Deterministic volatility models and dynamics of option returns

Takahiro Yamamoto¹ and Koichi Miyazaki¹

¹ Graduate School of Informatics and Engineering, The University of Electro-Communications, 1-5-1, Chogugakou, Chouh-shi, Tokyo 182-8585, Japan

E-mail y1030103@edu.cc.ucc.ac.jp

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1. Introduction

One extension of the famous BS equity model (geometric Brownian motion) is deterministic volatility model (for short, DVM), whose volatility is deterministic functional form of equity price (Dupire (1994) [1] among others). Mawaribuchi, Miyazaki and Okamoto (2009) [2] calibrates the newly introduced 5-parameter DVM to cross-sectional options market prices on an evaluation date and reports that the model prices of options derived from their 5-parameter DVM are quite close to their corresponding market prices on the date. The purpose of this study is to discuss whether the model prices of options derived from the DVMs are close to their corresponding market prices time-series-wise, in short, the DVMs could capture the dynamics of the market prices of options.

The preceding research Buraschi and Jackwerth (2001) [3] statistically examines the consistency of the pricing-kernel induced from the DVM to the time-series of returns of the S&P 500 options (ATM, OTM, ITM) by GMM (Generalized Method of Moments) technique. We revamp their approach, especially, in the derivation of the pricing-kernel and the data-handling technique and then empirically analyze the consistency of the DVMs introduced in Mawaribuchi, Miyazaki and Okamoto (2009) [2] to the dynamics of the cross-sectional option returns. The implication attained from our quantitative analyses is that even in the trending and volatile market, we could build the equity models that are rational to the dynamics of the cross-sectional option market prices within the framework of the complete model without incorporating the additional stochastic variable such as jump or stochastic volatility.

Keywords pricing kernel, deterministic volatility model, nikkei225 option

Research Activity Group Mathematical Finance

2. Quantitative methods

2.1 Framework of the quantitative analyses based on the pricing-kernel

The purpose of this research is to discuss whether the DVMs is able to capture the dynamics of the cross-sectional option market prices. To the end, we attempt to examine statistically whether the option returns from the DVMs is consistent to the realized returns of the options (ATM, OTM, ITM) time-series-wise. In the derivation of the realized returns from the ITM, the ATM and the OTM option market prices, we regard these options as the individual assets and compute the realized returns of the assets under the empirical measure. To make all the analyses proceed under the empirical measure, we introduce the pricing-kernel induced from the DVM and statistically examine whether the market prices of the options (ATM, OTM, ITM) multiplied by the pricing-kernel are all close to 1 time-series-wise by GMM technique.

The pricing-kernel \( m_{t,t+\Delta t} \) (the suffix indicates the time interval from time \( t + \Delta t \) to time \( t \)) satisfies (1) and the asset price \( S_t \) at time \( t \) is able to be evaluated by taking expectation of the multiplication of the asset price \( S_{t+\Delta t} \) and the pricing-kernel \( m_{t,t+\Delta t} \) at time \( t+\Delta t \) under the empirical measure \( E_t \) conditioned on \( S_t \).

\[
S_t = E_t[m_{t,t+\Delta t} S_{t+\Delta t}]. \quad (1)
\]

Transforming (1) by \( S_{t+\Delta t}/S_t = R_{t,t+\Delta t}^S \), we attain (2) for the pricing-kernel and the options gross returns.

\[
1 = E_t[m_{t,t+\Delta t} R_{t,t+\Delta t}^S]. \quad (2)
\]

where \( R_{t,t+\Delta t}^S \) is the gross return from time \( t \) to \( t + \Delta t \) of the asset \( S \).

