alpha_kx_k) = \sigma_k(u_k, u_k) = \sigma_k\delta_k(z_k, v_k) = \alpha_k\delta_k$ and $(\psi_k, z) = C/\alpha_k$ in (18), we have det($J(\alpha_kx_k)$) = $2(-1)^{n+1}\alpha_k^{m-2}\prod_{j\neq k}(\sigma_j^2 - \sigma_k^2)$. Since $\sigma_k \neq 0$ and $\sigma_j \neq \sigma_k$, it is obvious that the right hand side is not zero. Hence $J(\alpha_kx_k)$ is nonsingular. (QED)

Theorem 6 If the singular values of A are distinct and not zero, then there exists a positive constant $M_1$ such that

$$\|J(\alpha_kx_k)\|^{-1} \leq M_1.$$

(19)

It is well-known that Newton’s iterative method has the quadratic convergence under some regularity assumptions. The convergence theorem for (7) (or (14)) is as follows.

Theorem 7 Suppose that $(z, v_k) \neq 0, v^{(0)} \in \Gamma$ and $J(x^{(l)})$ are nonsingular for $l = 0, 1, \ldots$, where $x^{(0)}$ is close to $\alpha_kx_k$, then the vector sequence $\{x^{(l)}\}_{l=0}^{\infty}$ generated by (7) or (14) converges to $\alpha_kx_k$ quadratically as $l \to \infty$.

Proof From Theorem 3, the iteration (14) is equivalent to (7). We prove the convergence of (7).

Let $\Delta x = (\Delta u^T \Delta v^T)^T$. Using (6), $H(x + \Delta x)$ can be rewritten as

$$H(x + \Delta x) = \left(\begin{array}{c} A(v + \Delta v) - \sigma(u + \Delta u)(u + \Delta u) \\ A^T (u + \Delta u) - \sigma(u + \Delta u)(v + \Delta v) \end{array}\right).$$

Note that $\sigma(u + \Delta u) = \sigma(u) + \sigma(\Delta u)$ and $\sigma(\Delta u) = (\Delta u^T C^{-1} \Delta u)$. Then it follows from (10) that

$$H(x + \Delta x) = H(x) + J(x)\Delta x - \sigma(\Delta u)\Delta x.$$  (20)

Moreover, $J(x + \Delta x)$ is expressed as

$$J(x + \Delta x) = J(x) - J'(\Delta x),$$  (21)

$$J'(\Delta x) := \sigma(\Delta u)I_{m+n} + \Delta x(w^T C^{-1} O_{1,n}).$$  (22)

Eq. (9) immediately gives us

$$\Phi(x + \Delta x) = x + \Delta x - (J(x + \Delta x))^{-1}H(x + \Delta x).$$  (23)

From (21), the inverse of $J(x + \Delta x)$ becomes

$$J(x + \Delta x)^{-1} = [I_{m+n} - (J(x))^{-1}J'(\Delta x)]^{-1}(J(x))^{-1}.$$  (24)

The formula $(I - X)^{-1} = \sum_{i=0}^{\infty} X^i$ yields

$$(J(x + \Delta x))^{-1} = (J(x))^{-1} + (J(x))^{-1}J'(\Delta x)(J(x))^{-1} + [(J(x))^{-1}J'(\Delta x)]^2(J(x))^{-1} + O(\|\Delta x\|^3).$$

Here, $O(\|\Delta x\|^k)$ denotes an $(m+n)$ dimensional vector whose entries are $O(\|\Delta x\|^k)$. Substituting (20) and (24) into (23), we derive

$$\Phi(x + \Delta x) = \Phi(x) - (J(x))^{-1}J'(\Delta x)(J(x))^{-1}H(x) + \sigma(\Delta u)(J(x))^{-1}\Delta x - (J(x))^{-1}J'(\Delta x)\Delta x - [(J(x))^{-1}J'(\Delta x)]^2(J(x))^{-1}H(x) + O(\|\Delta x\|^3).$$

