A modified Calogero-Bogoyavlenskii-Schiff equation with variable coefficients and its non-isospectral Lax pair

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Abstract

In this paper, we present a modified version with variable coefficients of the $(2 + 1)$ dimensional Korteweg-de Vries, or Calogero-Bogoyavlenskii-Schiff equation, derived by applying the Painlevé test. Its Lax pair with a non-isospectral condition in $(2 + 1)$ dimensions is also given. Moreover a transformation which links the form with variable coefficients to the canonical one is shown.

Keywords integrable equation with variable coefficients, Painlevé property, Lax pair

Research Activity Group Applied Integrable Systems

1. Introduction

Over the last three decades many mathematicians and physicists study the nonlinear integrable systems from various perspectives. They have remarkable applications to many physical systems such as hydrodynamics, nonlinear optics, plasma physics, field theories and so on [1–3]. Generally the notion of the nonlinear integrable systems is characterized by several features: solitons [4–8], Lax pairs [9–11], Painlevé tests [12–18] and so on. The integrable system has “good” nature as previously described. Moreover, solitons are a major attractive issue in mechanical and engineering sciences as well as mathematical and physical ones. For instance, a real ocean is inhomogeneous, and the dynamics of nonlinear waves is strongly influenced by refraction, geometric divergence and so on.

The physical phenomena in which many nonlinear integrable equations with constant coefficients arise tend to be very highly idealized. Therefore, equations with variable coefficients may provide various models for real physical phenomena, for example, in the propagation of small-amplitude surface waves, which run on straits or large channels of slowly varying depth and width. On one hand, the variable-coefficient generalizations of nonlinear integrable equations are a currently exciting subject [19–22] (and also [21, Refs. [24–45]]). Many researchers have mainly investigated $(1 + 1)$ dimensional nonlinear integrable systems with constant coefficients for discovery of new nonlinear integrable systems. On the other hand, there are few research studies to find nonlinear integrable systems with variable coefficients, since they are essentially complicated. Analysis of higher dimensional systems is also an active issue in nonlinear integrable systems. Since the study of nonlinear integrable equations in higher dimensions with variable coefficients has attracted much more attention. So the main aim of this paper is to construct a $(2 + 1)$ dimensional integrable version of the modified Korteweg-de Vries (KdV) equations with variable coefficients.

It is widely known that the Painlevé test in the sense of the Weiss–Tabor–Carnevale (WTC) method [13] is a powerful tool to investigate integrable equations with variable coefficients. We will discuss the following higher dimensional nonlinear evolution equation with variable coefficients for $q = q(x, z, t)$:

$$ q_t + a(x, z, t)q_{xxx} + b(x, z, t)q^2q_x + c(x, z, t)q_x\partial_x^{-1}(q^2)_x $$

$$ + d(x, z, t)q + e(x, z, t)q_x + f(x, z, t)q_z = 0, \quad (1) $$

where $a(x, z, t) \neq 0$, $b(x, z, t) \neq 0$, $c(x, z, t) \neq 0$, $b(x, z, t) + c(x, z, t) \neq 0$ and subscripts with respect to independent variables denote their partial derivatives and $\partial_x^{-1}$ is the integral operator, $\partial_x^{-1}q := \int q(X)dX$. Here (and hereafter) $a(x, z, t)$, $b(x, z, t)$, $c(x, z, t)$, $f(x, z, t)$ are coefficient functions of two spatial variables $x$, $z$ and one temporal one $t$. We will carry out the WTC method for (1), and present a set of the coefficient functions. Equation of the form (1) includes one of the integrable higher dimensional modified KdV equations:

$$ q_t - 1 \frac{1}{4} q^2_q_x - 1 \frac{1}{8} q_x \partial_x^{-1}(q^2)_x + 1 \frac{1}{4} q_{xxx} = 0, \quad (2) $$

which is called the modified Calogero-Bogoyavlenskii-Schiff (CBS) equation [23]. Eq. (2) can be dimensionally reduced to the standard modified KdV equation for $q = q(x, t)$:

$$ q_t - 3 \frac{3}{8} q^2 q_x + 1 \frac{1}{4} q_{xxx} = 0, \quad (3) $$
by a dimensional reduction $\partial_x = \partial_z$. Here (and hereafter) $\partial_x = \partial/\partial z$ and so on.

The plan of this paper is as follows. In Section 2, we will review the process of the WTC method of (1) in brief. Next we will construct its Lax pair. In Section 4, we prove that the equation with variable coefficients can be reduced to the canonical form by a certain transformation. Section 5 will be devoted to conclusions.

