Affine term structure as multi-soliton

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Abstract

In the real market, the term structure of forward rates exhibits some humps. The quadratic Gaussian term structure models or affine term structure models well explain this phenomena. In this research, we give a new insight, where we understand the humps as multi-soliton that are related to KdV solitons.

Keywords term structure of interest rates, humps, affine class, quadratic Gaussian model, solitons

Research Activity Group Mathematical Finance

1. Introduction

The spot interest rate $r(t, T)$ is the rate per unit of time (normally it is one year) at which one can (in practice, the rate can vary depending on who they are and how it is agreed but we ignore such credit risks/counter party risks here) borrow (lend) cash at time t and repay (be repaid) at time T . Theoretically it is related to the price $P(t, T)$ of the zero-coupon bond maturing at T as

$$r(t, T) = -\frac{1}{T-t} \log P(t, T).$$

In practice, the rate so defined is called *zero rate*. The function

$$T \mapsto r(t, T)$$

is what we call *term structure of spot rates*, or in practice it is rather function in $x = T - t$;

$$x \mapsto r(t, t+x),$$

which is often referred to as *yield curve*.

In theoretical finance, one rather work on the term structure of (the instantaneous) forward rates, which is given by

$$T \mapsto f(t, T) = -\frac{\partial}{\partial T} \log P(t, T),$$

or

$$x \mapsto f(t, t+x) = -\frac{\partial}{\partial T} \log P(t, T) \Big|_{T=t+x}.$$

This is because the forward rate is easier to handle *mathematically*. In particular to impose arbitrage-free property to the term structure.

In real market, however, the term structure of spot rates behaves nicer. According to the series of studies by N.L.Liu and her collaborators [1–3], from the term structure of spot rates only two or three factors up to almost 99% are detected when applied a principal component analysis (or its variants), while that of forward

Fig. 1. Typical forward rate movement: EU zero rate.

Fig. 2. Spot rate movement of the same data as Fig. 1.

rates exhibits more than 10, sometimes 15, or even more factors. Much more straightforward peculiarity is that the samples of the term structure of forward rates often have more *humps* than those of spot rates.

The main aim of the present paper is to propose a new point of view where the humps are understood as a kind of *solitons*.

The rest of the paper is organized as follows. In Section

2, we illustrate our idea by a primitive one dimensional example. In Section 2.1, we present a brief introduction to *solitons*. In Section 3, we give a multi-dimensional version of the observation made in Section 2. We emphasize that a class of affine (quadratic Gaussian) models exhibits multi-soliton shape term structures. Finally in Section 4, we remark that the solitons appearing in the term structure models are related to a non-linear partial differential equations called *KdV equations*.

2. A primitive example

To explain the idea, we start with a primitive example. Let

$$P(t, T) = \mathbf{E} \left[e^{-\frac{1}{2} \int_t^T c^2 |W_s|^2 ds} \middle| W_t \right], \quad 0 \leq t \leq T, \quad (1)$$

where W is a 1-dimensional Brownian motion. This formula defines an arbitrage-free bond market, which is a simplest example of the quadratic Gaussian model, and at the same time, an affine term structure model (see e.g. [4]) where we consider $|W|^2$ to be a state variable. In fact, we have an explicit expression as

$$P(t, T) = (\cosh(c(T-t)))^{-\frac{1}{2}} e^{-\frac{c}{2} \tanh(c(T-t)) |W_t|^2},$$

and the (instantaneous) forward rate $f(t, T) = -\partial_T \log P(t, T)$ is then expressed as

$$f(t, T) = \frac{c}{2} \tanh(c(T-t)) + \frac{c^2 |W_t|^2}{2} \operatorname{sech}^2(c(T-t)), \quad (2)$$

which is an affine function in the state variable.

By (1), we know that

$$T \mapsto -\log P(t, T)$$

is increasing, and therefore the term structure of spot rates under this model behaves nicely, while one notices that

$$T \mapsto f(t, T)$$

is a rational function of $e^{c(T-t)}$ and $e^{-c(T-t)}$, which is, what we will call in local terminology, a *soliton*.

Fig. 3 exemplifies a sample path of the affine forward rate.

