Remarks on numerical integration of $L^1$ norm

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Abstract

Non-differentiability of the absolute value function at the origin can affect the accuracy of numerical computations for the $L^1$ norm. We present an example in which the accuracy does deteriorate, and we provide a convergence order for such situations. We propose a simple algorithm to improve the convergence order, confirming its effectiveness as in the example described above. Mesh-dependent integrands and applications to finite element method are also considered.

Keywords numerical integration, non-smooth integrand, error estimate

Research Activity Group Scientific Computation and Numerical Analysis

1. Introduction

We consider the one-dimensional integral of a continuous function $f$ on $[a, b]$, that is,

$$I(f; a, b) = \int_a^b f(x) \, dx,$$

and its approximation as

$$I_n(f; a, b) = (b - a) \sum_{j=1}^{n} w_j f(x_j),$$

where $w_j$ and $x_j = (1 - t_j)a + t_jb$, $0 \leq t_1 < \cdots < t_n \leq 1$ denote the weights and integral points respectively. For example, Newton–Cotes and Gaussian quadrature rules can be written with the form shown above (see Table 1).

We refer to the exactness of $I_n$ as $r$, which implies that $I_n$ is exact for polynomials of degree $\leq r$. Throughout this paper, the weights are assumed to be positive, i.e.,

$$w_j > 0, \quad j = 1, \ldots, n.$$

Let $\Delta_h : a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$ be a mesh of $[a, b]$, with $h = \max_{0 \leq i \leq n} [a_{i+1} - a_i]$. The composite formula based on (1) is defined as

$$I_n^r(f; a, b) = \sum_{i=0}^{N} I_n(f; a_i, a_{i+1}).$$

In what follows, we simply write $I(f)$ instead of $I(f; a, b)$ etc. when there is no possibility of confusion. Convergence order estimates for $E_n(f) = I_n(f) - I(f)$ and $E_n^r(f) = I_n^r(f) - I(f)$ are well known if $f$ is sufficiently smooth.

Proposition 1 Let $I_n$ have exactness $r \geq 0$.

(i) For all $f \in C^{r+1}([a, b])$,

$$|E(f; a, b)| \leq \frac{2}{(r + 1)!} (b - a)^{r+2} \max_{0 \leq x \leq b} |f^{(r+1)}|. \quad (2)$$

(ii) For all $f \in C^{r+1}([a, b])$,

$$|E_n^r(f; a, b)| \leq \frac{2}{(r + 1)!} h^{r+1} \max_{a \leq x \leq b} |f^{(r+1)}|.$$
First, let us prove the $O(h^2)$ result for $E_n^c(|f|)$ under $f \in W^{1,1}([a,b])$. This is, in fact, valid not only for $|f|$ but also for a composite function $\rho(f)$, where $\rho: \mathbb{R} \to \mathbb{R}$ is any Lipschitz continuous function.

**Theorem 2** Assume that $I_n$ has exactness $\geq 0$, i.e., $\sum_{j=1}^n |w_j| = 1$. If $f \in W^{1,1}([a,b])$ then we have

$$|E_n^c(\rho(f); a, b)| \leq 2Lh\|f''\|_{L^1([a,b])},$$

where $L$ is a Lipschitz constant of $\rho$.

**Proof** First, it is noteworthy that every function in $W^{1,1}([a,b])$ is continuous on $[a,b]$. We define a piecewise constant function $q_n f$ on $[a,b]$ as

$$q_n f(x) = f(y_i), \quad x \in [a_i, a_{i+1}], \quad i = 0, \ldots, N,$$

where $y_i$ is any point in $[a_i, a_{i+1}]$, say $y_i = a_i$.