As the assets to be examined, we adopt four kinds of...
assets such as the NIKKEI1225 index, the ATM option, the OTM option and the ITM option. Denoting the vector consisting of the gross returns of the four assets by \( R_{t, t+\Delta t} = [R^S_{t, t+\Delta t}, R^{ATM}_{t, t+\Delta t}, R^{OTM}_{t, t+\Delta t}, R^{ITM}_{t, t+\Delta t}] \) (for example, \( R^{ATM}_{t, t+\Delta t} \) indicates the gross return from time \( t \) to \( t + \Delta t \) of the ATM option), we statistically examine whether all of the components in the expectation of the gross return vector multiplied by the DVM pricing-kernel are close to 1 (convergence in (3)) using GMM technique.

\[
h_t = 1 - E_t[m_{t, t+\Delta t} R_{t, t+\Delta t}] \to 0. \quad (3)
\]

### 2.2 Construction of the pricing-kernel

Assuming that the equity process follows the DVM in (4) and the risk-free interest rate \( r \) is not equal to 0, we introduce the pricing-kernel \( m_{t, t+\Delta t} \) that is able to discount both of the bond and the equity returns.

\[
dS_t = \mu S_t dt + \sigma(S_t, t)S_t dW_t,
\]

\[
S_t = S_0 \exp \left[ \mu - \frac{\sigma(S_t, t)^2}{2} \right] t + \sigma(S_t, t)W_t \]. \quad (4)

where \( W_t \) is Wiener process and \( \sigma(S_t, t) \) is volatility. The pricing-kernel should satisfy (5).

\[
\begin{bmatrix}
S_t \\
B_t
\end{bmatrix} = E_t \begin{bmatrix}
m_{t, t+\Delta t} \\
S_{t+\Delta t} \\
B_{t+\Delta t}
\end{bmatrix} \quad (5)
\]

Eq. (5) is the extension of the pricing-kernel (derived assuming that the risk-free interest rate is equal to 0) in the preceding research that could discount only the equity return.

The stochastic process \( \xi_t \) in (6) satisfies \( \xi_0 = 1, \xi_T > 0 \), \( \xi_t = E_t[\xi_{t+\Delta t}] \) and the \( m_{t, t+\Delta t} = \xi_{t+\Delta t}/\xi_t \) also satisfies (5) and is found to be the pricing-kernel.

\[
\xi_t = e^{-rt} \exp \left[ -\frac{(\mu - r)^2}{2\sigma(S_t, t)^2} t - \frac{\mu - r}{\sigma(S_t, t)} W_t \right]. \quad (6)
\]

Replacing the small time interval with unit time interval 1 and taking logarithms of the pricing-kernel, we get (7).

\[
\ln m_{t, t+1} = -r - \frac{(\mu - r)^2}{2\sigma(S_t, t)^2} - \frac{\mu - r}{\sigma(S_t, t)} (W_{t+1} - W_t). \quad (7)
\]

Removing the Wiener process in (7) by way of (4), we could derive (8).

\[
\ln m_{t, t+1} = -r - \frac{(\mu - r)^2}{2\sigma(S_t, t)^2} + \frac{\mu - r}{\sigma(S_t, t)} \left( \frac{\mu - \sigma(S_t, t)^2}{2} \right) \ln \frac{S_{t+1}}{S_t}.
\]

Setting the risk-free interest rate to be 0% in (8), the pricing-kernel is reduced to (9).

\[
\ln m_{t, t+1} = \frac{\mu - \sigma(S_t, t)^2}{2\sigma(S_t, t)^2} - \frac{\mu}{\sigma(S_t, t)^2} \ln \frac{S_{t+1}}{S_t}.
\]

Due to the market environment such that the Japanese risk-free interest rate is equal to around 0 in most of the period, we adopt (9) as the DVM pricing-kernel. Regarding the specific functional form of the volatility \( \sigma(S_t, t) \), we examine the 2-parameter DVM, the 3-parameter DVM and the 5-parameter DVM in Mawaribuchi, Miyazaki and Okamoto (2009) [2] and list them in Table 1.

### 2.3 Setting, data-handling and statistical method

#### 2.3.1 Setting

In this quantitative analysis, the most important thing is to decide the period of the analysis appropriately. When we test the hypothesis that the pricing-kernel composed of the equity return is able to discount the equity option returns properly, we should take the maturities of the options into consideration. The underlying equity does not have its maturity, whereas the equity option has its own maturity and thus we should distinguish one option from the other if the maturities of the options are different from each other. Investors price the equity option incorporating the forecast of the underlying equity dynamics and the risk premium up to the maturity of the option and thus the dynamics of the option return that could be related to that of the underlying equity is only up to the maturity of the option. When the parameters of the equity model are estimated from the data that does not fall on the period from the issuing date of the option to its maturity, we could not identify whether the equity model to derive the pricing-kernel is not appropriate or the data period is not appropriate in the rejection of the hypothesis testing. Therefore, each quantitative analysis should be attempted for each option in the period from the issuing date of the option to its maturity.