2.1 Solitons

In general, a traveling wave solution to a non-linear (evolution-type) differential equation is not stable; it collapses from the top. The soliton solutions are exceptions. They have (sometimes more than two) solitary waves=humps, and the humps are quite stable even after the “collisions”. Somehow they behave like particles, and that is why they are called “solitons”, which are shown in Fig. 4. Mathematically, solitons can be defined as some rational functions of exponentials (see [5]). More precisely, it is something like

$$u(t, x) = \frac{f}{g} = \frac{\sum_i K_i e^{A_i t - B_i x}}{\sum_i L_i e^{C_i t - D_i x}}, \quad (3)$$

where A_i, B_i, C_i, D_i, K_i and L_i are constants, and the summations are finite ones. Here we assume $\max_i C_i \geq \max_i A_i$ and $\min_i C_i \leq \min_i A_i$ to ensure the existence of

Fig. 3. A sample path of the forward rate given by (2) with $W_0 = 8$, $c = 0.1$.

Fig. 4. Solitons.

the limits at $x \rightarrow \pm\infty$. If we require the inequality to be strict, then the graph $x \mapsto u(t, x)$ is hump-shaped. Note that solitons of this definition are stable under summation, multiplication, and differentiations. Note that the forward rate (2) in the previous section is a soliton in T or $x = T - t$ in this sense.

3. Affine term structure as multi-soliton

We generalize the observation made in Section 2. Let $W = (W^1, \dots, W^n)$ be an n -dimensional Brownian motion starting at $x = (x^1, \dots, x^n) \in \mathbf{R}^n$, defined on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ with for each $\lambda_i \in \mathbf{R}$ ($i = 1, 2, \dots, n$), and $C \in M(n)$ be a positive definite matrix.

Let

$$P(t, T) := e^{\langle CW_t, W_t \rangle} \times \mathbf{E} \left[e^{-\frac{1}{2} \int_t^T |\Lambda W_s|^2 ds - \langle CW_T, W_T \rangle} \middle| W_t \right]. \quad (4)$$

Then $\{P(\cdot, T)\}$ defines an arbitrage-free bond market with

$$\pi_t = e^{-\frac{1}{2} \int_0^t |\Lambda W_s|^2 ds - \langle CW_t, W_t \rangle},$$

being a state price density.

Proposition 1 *Under the model (4), the forward rate is an n -soliton; a rational function in $e^{\pm(T-t)\lambda_i}$ ($i = 1, 2, \dots, n$), of degree at most $2n$ for any state W_t .*

Proof Let

$$K(t) = -\cosh(t\Lambda)C - \frac{1}{2}\Lambda \sinh(t\Lambda), \quad (5)$$

$$L(t) = 2\sinh(t\Lambda)\Lambda^{-1}C + \cosh(t\Lambda),$$

and

$$H(t) = K(t) \cdot L(t)^{-1}. \quad (6)$$

Note that

$$K'(t) = -\frac{1}{2}\Lambda^2 L(t) \quad (7)$$

and

$$L'(t) = -2K(t). \quad (8)$$

We will show that

$$P(t, T) = (\det(L(T-t)))^{-\frac{1}{2}} e^{\langle (H(T-t)+C)W_t, W_t \rangle}. \quad (9)$$

By the Feynman-Kac formula,

$$u(t, x) := \mathbf{E} \left[e^{-\frac{1}{2} \int_0^t |\Lambda W_s|^2 ds - \langle CW_t, W_t \rangle} \middle| W_0 = x \right],$$

where $x = (x^1, \dots, x^n)$, satisfies the following differential equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \frac{1}{2}\langle \Lambda^2 x, x \rangle u, \\ u(0, x) = e^{-\langle Cx, x \rangle}, \end{cases} \quad (10)$$

where Δ is the Laplacian. Note that

$$P(t, T) = e^{\langle CW_t, W_t \rangle} u(T-t, W_t). \quad (11)$$

It is well-recognized that the solution u to (10) is expressed by

$$e^{H_0(t) + \langle H(t)x, x \rangle}, \quad (12)$$

where H is a symmetric-matrix valued differentiable function satisfying

$$\frac{dH}{dt}(t) = 2H(t)^2 - \frac{1}{2}\Lambda^2, \quad H(0) = -C, \quad (13)$$

and H_0 is given by

$$\frac{dH_0}{dt}(t) = \text{tr} H(t), \quad H_0(0) = 0. \quad (14)$$

Now we see that H given by (5) and (6) is the unique solution to (13). In fact, by (7) and (8), we have

$$\begin{aligned} H' &= (KL^{-1})' = -KL^{-1}L'L^{-1} + K'L^{-1} \\ &= 2(KL^{-1})^2 - \frac{1}{2}\Lambda^2 = 2H^2 - \frac{1}{2}\Lambda^2, \end{aligned}$$

and also $L(0) = I$ and $K(0) = -C$, which imply $H(0) = -C$.