From a triangle inequality, it is clear that

$$|E_n^c(\rho(f))| \leq |I_n^c(\rho(f) - \rho(q_n f))| + |I_n^c(\rho(q_n f)) - I(\rho(q_n f) - \rho(f))|.$$ 

Because $I_n$ is exact for constants, the second term in the right-hand side vanishes. The first term is bounded by

$$\sum_{i=0}^N |a_{i+1} - a_i| \sum_{j=1}^n |w_j| |\rho(f(x_{i,j})) - \rho(f(y_i))| \leq \sum_{i=0}^N h \sum_{j=1}^n |w_j| |f''(x_{i,j})| dx \leq Lh \sum_{i=0}^N \sum_{j=1}^n |w_j| |f''(x)|_{L^1([a_i, a_{i+1}])} = Lh\|f''(x)|_{L^1([a,b])}.$$ 

A similar technique enables us to bound the third term by $Lh\|f''|_{L^1([a,b])}$. Combining these estimates, we obtain the conclusion.

(QED)

Next, restricting our attention to the case $\rho = |\cdot|$, we specifically examine higher order estimates. To do so, we introduce the following terminology.

**Definition 3** The subinterval $[a_i, a_{i+1}]$ is said to be $|f|$-regular (resp. $|f|$-singular) if

$$f(x)f(y) \geq 0 \text{ for all } x, y \in [a_i, a_{i+1}].$$

(resp. $f(x)f(y) < 0 \text{ for some } x, y \in [a_i, a_{i+1}]$)

We set

$$R_h^l = \{i : a_i, a_{i+1} \text{ is } |f|$-regular$, \}

$$S_h^l = \{i : a_i, a_{i+1} \text{ is } |f|$-singular$.\}

We say a mesh $\Delta_h$ is $|f|$-stable if the cardinality of $S_h^l$ is bounded by a constant $M$ independently of $h$.

**Theorem 4** Let $f \in C^2([a,b])$ and $I_n$ have exactness $\geq 1$. If $\Delta_h$ is $|f|$-stable, then

$$|E_n^c(|f|; a, b)| \leq \left(\frac{b-a}{3} + 2M\right) h^2 \max_{a \leq x \leq b} (|f'| + |f''|).$$

**Proof** It is clear that

$$E_n^c(|f|) = \sum_{i \in R_h^l} E_n(|f|; a_i, a_{i+1}) + \sum_{i \in S_h^l} E_n(|f|; a_i, a_{i+1}).$$

Fig. 1. Convergence behavior of $E_n^c(|f|; 0, 1)$ for $f_1$ and $f_2$. (top, Simpson’s rule; bottom, three-point Gaussian rule.)

$O(h^2)$, whereas $E_n^c(|f_2|)$ decreases monotonically at the optimal rate $O(h^4)$ or $O(h^6)$ given by Proposition 1.

The purpose of this paper is to provide a theoretical analysis for $E_n^c(|f|)$. In Section 2, we establish several convergence-order estimates depending on assumptions related to the mesh or regularity of $f$. Particularly, a sufficient condition to recover the optimal order is described.

To achieve that condition, we propose a numerical implementation in Section 3 using a zero-point search based on Newton’s method. Unfortunately, such a strategy might not work when the integrand depends on the mesh, as discussed in Section 4.

Section 5 presents several applications to problems appearing in finite element method. Finally, we describe perspectives for future works in Section 6.

2. Convergence analysis for $E_n^c(|f|)$

Here and hereinafter we use the standard notation of Lebesgue and Sobolev spaces. The integral points used in the composite formula $I_n^c$ are denoted as

$$x_{i,j} = (1 - t_j)a_i + t_ja_{i+1}$$

for $i = 0, \ldots, N$, and $j = 1, \ldots, n$. Therefore, the composite summation formula $I_n^c$ is represented as

$$I_n^c(f; a, b) = \sum_{i=0}^N (a_{i+1} - a_i) \sum_{j=1}^n w_j f(x_{i,j}).$$

We set

$$R_h^l = \{i : a_i, a_{i+1} \text{ is } |f|$-regular$\},$$

$$S_h^l = \{i : a_i, a_{i+1} \text{ is } |f|$-singular$.\}$$

We say a mesh $\Delta_h$ is $|f|$-stable if the cardinality of $S_h^l$ is bounded by a constant $M$ independently of $h$.

