Discontinuous Galerkin FEM of hybrid type

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Abstract

Recently, the discontinuous Galerkin FEM's (DGFEM) are widely studied. They use discontinuous approximate functions, where the discontinuity is dealt with by the Lagrange multiplier and/or interior penalty techniques. Such methods has a merit that various types of approximate functions can be used besides the usual continuous piecewise polynomials, although the band-widths of arising matrices are often much larger than the conventional ones. We here propose a hybrid displacement type DGFEM for the 2D Poisson equation with some mathematical and numerical results. In particular, we can use element matrices and vectors similar to those in the classical FEM.

Keywords Discontinuous Galerkin FEM, hybrid method, stabilization, error analysis

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1. Introduction

Considerable attention has been drawn to the discontinuous Galerkin FEM's (DGFEM) [1–3], whose root is reported to be in neutron transportation problems. They use discontinuous approximate functions, where the discontinuity is dealt with by the Lagrange multiplier and/or interior penalty methods. Such methods have a merit that various approximate functions besides the usual piecewise polynomials can be used, and are expected to be robust to variation of element geometry. However, band-widths of the arising matrices can be much larger than those of the conventional FEM.

Actually, another origin can be traced to solid mechanics: the well-known non-conforming and hybrid FEM's use discontinuous approximate field functions. Typical examples of them are Pian's hybrid stress method [4] and Tong's hybrid displacement one [5, 6]. One of the authors also developed a variant of the hybrid displacement one, and applied it to plate problems [7, 8]. Such an approach enables the use of conventional element matrices and vectors, although it suffered from numerical instability and were not fully successful [9].

Stimulated by rapid development of DGFEM, we propose a DGFEM of hybrid displacement type by stabilizing our old approach. We will show the idea with outline of theoretical analysis for the 2D Poisson equation as a model problem, and then give some concrete finite element models with a few numerical results and observations. Application of our approach to linear elasticity is given in [10], and a closely related approach can be found in [11]. Details of theoretical analysis and modification of the present approach will be reported in due course.

2. Hybrid displacement formulation

2.1 Model problem

Let us consider the 2D Poisson equation over a bounded convex polygonal domain Ω with the homo-

geneous Dirichlet condition on the boundary $\partial\Omega$:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1)$$

where Δ is the Laplacian, and u and f are respectively an unknown and a given real-valued functions defined in Ω . The most popular weak formulation for (1) is to use $H_0^1(\Omega)$ and to find $u \in H_0^1(\Omega)$ s.t., for a given $f \in L_2(\Omega)$,

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega; \quad \forall v \in H_0^1(\Omega), \quad (2)$$

where ∇ denotes the gradient, and $(\cdot, \cdot)_\Omega$ does the inner products of both $L_2(\Omega)$ and $L_2(\Omega)^2$, with the associated norms designated by $\|\cdot\|_\Omega$. Since Ω is convex, $u \in H^2(\Omega) \cap H_0^1(\Omega)$. For the definitions of $L_2(\Omega)$, $H_0^1(\Omega)$, $H^2(\Omega)$ and various Hilbertian Sobolev spaces, see [2, 12].

2.2 Definitions and notations

We first construct a family of triangulations $\{\mathcal{T}^h\}_{h>0}$ of Ω by polygonal finite elements: each $K \in \mathcal{T}^h$ is an m -polygonal domain (Fig. 1), where m is an integer ≥ 3 and can differ with K . Thus the boundary ∂K of $K \in \mathcal{T}^h$ is composed of m edges. We assume that m is bounded from above independently of $\{\mathcal{T}^h\}_{h>0}$, K is not “too thin”, and ∂K does not intersect with itself. The diameter and measure of K are denoted by h_K and $|K|$, respectively, while the length of an edge $e \subset \partial K$ by $|e|$. Furthermore, $h := \max_{K \in \mathcal{T}^h} h_K$. The L_2 and L_2^2 inner product and norm for K are written as $(\cdot, \cdot)_K$ and $\|\cdot\|_K$. We also define the following forms for $\hat{u}, \hat{v} \in L_2(\partial K)$:

$$\langle \hat{u}, \hat{v} \rangle_{\partial K} = \int_{\partial K} \hat{u} \hat{v} ds, \quad |\hat{v}|_{\partial K} = \langle \hat{v}, \hat{v} \rangle_{\partial K}^{1/2},$$

where ds is the infinitesimal line element on ∂K . Forms $\langle \cdot, \cdot \rangle_e$ and $|\cdot|_e$ for each edge $e \in \partial K$ are given similarly.

