Analysis of downgrade risk in credit portfolios with self-exciting intensity model

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Abstract

We present an intensity based credit rating migration model and execute empirical analyses on forecasting the number of downgrades in some credit portfolios. The framework of the model is based on so-called top-down approach. We firstly model economy-wide rating migration intensity with a self-exciting stochastic process. Next, we characterize the downgrade intensity for the underlying sub-portfolio with some thinning model specified by the distribution of credit ratings in the sub-portfolio. The results of empirical analyses indicate that the model is to some extent consistent with downgrade data of Japanese firms in a sample period.

Keywords credit risk, rating migration, self-exciting intensity

Research Activity Group Mathematical Finance

1. Introduction

In credit portfolio risk management, we quantify credit risks with some model of credit event occurrences such as credit rating migrations and defaults. In this paper, we introduce an intensity based credit rating migration model for risk analyses of credit portfolios and perform statistical test for model validation with credit migration samples of Japanese firms.

Our modeling framework is based on the top-down approach studied in [1,2]. Namely, our model is constituted by two parts, top-part and down-part. In the top-part, we model rating migration in whole economy with event intensities. In this paper, we use a self-exciting process for the intensity model, where the term “self-exciting” means that the intensity increases when an event occurs. Several self-exciting type intensity models have been recently used in credit risk modeling to capture credit event clusters (see [2–5]). Credit event cluster is well-known feature of credit events. For example, the historical data of the monthly downgrade number in Fig. 1 show that there are downgrade clusters, from 1998 to 2000, from 2001 to 2003 and from 2008 to 2010. Specifically, we use the self-exciting model proposed in [5].

In the down-part, we obtain intensity models of sub-portfolios with a thinning model. Our thinning model is specified by some factors which represent characteristics of sub-portfolios. We adopt rating distribution for one of the factors to consider the size of credit portfolios, in common with the thinning model of [5].

To check the adequacy of the model, we perform empirical analyses on downgrade forecast. First, we specify our model by maximum likelihood approach and perform statistical test for in-sample fit. Second, we perform statistical test for out-of-sample forecast with the fitted model. Specifically, with the estimated intensity model and thinning model, we derive the distribution of the downgrades number in a reference bond portfolio underlying a collateralized bond obligation, and test the validity of the distribution with realized downgrade number.

The organization of this paper is as follows. Section 2 provides a rating migration model for credit portfo-
2. Model

In this section, we introduce an intensity model of economy-wide rating migrations. In addition, we specify intensities of rating migration in sub-portfolios by a thinning model.

2.1 Intensity model for economy-wide events

We model the uncertainty in the economy by a filtered complete probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})\), where \(\mathbb{P}\) is the actual probability measure and \(\{\mathcal{F}_t\}\) is a right-continuous and complete filtration. For each type of the credit event, consider an increasing sequence of totally inaccessible \(\{\mathcal{F}_t\}\)-stopping times \(0 < T_1 < T_2 < \cdots\), which represents the ordered event times in the whole economy. We denote the counting process of the event by \(N_t = \sum_{n \geq 1} 1_{(T_n \leq t)}\).

Suppose \(N_t\) has intensity process \(\lambda_t\). Namely, \(\lambda_t\) is a \(\{\mathcal{F}_t\}\)-progressively measurable non-negative process, and the process \(N_t - \int_0^t \lambda_s ds\) is an \(\{\mathcal{F}_t\}\)-local martingale. Let \(\lambda_t\) be the self-exciting stochastic process:

\[
\frac{d\lambda_t}{\lambda_t} = \kappa (c_t - \lambda_t) dt + dJ_t,
\]

\[
J_t = \sum_{n \geq 1} \min(\delta \lambda_{T_n-}, \gamma) 1_{(T_n \leq t)} = \kappa \lambda_{T_{N_t}},
\]

where \(\lambda_{T_{N_t}} := \lim_{s \uparrow T_{N_t}} \lambda_s\) and the constants \(\kappa > 0, c \in (0, 1), \delta > 0, \gamma \geq 0, \lambda_0 > 0\) are parameters.

2.2 Thinning model

In down-part, we decompose the economy-wide event intensity into sub-portfolio event intensity with a thinning model based on rating distributions.

Suppose each firm in the economy is associated with a credit rating. There are \(K\) credit ratings and we denote credit ratings by \(1, 2, \ldots, K\), in order of credit quality. Let \(S^k_{\mathbb{I}}\) denote the set of \(k\) rated firms in portfolio \(S_i\) \((i = 1, 2, \ldots, I, k = 1, 2, \ldots, K)\). At each time, each rated firm belongs to one of sub-portfolios \(S^k_{\mathbb{I}}\). Let \(N^k_{\mathbb{I}}(t)\) be the counting process of credit events in sub-portfolio \(S^k_{\mathbb{I}}\). The counting process is given by

\[
N^k_{\mathbb{I}}(t) = \sum_{n \geq 1} 1_{(T_n \leq t)} 1_{(T_n \in \tau(S^k_{\mathbb{I}}))},
\]

where \(\tau(S)\) denotes the set of event times in portfolio \(S\).