#### 2.3.2 Data handling technique

We mention the way to measure the option return. For the options data, we adopt three kinds of the three-month call options such as the ATM (the strike price is equal to the current equity price), the OTM500 (the strike price is 500 yen higher than the current equity price) and the ITM500 (the strike price is 500 yen lower than the current equity price). The strike prices of the listed options are fixed by 500 yen interval and thus above options actually do not exist except for the case that the current equity price exactly falls on a multiple of 500 yen. Thus, we have to infer the prices of the above options from the market prices of the listed options. We adopt the approach to interpolate the implied volatility (hereafter, we call it IV) using spline-function. We select six kinds of options close to the current equity price. The three options are put options whose strike prices are 500 yen, 1000 yen and 1500 yen lower than the current equity price and other three options are call options whose strike prices are 500 yen, 1000 yen and 1500 yen higher than the current equity price. We compute the IVs of the six options by inverting the options market prices by way of BS model and spline-interpolate the six IVs and pick up the IVs corresponding to the

### Table 1. Three kinds of the DVMs.

<table>
<thead>
<tr>
<th>DVM Type</th>
<th>Equation</th>
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<tbody>
<tr>
<td>2P-DVM</td>
<td>( \sigma(S_t, t) = aS_t^b )</td>
</tr>
<tr>
<td>3P-DVM</td>
<td>( \sigma(S_t, t) = a + b \left[ 1 - \tanh \left( \frac{S_t - S_0}{S_0} \right) \right] )</td>
</tr>
<tr>
<td>5P-DVM</td>
<td>( \sigma(S_t, t) = a + b \left[ 1 - \tanh \left( \frac{S_t - S_0}{S_0} \right) \right] + d \left[ 1 - \text{sech} \left( \frac{S_t - S_0}{S_0} \right) \right] )</td>
</tr>
</tbody>
</table>
strike prices of ATM, OTM500, ITM500 options. Then, we compute the prices of ATM, OTM500, ITM500 options by putting the spline-estimated IVs for the three into the BS model. Once we attain the daily prices of ATM, OTM500, ITM500 options, it is easy to compute the daily returns of the three options.

2.3.3 Statistical method (Generalized Method of Moments; GMM)

We statistically test the hypothesis that all the four components in the expectation of the gross return vector \( \mathbf{R}_{t+\Delta t} = [R^\text{ATM}_{t+\Delta t}, R^\text{OTM}_{t+\Delta t}, R^\text{ITM}_{t+\Delta t}]^\prime \) multiplied by the pricing-kernel \( m_{t+\Delta t} \) are all close to 1 in (3) using GMM technique. As the moment conditions of the GMM, we adopt (10) and (11).

\[
g(\theta) = \frac{1}{N} \sum_{t=1}^{N} h_t, \tag{10}
\]

\[
h_t = \begin{bmatrix}
1 - m_{t+\Delta t} \frac{S^\text{ATM}_{t+\Delta t}}{S_t} \\
1 - m_{t+\Delta t} \frac{S^\text{OTM}_{t+\Delta t}}{S_t} \\
1 - m_{t+\Delta t} \frac{S^\text{ITM}_{t+\Delta t}}{S_t} \\
S_t (1 - m_{t+\Delta t} \frac{S^\text{ATM}_{t+\Delta t}}{S_t}) \\
S_t (1 - m_{t+\Delta t} \frac{S^\text{OTM}_{t+\Delta t}}{S_t}) \\
S_t (1 - m_{t+\Delta t} \frac{S^\text{ITM}_{t+\Delta t}}{S_t})
\end{bmatrix}. \tag{11}
\]

Using the moment condition, we construct \( J_N(\theta) \) in (12) and minimize it to estimate the parameter set \( \hat{\theta} \) of the DVM.