**Theorem 4** Let $f \in C^2([a,b])$ and $I_n$ have exactness $\geq 1$. If $\Delta_h$ is $|f|$-stable, then

$$|E_n^c(|f|; a, b)| \leq \left(\frac{b-a}{3} + 2M\right) h^2 \max_{a \leq x \leq b} (|f'| + |f''|).$$

**Proof** It is clear that

$$E_n^c(|f|) = \sum_{i \in R_h^l} E_n(|f|; a_i, a_{i+1}) + \sum_{i \in S_h^l} E_n(|f|; a_i, a_{i+1}).$$
By the definition of $R_b^h$ and positivity of $w_j$'s,
\[ \sum_{i \in R_b^h} E_n([f]; a_i, a_{i+1}) = \sum_{i \in R_b^h} |E_n(f; a_i, a_{i+1})| \leq \sum_{i=1}^N |E_n(f; a_i, a_{i+1})| \leq \frac{b-a}{3} h^2 \max_{a \leq x \leq b} |f''|, \]
where we have used (2) in the last line.

Related to the second term, because $S_h^f$ contains at least one zero-point of $f$, Taylor's theorem leads to
\[ |f(x)| \leq h \max_{a \leq x \leq a_{i+1}} |f'| \quad (i \in S_h^f, x \in [a_i, a_{i+1}] ). \]

Consequently,
\[ \left| \sum_{i \in S_h^f} E_n([f]; a_i, a_{i+1}) \right| \leq \sum_{i \in S_h^f} \left( |I_n([f]; a_i, a_{i+1})| + |I([f]; a_i, a_{i+1})| \right) \leq \sum_{i \in S_h^f} |a_i + 1 - a_i| \left( \sum_{j=1}^n |w_j| |f(x_{ij})| + \max_{a \leq x \leq a_{i+1}} |f| \right) \leq 2Mh^2 \max_{a \leq x \leq b} |f'|. \]

Adding (4) and (5) yields (3).

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For example, if $f$ has finitely many zero-points, then Theorem 4 holds, which is consistent with the numerical result for $f_1$ ($M = 17$ in this case) shown in Fig. 1.

Finally, we describe a sufficient condition to recover the optimal convergence order.

**Definition 5** A mesh $\Delta h$ is said to be $|f|$-fitted with order $r$ if every $|f|$-singular subinterval $[a_i, a_{i+1}]$ contains just one zero-point $x_*$ of $f$, and either of
\[ \begin{align*}
& \bullet \quad a_i \leq x_* \leq x_{i+1} \text{ and } |a_i - x_*| \leq \alpha h^r, \\
& \bullet \quad x_{i+1} - x_* \leq a_{i+1} \text{ and } |a_{i+1} - x_*| \leq \alpha h^r,
\end{align*} \]
is valid. Here, $\alpha$ is independent of $h, i$, and $x_*$. We say $\Delta h$ is exactly $|f|$-fitted if $\alpha = 0$.

**Theorem 6** Let $f \in C^{r+1}([a, b])$ and $I_n$ have exactness $r \geq 2$. If $\Delta h$ is $|f|$-stable and $|f|$-fitted with order $r$, then
\[ E_n^r([f]; a, b) \leq C r^{r+1} h^r \max_{a \leq x \leq b} |f^{(r+1)}|, \]
where constant $C$ depends only on $a, b, r, \alpha, M$.

**Proof** As in Theorem 4, we obtain
\[ \sum_{i \in S_h^f} E_n([f]; a_i, a_{i+1}) \leq \frac{2(b-a)}{(r+1)!} h^{r+1} \max_{a \leq x \leq b} |f^{(r+1)}| \]
Next let $i \in S_h^f$. We consider only the case of $f \geq 0$ on $[x_i, a_{i+1}]$; the other cases ($f \leq 0$ on $[x_i, a_{i+1}]$, $f \geq 0$ on $[a_{i+1}, x_i]$, $f \leq 0$ on $[a_{i+1}, x_i]$) can be treated similarly. By assumption and Taylor's theorem,
\[ \max_{a \leq x \leq b} |f| \leq \alpha h^r \max_{a \leq x \leq b} |f'|. \]