Over $\mathcal{T}^h$, we consider the spaces ($k = 0, 1, 2, \dots$):

$$H^k(\mathcal{T}^h) = \{v \in L_2(\Omega); v|_K \in H^k(K) \ (\forall K \in \mathcal{T}^h)\}.$$

For $v \in H^1(\mathcal{T}^h)$ and $K \in \mathcal{T}^h$, its trace to ∂K is well defined as an element of $L_2(\partial K)$ and is denoted by $v|_{\partial K}$ or

Fig. 1. m -polygonal element K ; non-convex case.

simply v , which can be double-valued on edges shared by two elements [1, 2]. For $v \in H^2(\mathcal{T}^h)$, we can also define its normal derivative $\partial v / \partial n$ as an element of $L_2(\partial K)$.

On the union Γ^h of edges in $\mathcal{T}^h$, we consider a kind of flux $\hat{u} \in L_2(\Gamma^h)$, which is single-valued on each edge shared by two elements, unlike various double-valued fluxes in some DGFEM's [1, 2]. To deal with the boundary condition in (1), define a subspace of $L_2(\Gamma^h)$ by

$$L_2^0(\Gamma^h) = \{\hat{v} \in L_2(\Gamma^h); \hat{v}|_{\partial\Omega} = 0\}.$$

2.3 Hybrid displacement-type DGFEM

Define a bilinear form $B_h^\pm(\cdot, \cdot; \cdot, \cdot)$ by

$$\begin{aligned} B_h^\pm(u, \hat{u}; v, \hat{v}) &= \sum_{K \in \mathcal{T}^h} \left[(\nabla u, \nabla v)_K + \left\langle \frac{\partial u}{\partial n}, \hat{v} - v \right\rangle_{\partial K} \pm \left\langle \hat{u} - u, \frac{\partial v}{\partial n} \right\rangle_{\partial K} \right. \\ &\quad \left. + \sum_{e \subset \partial K} \frac{\eta_{K,e}}{h_{K,e}} \langle \hat{u} - u, \hat{v} - v \rangle_e \right]; \end{aligned} \quad (3)$$

$\forall \{u, \hat{u}\}, \{v, \hat{v}\} \in H^2(\mathcal{T}^h) \times L_2(\Gamma^h)$, where $\eta_{K,e} > 0$ is the stabilization or interior penalty parameter for $e \subset \partial K$, $h_{K,e}$ is an edge length parameter such as $|e|$ and h_K , and the suffixes $+$ and $-$ of $\pm$ denote symmetric and asymmetric forms, respectively. Our old symmetric formulation [7, 8] lacked the penalty term and suffered from numerical instability [9].

In our DGFEM, we prepare a finite element space V^h of the form:

$$V^h = U^h \times \hat{U}^h,$$

where U^h and $\hat{U}^h$ are finite-dimensional subspaces of $H^2(\mathcal{T}^h)$ and $L_2^0(\Gamma^h)$, respectively. We often use $\hat{U}^h$ such that $\hat{U}^h \subset L_2^0(\Gamma^h) \cap C(\Gamma^h)$ to reduce number of unknowns associated to $\hat{U}^h$.

Then our finite element approximation is: Given $f \in L_2(\Omega)$, find $\{u_h, \hat{u}_h\} \in V^h$ s.t.

$$B_h^\pm(u_h, \hat{u}_h; v_h, \hat{v}_h) = (f, v_h)_\Omega; \quad \forall \{v_h, \hat{v}_h\} \in V^h. \quad (4)$$

Fundamental properties of the above formulation such as the existence and uniqueness of the approximate solutions, error estimates, etc. will be discussed later.

2.4 Linear simultaneous equations

From (4), we have linear simultaneous equations exactly as in the classical FEM. Although we can deal with it as a whole, interior element unknowns associated to U^h can be usually a priori eliminated elementwise (i.e., by the so-called static condensation) to obtain matrices and vectors similar to the element stiffness matrices and

load vectors of the conventional FEM, cf. [4, 9, 10]. Thus we can first construct linear simultaneous equations for element boundary unknowns to be solved by appropriate FEM codes. Then the interior unknowns are obtained by the post-processing. On the other hand, in the usual DGFEM's where the element boundary flux $\hat{u}_h$ is not used, the interior element function u_h can be highly coupled with that of neighboring elements, so that the linear simultaneous equations are often more dense than those of our hybrid DGFEM.