To obtain intensity of the counting process \(N^k_{\mathbb{I}}(t)\), we introduce \(\{\mathcal{F}_t\}\)-adapted process \(\{Z^k_{\mathbb{I}}(t)\}\). \(Z^k_{\mathbb{I}}(t)\) represents the conditional probability that the credit event is the event in the portfolio \(S^k_{\mathbb{I}}\), given that an event occurs in the economy. \(Z^k_{\mathbb{I}}(t)\) satisfy the following properties: (a) \(Z^k_{\mathbb{I}}(t)\) takes values in the unit interval \([0, 1]\), (b) \(\sum_{i,k} Z^i_{\mathbb{I}}(t) = 1\). From [4, Proposition 2.1], we obtain the intensity associated with counting process \(N^k_{\mathbb{I}}(t)\) as follows:

\[
\lambda^k_{\mathbb{I}}(t) = Z^k_{\mathbb{I}}(t) \lambda_t.
\]

To analyze credit risk in sub-portfolios, we introduce a thinning model characterized by the distribution of credit ratings in the sub-portfolios. In particular, we specify thinning model for downgraded intensity as follows:

\[
Z^k_{\mathbb{I}}(t) = \zeta^k \tilde{Z}^k_{\mathbb{I}}(t)
\]

where

\[
\tilde{Z}^k_{\mathbb{I}}(t) = \frac{X^k(t)}{\sum_{k=1}^{K-1} X^k(t)} 1_{(\sum_{k=1}^{K-1} X^k(t) > 0)},
\]

and the quantity \(\tilde{Z}^k_{\mathbb{I}}(t)\) denotes the rating distribution of portfolio. \(X^k(t)\) denotes the number of firms in the portfolio \(S^k\) at time \(t\). \(X^k(t)\) denotes the number of \(k\)-rated firms in the whole economy at time \(t\). The denominator in the thinning model (2) represents the number of firms with downgraded possibility. The quotients in (2) is taken to be 0 when the denominator vanishes. The quantity \(\zeta^k(t)\) represents the portfolio characteristic that the rating distribution of portfolio can not capture. While we use thinning models (1) with two factors, we can consider some additional factors in thinning models to obtain more specific sub-portfolio intensities.

3. Empirical analyses

In this section, we estimate the intensity model with the downgrades samples of Japanese firms. Then, we estimate the thinning model for a reference portfolio underlying the collateralized bond obligation called J-Bond Limited. In addition, we perform validation test on in-sample fitness and out-sample downgrade forecast. Specifically, we firstly divide downgrade samples into first half period and second half period. Next, we estimate the model with the first half period and perform statistical tests on fitness. Then, we derive the downgrade distribution in second period with the model, and compare it with the realized downgrades in the second period. As our validation test is statistical one, the results of our test indicate whether the model is rejected or not. In other words, the testing methods in this paper do not necessary give active support on model validity.

3.1 Data

The data for parameter estimation are the sample records on rating changes of Japanese firms from April 1, 1999 to March 31, 2004. The ratings are announced by Rating and Investment Information, Inc. (R&I). During the sample period of 1243 working days, there are 509 downgrades and 55 up-grades. We focus on downgrades, because the number of up-grades are too small to estimate up-grade intensity and to discuss the model adequacy. Excluding no-working days, we transformed calendar times April 1, 1999, April 1, 2000, … to \(t = 0, 1, \ldots\) There are a lot of events in the same day, so we slide the event times with uniform random number so as to make every event times different. We employ the reference credit portfolio underlying J-Bond Limited as a target sub-portfolio (corresponding index number is \(i = 1\)). J-Bond Limited is a collateralized bond obligation with the reference portfolio consisted by 67 corporate bonds. J-Bond Limited was issued in 1999 and re-
demption date of tranches were from 2002 to 2003. The
details of J-Bond Limited are described in [6].

For testing in-sample fit and out-sample forecast, we
divide the samples into first half period, from April 1,
1999 to September 27, 2001 ([0, 2.5]) and the second half
period, from September 28 to March 31, 2004 ([2.5, 5.0]).
The downgrade samples in the first half period are used
for estimating models and testing in-sample fit. The
downgrade samples in the second half period are used for
testing out-sample forecast.

### 3.2 Estimation procedure

For estimating event intensity models, we apply the
maximum likelihood method performed in [3]. Suppose
that we have event time samples \(0 < T_1 < T_2 < \cdots < T_N(\leq H)\). Then the log-likelihood function of the intensity
is following:

\[
\sum_{n=1}^{N} \log \lambda_{T_n} = - \int_{0}^{H} \lambda_{s} ds. \tag{3}
\]

We specify the parameters that maximize (3).

To test the validity of the estimated intensity model to
the data, we apply the Kolmogorov-Smirnov test, that [3] performed as follows. First, we transform the event
times \(\{T_n\}_{n=1}^{N}\) into \(A_n\) by

\[
A_n := \int_{0}^{T_n} \lambda_{s} ds.
\]

We perform the Kolmogorov-Smirnov test using the fact
that \(\{A_n\}_{n=1}^{N}\) will be the jump times of the standard
Poisson process in the case of \(\{T_n\}_{n=1}^{N}\) are generated by \(\lambda_t\). Thus, the null hypothesis is that \(\{A_{n+1} - A_n\}_{n=1}^{N-1}\)
are independent and exponentially-distributed (parameter 1).