Here, it is clear that
\[ E_n([f]; a_i, a_{i+1}) \leq |I_n([f]) - I_n(f)| + |I_n(f) - I(f)| + |I(f) - I([f])|. \]
The first term vanishes if $x_{i+1} > a_i$. It is estimated as
\[ 2(a_{i+1} - a_i)w_1 f(a_i) \leq 2a_i h^{r+1} \max_{a \leq x \leq b} |f'|, \]
if $x_{i+1} = a_i$. The second and third terms are bounded respectively by
\[ \frac{2}{r!} h^{r+1} \max_{a \leq x \leq b} |f^{(r)}| \]
and
\[ 2 \int_{a_i}^{x_{i+1}} |f(x)| dx \leq 2a_i h^{r+1} \max_{a \leq x \leq b} |f'|. \]
Therefore,
\[ E_n([f]; a_i, a_{i+1}) \leq C r^{r+1} h^r \max_{a \leq x \leq b} |f^{(r+1)}|, \]
Noting that the number of $i$ such that $i \in S_h^f$ is $M$ at most, we deduce the desired estimate.

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**3. Numerical implementation**

In view of Theorem 6, we need an $|f|$-fitted mesh for the optimal convergence. A natural idea to achieve this requirement is to combine a zero-point search by Newton's method with composite quadratures. Thereby we propose the following algorithm:

**Algorithm 1**

Set $I = 0$

for $i = 0$ to $N$ do:

Set $x_i^0 = a_i$

for $l = 1$ to $k$ do:

$x_i^l = x_i^{l-1} - \frac{f(a_i^{l-1})}{f'(x_i^{l-1})}$

end for

if $x_i^k \leq a_i$ or $x_i^k \geq a_{i+1}$ then

$I \leftarrow I + I_n([f]; a_i, a_{i+1})$

else

$I \leftarrow I + I_n([f]; a_i, x_i^k) + I_n([f]; x_i^k, a_{i+1})$

end if

end for

Presuming that $x_i^k$ converges to a simple zero $x_*$ as $k \to \infty$ in the above notation, then because the convergence order of Newton's method is quadratic, if $k \geq r/2$, then it follows that
\[ |x_i^k - x_*| \leq \alpha f/a_i - x_*)^r \leq \alpha f h^r, \]
where $\alpha f$ is a coefficient that depends on $f'$ and $f''$. This fact implies that if we make subdivision $\Delta h$ of $\Delta h$ by choosing $x_i^k$ as new division points for each $i \in S_h^f$, then $\Delta h$ becomes $|f|$-fitted with order $r$. Therefore, we can expect recovery of the optimal convergence for $E_n^r([f]; a, b)$.

Here, employing Algorithm 1, we again compute $I([f_j]; 0, 1)$ on the uniform mesh. Simpson’s and 3-point Gaussian rules, which are designated as “Newton–Simpson” and “Newton–Gauss3” in the legend of Fig.
2. are used for \( I_n \). The number of Newton iterations \( k \) is fixed to \( k = 3 \). Furthermore, our method is compared with one of the standard adaptive-quadrature routines: DQAG (with the 15-point Gauss–Kronrod rule) provided in QUADPACK [2].

Fig. 2 shows that \( E_n^0([f_i]; 0, 1) \) obtained in Algorithm 1 for \( N \geq 60 \) decreases monotonically at the optimal convergence rate. Consequently, our method drastically improves the behavior of the error if compared with the situation shown in Fig. 1. Although the result obtained by DQAG seems satisfactory for small \( N \), the error stops to decrease for \( N \geq 200 \). This happens because DQAG fails to estimate the error accurately; the estimated error is about \( 10^{-15} \) although the true one is only about \( 10^{-9} \).