2. Painlevé test of (1)

Weiss et al. claimed in [13] that a partial differential equation (PDE) has the Painlevé property if the solutions of the PDE are single-valued about the movable singularity manifold. They have proposed a technique, which determines whether a given PDE is integrable or not. This technique is called the WTC method. Now we show the WTC method for (1). In order to eliminate the integral operator, we rewrite (1) in the form of coupled systems:

$$q_t + a(x, z, t)q_{xxz} + b(x, z, t)q^2q_z + c(x, z, t)q_x = 0, \quad r_z = (q^2)_z. \tag{4}$$

We are now looking for solutions of (4) and (5) in the Laurent series expansion with $\phi = \phi(x, z, t):$

$$q = \sum_{j=0}^{\infty} q_j \phi^{j-\alpha}, \quad r = \sum_{j=0}^{\infty} r_j \phi^{j-\beta}, \tag{6}$$

where $q_j = q_j(x, z, t)$ and $r_j = r_j(x, z, t)$ are analytic functions in a neighborhood of $\phi = 0$. In this case, the leading orders are $\alpha = 1$ and $\beta = 2$. Then

$$q_0 = i \sqrt{\frac{6a}{b+c}} \phi_x, \quad r_0 = - \frac{6a}{b+c} \phi_x \phi_z,$$

are obtained. Here $i^2 = -1$. To find the resonance we now substitute the Laurent series expansions (6) into (4) and (5). Rearranging (4) into terms of $\phi^{j-4}$ and the other higher powers of $\phi$, we obtain recurrence relations for $q_j$ and $r_j$:

$$((j-3)bq_0^2q_j\phi_x + \{ c [(j-1)ro_j - q_0r_j] + (j-1)(j-2)(j-3)aq_j\phi_x\phi_z \})\phi^{j-4} = \delta_j. \tag{7}$$

Similarly, rearranging (5) into terms of $\phi^{j-3}$ and higher powers of $\phi$, we have

$$(j-2)(2q_0q_j\phi_x - r_j\phi_x)\phi^{j-3} = \sigma_j. \tag{8}$$

Here $\sigma_j$ and $\delta_j$ are given in terms of $q_0$ and $r_1$ ($0 \leq \ell \leq j - 1$). Then we get the following recurrences:

$$j = -1, 2, 3, 4. \tag{9}$$

Let us note here that the resonance $j = -1$ in (9) corresponds to the arbitrary singularity manifold $\phi$. If the recurrence relations are consistently satisfied at the resonances then the differential equations are said to possess the Painlevé property.

Subsequent coefficients $q_j$ and $r_j$ are determined from (7) and (8). However, from the consistency condition, they must include arbitrary functions at the resonances.

Eqs. (7) and (8) must be satisfied in the respective powers of $\phi$. Requiring that every power of $\phi$ ($\phi^{4+k}$ and $\phi^{3+k}$ with positive integer $k$ in (7) and (8), respectively) should vanish, we obtain the consistency conditions as follows.

$$\phi^{-3}, \phi^{-2} :$$

$$q_1 = \frac{1}{2(b+c)q_0^2q_z} \{ bq_0^2q_0 - c(2q_0q_0 - q_0q_0 - q_0q_0) + 2a[\phi_x(q_0\phi_x + 2q_0\phi_x) + 4q_0\phi_x] \} + q_0\phi_x\phi_z, \tag{10}$$

$$r_1 = \frac{1}{(b+c)q_0q_x} \{ crq_0q_x - bq_0(q_0 - q_0q_0) + 2a[\phi_x(q_0\phi_x + 2q_0\phi_x) + 4q_0\phi_x] \} + q_0\phi_x\phi_z. \tag{11}$$

It follows from (10) and (11) that one of the two variables $(q_2, r_2)$ must be arbitrary,

$$4b^3a_{xx} - b^2 \{ a_x(8b_x + 11c_x) - 9ca_x + 4a(b_x + c_x) \}$$

$$- c \{ a(b_x + 2c_x)(b_x + c_x) - c^2a_{xx} + c[a_x(b_x + 2c_x) + a(b_x + c_x)] \} - b \{ -a(b_x + c_x)(8b_x + 11c_x) - 6c^2a_{xx} + c[a_x(7b_x + 13c_x) + 5a(b_x + c_x)] \} = 0, \tag{12}$$

$$2b^3a_{xx} - b^2 \{ a_x(2b + 5c) + a_x(2b + 5c) + 2a(b_x + c_x) \}$$

$$+ 4a(b_x + c_x) \} = 0, \tag{13}$$

$$2b^3a_{xx} - b^2 \{ a_x(4b_x + c_x) - 9ca_x + 2a(b_x + c_x) \}$$

$$+ c \{ a(13b_x + 10c_x)(b_x + c_x) + 5c^2a_{xx} - c[a_x(13b_x + 10c_x) + 5a(b_x + c_x)] \} + b \{ a_x(5b_x + c_x) + 12c^2a_{xx} - c[a_x(17b_x + 11c_x) + 7a(b_x + c_x)] \} = 0. \tag{14}$$

We take into account two cases, case (i) $c(x, z, t) =$
The initial condition, \( b(x,z,t) \) and case (ii) \( a(x,z,t) = a(x,t)(b(x,z,t) + c(x,z,t)) \) from (15).