3. Abstract error analysis

To analyze (4) referring to [1, 2], we should prepare some conditions for $B_h = B_h^\pm$ and V^h . To such an end, we need some semi-norms for $\{v, \hat{v}\} \in H^2(\mathcal{T}^h) \times L_2(\Gamma^h)$:

$$\begin{aligned} |v|_{1,h}^2 &= \sum_{K \in \mathcal{T}^h} |v|_{1,K}^2, \quad |\hat{v}|_*^2 = \sum_{K \in \mathcal{T}^h} \sum_{e \subset \partial K} \frac{1}{h_{K,e}} |\hat{v}|_e^2, \\ \|\{v, \hat{v}\}\|_h^2 &= |v|_{1,h}^2 + |\hat{v} - v|_{\Gamma^h}^2 + \sum_{K \in \mathcal{T}^h} h_K^2 |v|_{2,K}^2, \end{aligned} \quad (5)$$

where $|\cdot|_{k,K}$ ($k=1, 2$) are the usual semi-norm of $H^k(K)$ [2, 12]. Clearly, these strongly depend on the triangulations. Then let us present the following three conditions.

[Consistency] The exact solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ of (2) and its trace $\hat{u} \in L_2^0(\Gamma^h)$ to Γ^h satisfy

$$B_h(u, \hat{u}; v, \hat{v}) = (f, v)_\Omega; \quad \forall \{v, \hat{v}\} \in H^2(\mathcal{T}^h) \times L_2^0(\Gamma^h).$$

[Boundedness] There exists a positive constant C_b s.t.

$$|B_h(u, \hat{u}; v, \hat{v})| \leq C_b \|\{u, \hat{u}\}\|_h \|\{v, \hat{v}\}\|_h; \quad (6)$$

$$\forall h > 0 \text{ and } \forall \{u, \hat{u}\}, \{v, \hat{v}\} \in H^2(\mathcal{T}^h) \times L_2^0(\Gamma_0^h).$$

[Coerciveness] There exists a positive constant C_c s.t.

$$|B_h(v_h, \hat{v}_h; v_h, \hat{v}_h)| \geq C_c \|\{v_h, \hat{v}_h\}\|_h^2;$$

$$\forall h > 0 \text{ and } \forall \{v_h, \hat{v}_h\} \in V^h.$$

Under the above conditions, we can derive the following theorem essentially following the approach in [1, 2].

Theorem 1 The unique existence and uniform boundedness of the approximate solution $\{u_h, \hat{u}_h\}$ follow from the boundedness and coerciveness above. Moreover, utilizing the consistency condition as well, we have an error estimate in the semi-norm $\|\cdot\|_h$ ($\hat{u}$ = trace of u):

$$\begin{aligned} \|\{u - u_h, \hat{u} - \hat{u}_h\}\|_h &\leq \left(1 + \frac{C_b}{C_c}\right) \inf_{\{v_h, \hat{v}_h\} \in V^h} \|\{u - v_h, \hat{u} - \hat{v}_h\}\|_h. \end{aligned} \quad (7)$$

Unfortunately, the above estimate does not give any information on the L_2 error estimate $\|u - u_h\|_\Omega$ at least explicitly, so that we introduce one more condition:

[Adjoint consistency] The solution $\psi \in H_0^1(\Omega)$ of (2) for $g \in L_2(\Omega)$, instead of f , and its trace $\hat{\psi}$ satisfy

$$B_h(v, \hat{v}; \psi, \hat{\psi}) = (v, g)_\Omega; \quad \forall \{v, \hat{v}\} \in H^2(\mathcal{T}^h) \times L_2^0(\Gamma^h). \quad (8)$$

For the symmetric formulation based on B_h^+ , the present condition reduces to the consistency one, but must be considered independently in the asymmetric case.

Now we can use Nitsche's trick [1, 2, 12] to obtain the following results for the L_2 error estimation.

Theorem 2 *Under the adjoint consistency with ψ and $\hat{\psi}$ in (8) as well as the other three conditions, we have:*

$$\begin{aligned} & \|u - u_h\|_{\Omega} \\ & \leq C_b \|\{u - u_h, \hat{u} - \hat{u}_h\}\|_h \\ & \quad \times \sup_{g \in L_2(\Omega) \setminus \{0\}} \inf_{\{v_h, \hat{v}_h\} \in V^h} \frac{\|\{\psi - v_h, \hat{\psi} - \hat{v}_h\}\|_h}{\|g\|_{\Omega}}. \end{aligned} \quad (9)$$