Let $x = \alpha_kx_k, \Delta x = \Delta x^{(l)} := x^{(l)} - \alpha_kx_k$ and noting that $H(\alpha_kx_k) = 0, \Phi(\alpha_kx_k) = \alpha_kx_k$, then from (22), we have

$$\Phi(x^{(l)}) = \alpha_kx_k + \Delta \Phi^{(l)},$$  (25)

where $\Delta \Phi^{(l)} = -\sigma(\Delta u^{(l)})/J(\alpha_kx_k)\Delta x^{(l)} + O(\|\Delta x^{(l)}\|)$ and $\Delta \Phi^{(l)} = (\Delta u^{(l)})^T (\Delta u^{(l)})^T$. Note here that there exists a positive constant $M_2$ such that

$$\left|\sigma(\Delta u^{(l)})\right| = \left|\begin{array}{c} w^T \\ O_{1,n} \end{array}\right|^T \Delta \Phi^{(l)} C^{-1} \leq M_2\|\Delta x^{(l)}\|.$$  (26)

Therefore, from (7), (19), (25) and (26), it is concluded that $\|x^{(l+1)} - \alpha_kx_k\| \leq M_1M_2\|\Delta x^{(l)} - \alpha_kx_k\|^2$. This proves the quadratic convergence of (7).

(QED)

Let us recall that from Theorem 2, $v^{(l)}$ for all $l$ given in (7) exist on $\Gamma$. On computer, however, $v^{(l)}$ are not always on $\Gamma$ because of numerical error. Let us here introduce the following Newton type iteration

$$x^{(l+1)} = \Psi(\Phi(x^{(l)}) + e^{(l)}), \quad \ell = 0, 1, \ldots$$

where $e^{(l)} \in R^{m+n}$. In (27), $\Psi$ plays an important role for keeping $v^{(l)} \in \Gamma$. The definition of $\Psi$ immediately leads to the following theorem.

Theorem 8 The vectors $v^{(l)}$ for $\ell = 1, 2, \ldots$ in (27) satisfy $v^{(l)} \in \Gamma$.

It is remarkable that (27) does not have the same convergence as (7) or (14). Therefore, we obtain the following theorem for (27).

Theorem 9 Suppose that $(z, v_k) \neq 0$ and $J(x^{(l)})$ are nonsingular for $l = 0, 1, \ldots, \ell$. Let $M_3$ be a positive constant such that

$$\|I_{m+n} - \alpha_kx_k z^T C^{-1}\| \leq M_3,$$  (28)

where $z = (O_{m,1} z^T)^T$. If $x^{(0)}$ is close to $\alpha_kx_k$ and $e^{(l)}$ satisfies the condition that either (i) there exists a positive constant $M_4$ such that $\|e^{(l)}\| \leq M_4\|x^{(l)} - \alpha_kx_k\|$ for $\ell = 0, 1, \ldots, \ell$, or (ii) there exists a pos-
Proof Let us evaluate $\Psi(x + \Delta x)$. Since $(z, v + \Delta v) = (z, v)[1 + (z, \Delta v)/(z, v)]$, it holds that

$$
C = \frac{1}{(z, v)} \left[ 1 - \frac{(z, \Delta v)}{(z, v)} + O(||\Delta x||^2) \right].
$$

Then we obtain

$$
\Psi(x + \Delta x) = \frac{C}{(z, v)} \left[ x + \left( I_{m+n} - \frac{x}{(z, v)} \right) \Delta x \right] + O(||\Delta x||^2).
$$

Substituting $x = \alpha_k x_k$ and $\Delta x = \Delta \Phi^{(i)} + e^{(i)}$ into (29) and using (25), $(z, \alpha_k v_k) = C$ and the assumption for $e^{(i)}$, we derive

$$
\Psi(\Phi(x^{(i)}) + e^{(i)}) = \alpha_k x_k + \left( I_{m+n} - \alpha_k x_k \frac{\Delta \Phi^{(i)}}{C} \right)^{-1} \left[ -(J(\alpha_k x_k))^{-1} \Delta x^{(i)} \right]
\times \sigma(\Delta \Phi^{(i)}) + e^{(i)} + O(||\Delta x^{(i)}||^3 + ||e^{(i)}||^2).
$$

If $e^{(i)}$ satisfies either (i) or (ii), then it follows from (19), (26), (27), (28) and (30) that $||x^{(i)} - \alpha_k x_k|| \leq M_1 M_2 ||x^{(i)} - \sigma(u_k)||$, or $||x^{(i)} - \alpha_k x_k|| \leq M_3 (M_1 M_2 + M_5) ||x^{(i)} - \alpha_k x_k||^2$, which implies the linear or the quadratic convergence of (27).