Case (i): \( c(x,z,t) = -(2/5)b(x,z,t) \)

We obtain the constraints, \( a = a(z,t) \) and \( b = b(z,t) \), from (12)-(14). But a relation, \( ab^2 = 0 \), is appeared at the next calculation \((\phi^{-1}, \phi^0)\). This breaks the initial condition, \( a \neq 0 \) and \( b \neq 0 \). Namely we have failed the Painlevé test in this case.

Case (ii): \( a(x,z,t) = a(x,t)(b(x,z,t) + c(x,z,t)) \)

From (12)-(14), we obtain the following equations:

\[
4b^2a_x + c^2a_{xx} + 3a_xb_xc + 5bca_{xx} - 3a_xb_xc = 0, \quad (16)
\]

\[
a_x(b_xc - b_xc) = 0. \quad (17)
\]

From (17), we obtain the following two cases: case (ii-1) \( c(x,z,t) = c(x,t)b(x,z,t) \) and case (ii-2) \( a(x,t) = a(t) \).

Case (ii-1): \( a(x,z,t) = a(x,t)b(x,z,t) \)

We obtain a relation for \( c(x,t) \) from (12)-(14), \( c(x,t) = 3/(1 - c(t)^2)a_x(z,t) - 4 \). But a relation, \( b = 0 \), is appeared at the next calculation \((\phi^{-1}, \phi^0)\). This breaks the initial condition, \( b \neq 0 \). Namely we have failed the Painlevé test in this case.

Case (ii-2): \( a(x,t) = a(t) \)

The compatibility condition is satisfied in this case.

\[
(\phi^{-1}, \phi^0): \quad \begin{align*}
q_0 [r_2\phi_x - 2(q_0q_3 + q_1q_2)\phi] \\
- (2q_0q_2 - q_1^2)z &+ F = 0, \quad (18) \\
r_3\phi_x - 2(q_0q_3 + q_1q_2)\phi - (2q_0q_2 - q_1^2)z = 0, \quad (19) \\
(\phi^0, \phi^1): \\
q_0 [r_2\phi_x - 2(q_0q_4 + 2q_1q_3 + q_2^2)\phi] \\
- (q_0q_3 + q_1q_2)z &+ G = 0, \quad (20) \\
r_4\phi_x - 2(q_0q_4 + 2q_1q_3 + q_2^2)\phi \\
- (q_0q_3 + q_1q_2)z = 0. \quad (21)
\end{align*}
\]

It follows from (18)-(21) that one of the two variables must be arbitrary in both pairs \((q_3, r_3)\) and \((q_4, r_4)\), and similarly \( F \) in (18) and \( G \) in (20) must vanish. Hence the following equations are obtained from \( F = 0 \),

\[
\begin{align*}
b - 2c &= 0, \\
b_xc - b_xc &= 0, \\
c_xf - cf_x &= 0, \\
c_x &= 0, \quad c_a + 2a(c_d + c_xe - c_ex) &= 0. \quad (22)
\end{align*}
\]

where \( t \) denotes the ordinary derivative with respect to \( t \). Hence from (22),

\[
\begin{align*}
b(x,z,t) &= b(z,t), \quad c(x,z,t) = \frac{b(z,t)}{2}, \\
d(x,z,t) &= e_0(z,t) - \frac{1}{2}a'(t) = f(x,z,t) = f(z,t),
\end{align*}
\]

and then, from \( G = 0 \),

\[
\begin{align*}
e_{xx} = 0 \Rightarrow e(x,z,t) &= e_0(z,t) + e_1(z,t),
\end{align*}
\]

are obtained, respectively.

Therefore the equation given in the form:

\[
\begin{align*}
q_t + \frac{3}{2}a(t)b(z,t)q_{xxx} + b(z,t)q^2q_z \\
+ \frac{1}{2}b(z,t)q_{xx}^{-1}(q^2)_z + \left( e_0(z,t) - \frac{1}{2}a'(t) \right) q \\
+ \{ xe_0(z,t) + e_1(z,t) \} q + f(z,t)q_z = 0, \quad (23)
\end{align*}
\]

admits the sufficient number of arbitrary functions corresponding to the resonances and namely passes the Painlevé test in the sense of the WTC method. It means that we have succeeded in finding of the modified CBS equation with variable coefficients (23). We used the MATHEMATICA [24] to handle calculation for the existence of arbitrary functions at the above resonances.