4. Polygonal P_k - P_k finite elements

As the simplest DGFEM, let us consider the P_k ($k \in \mathbb{N}$) approximations for both U^h and $\hat{U}^h$ among various possible choices. Thus $v \in U^h$ is a single polynomial in each K and is a discontinuous piecewise polynomial over $\mathcal{T}^h$. On the other hand, $\hat{v} \in \hat{U}^h$ is a one-dimensional polynomial on each edge $e \subset \Gamma^h$, but we have two possibilities for $\hat{U}^h$: a continuous space $\hat{U}^h \subset C(\Gamma^h) \cap L_2^0(\Gamma^h)$ and a discontinuous one, i.e., just $\hat{U}^h \subset L_2^0(\Gamma^h)$. If desired, we can use vertices on Γ^h as nodes, where continuity is imposed for the continuous $\hat{U}^h$. We can sometimes consider nodes for U^h , which are used only for auxiliary purposes in computations unlike in the conventional FEM. In principle, interior functions in U^h are independent of edge functions in $\hat{U}^h$, and their restrictions to K are independent of their restrictions to other elements.

For the triangular element with $k = 1$ and $\hat{U}^h \subset C(\Gamma^h)$, we can prove that the statically condensed element matrix and vector coincide with those of the classical P_1 triangle, though the interior FE solution does not necessarily coincide with the classical P_1 solution. We can also consider arbitrary m -polygonal elements ($m \geq 3$), but larger m may yield poorer results for small fixed k .

5. Preliminary considerations on error analysis

To give concrete error estimates of the finite element schemes in Sec. 4, we must establish the former three conditions in Sec. 3, and the adjoint consistency if possible, as well as the estimation of the right-hand side of (7). Since the available spaces are insufficient to describe such processes in detail, we give only preliminary considerations and brief comments below. The theoretical analysis and the obtained results are essentially the same for the continuous and discontinuous $\hat{U}^h$'s.

5.1 Comments on the 4 conditions

As in [1, 2], the consistency condition is easy to prove for the present hybrid DGFEM by using the Green formula and noting that $\hat{v}$ in (3) is single-valued on Γ^h .

To establish the boundedness condition for the present concrete schemes, we must assume the boundedness of the stabilization parameter $\eta_{K,e}$: *there exists a positive constant $\bar{\eta}$ s.t.*

$$\eta_{K,e} \leq \bar{\eta}; \quad \forall h > 0, \forall K \in \mathcal{T}^h, \forall e \subset \partial K.$$

We also use some trace theorems associated to each element $K \in \mathcal{T}^h$ [1, 2], so that we need appropriate regularity conditions on the family of triangulations $\{\mathcal{T}^h\}_{h>0}$. In the cases of triangulations by triangles and quadri-

laterals, we can adopt the regularity conditions stated in [2, 12], but we must perform deeper analysis in other case, i.e., m -polygonal elements with $m \geq 5$. It appears, however, the convexity assumption on the element shape may be omitted for the present DGFEM [10]. Anyway, we must continue our study further on this issue, and we restrict our analysis to the established cases of triangular and quadrilateral elements, if necessary. As for $h_{K,e}$, the choice $h_{K,e} = |e|$ is acceptable under appropriate regularity conditions, but some other choice may be possible. In general, the obtained constant C_b in (6) depends on $\{\mathcal{T}^h\}_{h>0}$ and $\bar{\eta}$, but is independent of $h > 0$.

Unlike the preceding two conditions, the coerciveness is entirely inside the finite element space V^h . We need the regularity conditions of triangulations and the specification of $h_{K,e}$, but also require the lower boundedness of $\eta_{K,e}$: *there exists a positive constant $\underline{\eta}$ s.t.*

$$\eta_{K,e} \geq \underline{\eta}; \quad \forall h > 0, \forall K \in \mathcal{T}^h, \forall e \subset \partial K.$$

Just as in [1, 2], the existence of such a constant $\underline{\eta}$ is assured, but its concrete value is generally difficult to evaluate. In the asymmetric formulation, however, any positive value is available as $\underline{\eta}$ at least theoretically.

As was already mentioned, the adjoint consistency is trivial for the symmetric formulation, but has not been shown yet for the asymmetric one. In fact, it does not hold for some asymmetric DGFEM schemes [1].