4. Case of mesh-dependent integrand

Let \( I_n \) have exactness \( r \) and let \( f_h \) be a piecewise polynomial of degree \( \leq r \) with respect to the mesh \( \Delta_h \). Such a situation often arises when we want to compute the \( L^1 \) norm of some numerical solution. For simplicity, we assume that \( r = 2 \) and \( I_n \) is Simpson’s rule in the following. It is readily apparent that \( E_n^0([f_h]; a, b) = 0 \) provided \( \Delta_h \) is exactly \( [f_h] \)-fitted. We still have, if this is not the case,

**Theorem 7**

\[
|E_n^0([f_h]; a, b)| \leq C h^s \| f_h \|_{H^1(a, b)}, \quad 0 \leq s \leq 1.
\]

where constant \( C \) depends only on \( a, b \).

**Proof** This point is proved in [4, Lemma IV.1.3] (see also [5]). We remark that the estimate involving \( H^1 \) norm in their proof can also be derived from our Theorem 2.

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Theorem 7 provides the best estimate unless \( \Delta_h \) is \([f_h]\)-fitted in some sense. In fact, consider the uniform mesh on \([0, 1]\) and the piecewise quadratic function \( f_h \) such that for \( i = 0, \ldots, N \)

\[
f_h(a_i) = f_h(a_{i+1}) = \sqrt{\Delta_i}, \quad f_h((a_i + a_{i+1})/2) = -\sqrt{\Delta_i}.
\]

Then we see that \( f_h \rightarrow 0 \) uniformly and that

\[
\frac{\sqrt{\Delta}}{3} \leq |E_n^0([f_h]; 0, 1)| \leq \frac{2\sqrt{\Delta}}{3}.
\]

However, we have

\[
\|f_h\|_{L^2(0, 1)} \leq \sqrt{\Delta_i}, \quad \|f_h\|_{H^1(0, 1)} \leq \frac{5}{\sqrt{\Delta_i}},
\]

so that by interpolation between \( L^2 \) and \( H^1 \),

\[
\|f_h\|_{H^{1/2}(0, 1)} \leq C \|f_h\|_{L^2(0, 1)} \|f_h\|_{H^1(0, 1)}^{1/2} \leq C.
\]

Therefore, an error estimate of the following form:

\[
|E_n^0([f_h]; 0, 1)| \leq C h^s \| f_h \|_{H^{1/2}(0, 1)},
\]

is valid only for \( s \leq 1/2 \). It is noteworthy that a zero-point-search strategy given in Section 3 might not work because \( \Delta_h \) is not \([f_h]\)-stable and \( \|f_h\|_{L^1} \rightarrow \infty \) as \( h \rightarrow 0 \).

5. Applications

Algorithm 1 will be useful for computing a numerical solution of convection–diffusion equations using hybridized discontinuous Galerkin method [3]. For a convective term, one must compute a quantity such as

\[
\int_{\partial K} (u_h - \hat{u}_h)(b \cdot n) - v_h - [b \cdot n]_+ v_h \, ds,
\]

where \( \partial K \) denotes the 1D boundary of an element \( K \), say a triangle. Although we do not provide additional details here, one notices that (6) involves a kind of \( L^1 \) norm because \( |x| = (|x| \pm x)/2 \).

The estimate given in Theorem 7 is exploited directly in error analysis of the finite element method applied to some friction problems (see [4, Chapter 4] or [5]). In this case, the finite element solution itself is an integrand \( f_h \), which implies that \( f_h \) is a priori unknown. Consequently, Theorem 7, which holds with no \([f_h]\)-fitness of the mesh, is a crucial tool to derive a priori error estimates.

6. Concluding remarks

First, this paper has presented a specific examination of a priori estimates for \( E_n^0([f]) \), but our method is not well-suited for a function which has zero-points accumulating in a narrow region, e.g. \( f(x) = x^2 \sin(1/x) \). An adaptive strategy based on a posteriori estimates is necessary to address such cases.

Second, extension of our results to 2D integrals is not straightforward because zero-point sets which might affect the convergence behavior of \( E_n^0([f]) \) now become a 1D manifold, which cannot be captured easily through a finite number of discrete points. It would be important to specify, in 2D cases, a counterpart to the \([f]\)-fitness considered in this paper.

**References**