We performed the maximum likelihood estimation of
the parameters with the free statistical software package
R. Specifically, we used the intrinsic function “optim”
to maximize the objective function. We performed the
maximization for 30 sets of initial values, and finally
chosen the estimates that maximize the objective function
among the initial value sets. In addition, we performed
Kolmogorov-Smirnov test with R, using the intrinsic
function “ks.test”.

The log-likelihood function of thinning models is as follows:

\[
\log(L(\zeta^i | \mathcal{H}_t^i)) = \sum_{1^{i}(T_n)=1} \log(Z_{T_n}^i(k)) + \sum_{1^{i}(T_n)=0} \log(1 - Z_{T_n}^i(k)),
\]

where \(\mathcal{H}_t^i = \{(T_n, 1^{i}(T_n))\}_{n \leq N_i}\) and

\[
1^{i}(T_n) = \begin{cases} 1 & (T_n \in \sigma(S_i)), \\
0 & (T_n \notin \sigma(S_i)). \end{cases}
\]

We also performed the maximum likelihood estimation
of the parameters with R.

### 3.3 Testing in-sample fit

Table 1 shows estimation result for intensity model
obtained from downgrade samples in the first period. For estimation tractability, we restricted the value of

### 3.4 Testing out-of-sample forecast

The result of out-of-sample fit test, namely comparison
of the model obtained by the first data with second
data, is following. First, with Kolmogorov-Smirnov test
for out-of-sample fitness, we obtained P-values of 0.416,
indicating the intensity model is not rejected at standard
significant level. Table 3 shows comparison of the
model distribution of downgrade number and realized
downgrade number. P-values indicate that the model is consistent with the
realized number of downgrades. Namely, the estimation
for whole model (both top-part and down-part) worked well with the first period samples.
sample.

Now, we show one of features of our model, namely, the model can capture the risk contagion among several portfolios. As we considered economy wide self-exciting intensity for top-part model, the occurrence of an event in a sub-portfolio increase the possibility of event occurrence in the whole economy. That means the model captures event risk contagious among portfolios. In the following example, we see the model captures the risk contagion from the complement portfolio to J-Bond Limited in the second period. Fig. 2 shows the conditional distribution on downgrades number in J-Bond Limited, conditioned on that the downgrades number in the complement is under 95-percentile (298 downgrades) and over 95-percentile. Table 4 shows the averages and the maximums of both distribution in Fig. 2. Table 4 and Fig. 2 indicate that as the downgrade risk in the complement portfolio increases, the downgrade risk of J-Bond Limited increases.

4. Concluding remarks

We introduced the intensity based rating migration model and performed goodness of fit test. Our model is consisted by two parts, self-exciting intensity model for economy wide rating migrations and thinning model based on rating distributions. For testing model adequacy, we used downgrade samples of Japanese firms from 1999 to 2004. We divided the sample period into first and second periods, then we estimated the models with first period and performed in-sample fitness test. The result of fitness test indicates the model estimation worked well. Also, the result of out-of-sample downgrade forecast indicates that the model prediction is consistent with the downgrade out-of-samples.

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Table 3. Average number and maximum of downgrades obtained by the model and realized downgrade number, in the second span. “Percentiles” are the percentile of realized downgrade number in the model distribution. The “complement” means the complement portfolio of J-Bond Limited.

<table>
<thead>
<tr>
<th></th>
<th>Economy</th>
<th>J-Bond Limited</th>
<th>Complement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model Average</td>
<td>274.344</td>
<td>32.350</td>
<td>241.994</td>
</tr>
<tr>
<td>Max</td>
<td>442</td>
<td>68</td>
<td>392</td>
</tr>
<tr>
<td>Realized number</td>
<td>242</td>
<td>24</td>
<td>218</td>
</tr>
<tr>
<td>(Percentile)</td>
<td>(19.33%)</td>
<td>(11.23%)</td>
<td>(23.60%)</td>
</tr>
<tr>
<td>P-value</td>
<td>0.409</td>
<td>0.248</td>
<td>0.456</td>
</tr>
</tbody>
</table>

Table 4. Average number and maximum of downgrades in J-Bond Limited obtained by the model when the downgrade number in the complement portfolio is under 95% percentile (298 downgrades) and over 95% percentile.

<table>
<thead>
<tr>
<th>Downgrades number in the complement portfolio</th>
<th>Under 298</th>
<th>Over 298</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>31.978</td>
<td>39.290</td>
</tr>
<tr>
<td>Max</td>
<td>65</td>
<td>68</td>
</tr>
</tbody>
</table>

Fig. 2. Conditional distribution of downgrades number in J-Bond Limited over 2.5 years. The solid line indicates the conditional distribution of downgrades number in J-Bond Limited, on the condition that the number of down grades in the complement portfolio is under 298. The dash line indicates the conditional distribution of downgrades number in J-Bond Limited, on the condition that the number of down grades in the complement portfolio is over 298.

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