\[
J_N(\theta) = g(\theta)^\prime W_N g(\theta), \tag{12}
\]

where \( N \) is the number of the data, \( W_N \) is the variance-covariance matrix of the moment conditions, \( \theta \) is the parameter set \( \{ \mu, \sigma \} \) of the pricing-kernel \( m_{t+\Delta t} \). The maturities of the options in our analysis are three months and they have 60 business dates from the issuing date to the maturity. We use the daily option returns up to the 5 business date before the maturity to stay away from the relatively large noise included in the very short period option prices. Thus, for each quantitative analysis corresponding to each option contract month, the number of the data \( N \) is 55. We statistically test the hypothesis by GMM using the fact that the test statistics \( J_N(\hat{\theta}) \) with the estimated parameter \( \hat{\theta} \) follows the Chi-square distribution \( \chi^2(n) \) with \( n \) degrees of freedom (refer to Newey and West (1987) [4] for more detail).

\[
d_N = N[J_N(\theta)] \sim \chi^2(n), \tag{13}
\]

\[
H_0 : J_N(\theta) = 0. \tag{14}
\]

The rejection of the hypothesis testing implies that the equity model to derive the pricing-kernel is not consistent to the dynamics of the cross-sectional option market prices.

3. Quantitative analyses

3.1 Data and the equity model

The data in this analyses are daily option prices of the remaining maturities from 60 to 5 business days for the March, June, September and December contracts of the NIKKEI225 options (ATM, OTM500 and ITM500) in the period from June 2003 contract to December 2010 contract and the daily NIKKEI225 index data corresponding to the options data period. We test the risk-free interest rate to be equal to 0% due to the fact that the most of the period of the analyses is under the BOJ’s zero interest rate policy. We test the four equity models such as the 2-parameter model, the 3-parameter model and the 5-parameter model of Mawaribuchi, Miyazaki and Okamoto (2009) [2] as listed in Table 1 in addition to the BS model.

3.2 Results and their implications

First of all, from Fig. 1, we review the dynamics of the NIKKEI225 index in the period of the analyses (from June 2003 to December 2010). There are two notable periods. One is the period from the beginning of 2005 to the mid 2006 when the NIKKEI225 index surges due to the recovery of the economy (the period is called period (i)) and the other is the period from the end of 2007 to the beginning of 2009 when the NIKKEI225 index dives due to the global recession originated from the U.S. sub-prime loan problem (we call the period as period (ii)). Except for the two periods, the NIKKEI225 index moves almost in a range.

The results of testing the hypothesis that the pricing-kernels induced by the DVMs are rational to the dynamics of the cross-sectional option market prices are provided in Table 2. In Table 2, ** and * indicate 0.5% and 1% significance levels, respectively. From Table 2, we see that the BS pricing-kernel is rejected for 19 contract months (with 0.5% significance for 10 contract months and with 1% significance for 9 contract months) out of total 31 contract months and the 2-parameter DVM pricing-kernel is rejected for 15 contract months (with 0.5% significance for 3 contract months and with 1% significance for 12 contract months) out of total 31 contract months. The result suggests that the extension of the BS model to the 2-parameter DVM improves the rationality of the pricing-kernel to the dynamics of the cross-sectional option market prices, however, the effect is quite limited. On the contrary, regarding the 3-parameter DVM and the 5-parameter DVM, except for only the December 2008 contract just after the corruption of the Lehman-Brothers security, the rationality of the pricing-kernel derived from the two models to the dynamics of the cross-sectional option market prices is not rejected even with 1% significance level.
More closely examining the relation between the dynamics of the NIKKEI225 index and the result of the hypothesis testing, we find that the pricing-kernels induced from the BS model and the 2-parameter DVM are not so rejected in the periods when the NIKKEI225 index moves mostly in a range. They, however, are quite often rejected in the periods when the NIKKEI225 is downward trending. To investigate the background reason of the result, we provide the dynamics of the pricing-kernel induced from the BS model and 2-parameter DVM in Mawaribuchi, Miyazaki and Okamoto (2009) [2] that are rational to the dynamics of the cross-sectional option market prices within the framework of the complete model without incorporating the additional stochastic variable such as jump or stochastic volatility.

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