3. Lax pair of (23)

It is well known that the Lax pair plays a key role in the theory of integrable systems. Consider two operators \( L \) and \( T \) which are called the non-isospectral Lax pair and given by

\[
L\psi = \lambda \psi, \quad T\psi = 0,
\]

and \( \lambda \) being a non-isospectral parameter [23, 25, 26] independent of only \( x \). Then the commutation relation:

\[
[L, T] = LT - TL = 0, \quad (24)
\]

contains a nonlinear evolution equation for suitably chosen operators \( L \) and \( T \). Eq. (24) is called the Lax equation.

The Lax pair of (23) is as follows,

\[
L = i\sqrt{\frac{3a(t)}{2}} \partial_x^2 + q_x \\
T = i\sqrt{\frac{3a(t)}{2}} \partial_x^2 + \frac{1}{2}q_x^{-1}q_z + q_xq_x \partial_z \partial_x \\
+ \frac{i}{2\sqrt{6a(t)}b(z,t)} \left\{ 
\begin{align*}
xe_0(z,t) + e_1(z,t) \\
b(z,t)q_x^{-1}q_z - 2b(z,t)q_x^{-1}q_x^{-1}q_z \\
- 3i\sqrt{\frac{3a(t)}{2}}b(z,t)q_z \partial_x + f(z,t) \partial_z + \partial_t
\end{align*}
\right\}, \quad (25)
\]

with a constraint condition:

\[
\begin{align*}
\frac{a(t)}{a(t)} \partial_x^{-1} \left( \frac{4a(t)e_0(z,t) - a'(t)}{b(z,t)} \right) &= \frac{8a(t)e_0(z,t)f(z,t)}{3b(z,t)} \\
- \frac{2}{3} \partial_x^{-1} \left[ 4e_0(z,t)a'(t) - a''(t) + 4a(t)e_0(z,t) \right] b(z,t) \\
- \frac{2}{3} b(z,t) \left[ \frac{4a(t)e_0(z,t) - a'(t)}{b(z,t)} \right] \partial_t \\
+ \frac{2a'(t)f(z,t)}{3b(z,t)} &= 0. \quad (27)
\end{align*}
\]
Notice here that \( \lambda = \lambda(z,t) \) satisfies a non-isospectral condition:
\[
\lambda_t + \left[ f(z,t) - \frac{b(z,t)}{2a(t)} \partial_z^{-1} \left( 4a(t)e_0(z,t) - a(t) \right) \right] - 2i \sqrt{6a(t)b(z,t)} \lambda \right] \lambda_z = 0. \tag{28}
\]
From (27), we obtain a relation,
\[
e_0(z,t) = \frac{a(t)}{4a(t)}. \tag{29}
\]
Using the above, (23) is rewritten as
\[
 q_t + \frac{3}{2} a(t) b(z,t) q_{xx} + b(z,t) q_z^2 q_z + \frac{1}{2} b(z,t) q_x \partial_x^{-1}(q_z^2) z - \frac{a'(t)}{4a(t)} q + \frac{xa'(t)}{4a(t)} q_x + e_1(z,t) q_x + f(z,t) q_z = 0. \tag{30}
\]
Note that (30) possesses both the Painlevé property and the Lax pair.

4. Reducibility to the canonical form

We show that (30) can be transformed to the standard modified CBS equation (2) by suitable transformations. As an example, we set the following expressions:
\[
 X = xa(t)^{-\frac{1}{4}}, \quad Z = \partial_z^{-1} \left( \frac{1}{b(z,t)} \right), \quad T = \partial_z^{-1} a(t)^{\frac{1}{4}},
\]
\[
 Q(X,Z,T) = a(t)^{-\frac{1}{4}} q(x,z,t), \quad e_1(z,t) = 0,
\]
\[
 f(z,t) = -b(z,t) \partial_z^{-1} \left( \frac{1}{b(z,t)} \right), \tag{31}
\]
for (30). Via a change of the dependent and independent variables, (30) is transformed to the modified CBS for \( Q = Q(X,Z,T) \):
\[
 Q_t + \frac{3}{2} Q XXX + Q^2 Qz + \frac{1}{2} Qx \partial_x^{-1}(Q^2) z = 0,
\]
of which \( N \) soliton solutions were given in [26].

5. Concluding remarks

In this paper, we have presented a modified CBS equation with variable coefficients (30), which is integrable in the sense of the Painlevé test and the existence of the Lax pair. Moreover we can construct its hierarchy by using the Lax-pair Generating Technique [27] for the operator \( L \) (25).

Let us note here that taking \( \partial_z = \partial_x \) as another dimensional reduction can respectively reduce (2) and (30) to the standard form and its extension with variable-coefficients of the modified version of the Ablowitz-Kaup-Newell-Segur equation in \( (2+1) \) dimensions.

By applying the (weak) Painlevé test, we are studying higher dimensional forms with variable coefficients of the nonlinear Schrödinger, Camassa-Holm and Degasperis-Procesi equations in \( (2+1) \) dimensions and so on.

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