5.2 Error estimates

Under appropriate regularity conditions on $\{\mathcal{T}^h\}_{h>0}$, we can expect the following estimate for sufficiently smooth v [1, 2, 12]: *there exist positive constants $C_{k,s}$ s.t., $\forall h > 0, \forall K \in \mathcal{T}^h, \forall v \in H^{k+1}(K), k = 1, 2, \dots$ and $s = 1, 2$,*

$$\inf_{v_h \in U^h} |v - v_h|_{s,K} \leq C_{k,s} h_K^{k+1-s} |v|_{k+1,K}. \quad (10)$$

Similarly we can expect: *there exists a positive constant C_0 s.t., $\forall h > 0, \forall K \in \mathcal{T}^h, \forall v \in H^k(K)$ with $k = 1, 2, \dots$,*

$$\inf_{\{v_h, \hat{v}_h\} \in V^h} \max_{e \subset \partial K} (|v - v_h|_e + |v - \hat{v}_h|_e) \leq C_0 h_K^{k-\frac{1}{2}} |v|_{k,K}, \quad (11)$$

and $\forall v \in H^{k+1}(K)$ with $k = 1, 2, \dots$,

$$\inf_{v_h \in U^h} \max_{e \subset \partial K} \left| \frac{\partial v}{\partial n} - \frac{\partial v_h}{\partial n} \right|_e \leq C_0 h_K^{k-\frac{1}{2}} |v|_{k+1,K}. \quad (12)$$

Now by noting (5), we can estimate the right-hand sides of (7) and (9) concretely as follows.

Theorem 3 *Under the first three conditions in Sec. 3 and estimates (10), (11) and (12), we have, for a smooth solution $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ ($k = 1, 2, \dots$),*

$$\|\{u - u_h, \hat{u} - \hat{u}_h\}\|_h \leq C_1 h^k \|u\|_{k+1,\Omega},$$

where C_1 is a positive constant independent of u and h (but may be a function of various other constants), and $\|\cdot\|_{k+1,\Omega}$ is the norm of $H^{k+1}(\Omega)$. Furthermore, if the adjoint consistency also holds, we have, with a positive constant C_2 similar to C_1 ,

$$\|u - u_h\|_{\Omega} \leq C_2 h^{k+1} \|u\|_{k+1,\Omega}.$$

Table 1. Observed orders of errors.

Formulations	k for V^h	Observed orders ^{*)}	
		$ u - u_h _{1,h}$	$\ u - u_h\ _\Omega$
Symmetric	1, 2	$O(h^k)$	$O(h^{k+1})$
Asymmetric	1, 2	$O(h^k)$	$O(h^2)$

*) The integers k in the observed orders above are only approximate values for the actual slopes. From Figs. 2 and 3, we can see that the slopes for larger N are actually close to integral values.

Fig. 2. $\|u - u_h\|_\Omega$ vs. N for P_1 - P_1 rectangles. Fig. 3. $\|u - u_h\|_\Omega$ vs. N for P_2 - P_2 rectangles.

6. Numerical results

We will show some numerical results for a very special case of the model problem: $\Omega =]0, 1[^2$ (unit square) and

$$f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y).$$

Then we find that $u(x, y) = \sin(\pi x) \sin(\pi y)$. We consider two cases for the polynomial degrees: $k = 1, 2$, and both the symmetric and asymmetric formulations. The shapes of finite elements are restricted to triangles and rectangles, and the triangulations are all uniform: $N \times N$ ($N \in \mathbb{N} \setminus \{2\}$) square and Friedrichs-Keller ones for rectangular and triangular elements, respectively. As for the interior penalty terms, we take $h_{K,e} = |e|$ or $h_{K,e} = |K|/h_K$, and $\eta_{K,e} = \eta_0 > 0$. We calculated the finite element solutions for various values of N and η_0 .

Table 1 summarizes the numerically observed error behaviors with $h = 1/N$. It is to be noted that the theory essentially predicts the orders of errors correctly, but the observed L_2 errors for the asymmetric formulation

with $k = 1$ appear one order higher than the theoretical one. Similar results are also reported in many literatures such as [1, 2], but recent numerical experiments for some DGFEM's in [13] show that such a phenomenon is probably attributed to the uniformness of the meshes.

Figs. 2 and 3 illustrate observed errors in $\|\cdot\|_\Omega$ versus N for P_1 - P_1 and P_2 - P_2 rectangular elements, where the penalty terms $\eta_{K,e}/h_{K,e}$ are $8N$ and N for the symmetric and asymmetric formulations, respectively. We cannot discuss here the desirable values of the penalty terms numerically, but a few results were reported in [10].

7. Concluding remarks

We have presented a hybrid-type DGFEM and shown some numerical results. The essential points of error analysis were also shown, but we must make clear the regularity conditions of triangulations to discuss the dependence of various error constants on the element geometries. We also wish to analyze the adjoint consistency in the case of the asymmetric formulation. Application to more practical problems is a subject of future studies, and we will also formulate and analyze a slightly different formulation based on the “lifting operator”, which is already used in some other DGFEM's [1, 2].

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