(\text{QED})

3. The V-type hyperplane constrained method

In this section, we show the convergence of the SVD method by using the V-type Newton type iteration sequentially.

According to the following theorem, it turns out that the normal vector $z$ of the hyperplane $\Gamma$ is an important parameter to determine the existing range of the solutions of (4).

Theorem 10 If $z$ satisfies $(z, v_i) = 0$ for $i = 1, \ldots, k$, then the solutions of (4) are $(\sigma, u, v) = (\sigma_i, \alpha_i u_i, \alpha_i v_i)$ for $i = k + 1, \ldots, m$.

Proof Suppose that $z$ satisfies $(z, v_i) = 0$, then the pairs $(\sigma, u, v) = (\sigma_i, \alpha_i u_i, \alpha_i v_i)$ do not satisfy (4) for $i = 1, \ldots, k$, since $C \neq 0$. Where $(\sigma, u, v) = (\sigma_i, \alpha_i u_i, \alpha_i v_i)$ for $i = k + 1, \ldots, m$ are the solutions of (4).

(\text{QED})

In general, for Newton’s iterative method, it is not easy to determine the initial values for computing all solutions. Theorem 10 implies that if we select $z$ from a suitable subspace, then we are able to restrict the limit of the vector sequence generated by the V-type Newton type iteration.

Theorem 11 Suppose that there exists the limit $x^* = ((u^*)^T (v^*)^T)^T$ of $\{x^{(i)}\}_{i=0}^\infty$ generated by either (14) or (27), for $z \in V^\perp_k$, then $v^* \notin V_k$ for any initial value $x^{(0)}$.

Proof Since $z \in V^\perp_k$ and $C \neq 0$, it holds that $\Gamma \cap V_k = \emptyset$. Let us recall that $v^{(i)} \in \Gamma$ in either Theorem 2 or 8. Then the completeness of $\Gamma$ gives $v^* \in \Gamma$. Hence it is concluded that $v^* \notin V_k$.

(\text{QED})

In other words, all singular pairs are computable by the V-type Newton type iteration if $z$ is given from the orthogonal complement of the subspace spanned by the already computed right singular vectors. Accordingly, an algorithm of the V-type hyperplane constrained method is as follows.

1. for $k = 1, \ldots, m$
2. Randomly select $u^{(0)} \in R^m$, $(v^{(0)}) \in R^n$, $z \in V^\perp_{k-1}$.
3. Compute a solution $u^*, v^*$ of (5) by the V-type Newton type iteration.
4. If the V-type Newton type iteration does not converge, then go to 2.
5. Set $(\sigma_k, u_k, v_k) = (\sigma(u^*), u^*/\|u^*\|_2, v^*/\|v^*\|_2)$.
6. end for
7. Compute the orthonormal basis of $V^\perp_k$ and adopt them as $v_{m+1}, \ldots, v_n$.

With the help of Theorem 11, we obtain a theorem for the convergence of the V-type hyperplane constrained method for SVD.

Theorem 12 If all the V-type Newton type iterations converge, then the V-type hyperplane constrained method for SVD computes the SVD of A.

4. Conclusion

In this paper, we have discussed the convergence of the V-type hyperplane constrained method for SVD, which employs the V-type Newton type iteration for computing each singular pair. First, we have clarified the nonsingularity of the Jacobian matrix appearing in the Newton type iteration, and then showed the convergence of the Newton type iteration without and with numerical error.

Next, we have proved that the computed right singular vector differs from any of those obtained in the preceding iteration steps, if the normal vector of the hyperplane is selected from the orthogonal complement of the subspace spanned by the right singular vectors from the preceding steps. Finally, we have reached the conclusion that SVD is computable by the V-type hyperplane constrained method.

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