Further, by (14),

$$\begin{aligned} e^{H_0(t)} &= e^{\text{tr}(-\frac{1}{2} \int_0^t L'(s)L(s)^{-1} ds)} \\ &= \det \left(e^{-\frac{1}{2} \int_0^t L'(s)L(s)^{-1} ds} \right) \\ &= \left(\det \left(e^{\int_0^t L'(s)L(s)^{-1} ds} \right) \right)^{-\frac{1}{2}} \\ &= (\det L)^{-\frac{1}{2}}. \end{aligned}$$

The last line needs some more lines of explanations, which we omit here.

Thus we have confirmed (9), at the same time (11) with (12), by which we have

$$f(t, T) = -\frac{\partial}{\partial T} H_0(T-t) + \frac{\partial}{\partial T} \langle H(T-t)W_t, W_t \rangle.$$

Then, by substituting (13) and (14), we get

$$\begin{aligned} f(t, T) &= -\text{tr} H(T-t) \\ &\quad - \frac{1}{2} \langle (4H(T-t)^2 - \Lambda^2)W_t, W_t \rangle. \end{aligned} \quad (15)$$

We note that the (i, j) -th entries k_{ij} and l_{ij} of $K(t)$ and $L(t)$ are given by

$$k_{ij} = -\cosh(t\lambda_i)c_{ij} - \frac{1}{2}\delta_{ij} \sinh(t\lambda_i),$$

and

$$l_{ij} = 2\sinh(t\lambda_i)\lambda_i^{-1}c_{ij} + \delta_{ij} \cosh(t\lambda_i),$$

and thus they are polynomials in $e^{\pm t\lambda_i}$. Since

$$H(t) = K(t)L(t)^{-1} = K(t)\tilde{L}(t)(\det(L(t)))^{-1},$$

where $\tilde{L}(t)$ is the cofactor matrix of $L(t)$, we see that each entry of $H(t)$ is a rational function in $e^{\pm t\lambda_i}$ ($i = 1, \dots, n$), with degree n . Hence, by the expression (15), we have the assertion.

(QED)

Remark 2 It is known that the forward rates stay positive if π is a strict supermartingale. In fact, for $T_1 > T_2$ we have

$$\mathbf{E}[\pi_{T_1} | \mathcal{F}_t] < \mathbf{E}[\pi_{T_2} | \mathcal{F}_t]$$

by the supermartingale property of π , and the formula reads

$$P(t, T_1) = \frac{\mathbf{E}[\pi_{T_1} | \mathcal{F}_t]}{\pi_t} \leq \frac{\mathbf{E}[\pi_{T_2} | \mathcal{F}_t]}{\pi_t} = P(t, T_2),$$

meaning that $P(t, \cdot)$ and hence $\log P(t, \cdot)$ is decreasing. This in turn implies that $f(t, T) = -\partial_T \log P(t, T)$ is positive.

We give a sufficient condition that ensures the positivity. Since

$$\begin{aligned} d\pi_t &= \pi_t(-d\langle CW_t, W_t \rangle - \frac{1}{2}|\Lambda W_t|^2 dt + \frac{1}{2}d[\langle CW_t, W_t \rangle]_t) \\ &= -2\langle CW_t, dW_t \rangle - \text{tr} C dt \\ &\quad - \frac{1}{2}|\Lambda W_t|^2 dt + \frac{2^2}{2}|CW_t|^2 dt, \end{aligned}$$

we see that π is a supermartingale, and hence the forward rates stay positive, if

$$\Lambda^2 - 4C^2 > 0$$

since $C > 0$ is already assumed.

4. Remarks on a relation with KdV equation

Let $\tilde{f}(t, T) := f(\frac{c^2}{24}t, \frac{1}{2}T)$. Then, we have

$$\tilde{f}(t, T) = \frac{c}{2^3} \tanh \left(\frac{1}{2} \left(\frac{c}{2}T - \frac{c^3}{2^3}t \right) \right)$$

$$+ \frac{c^2 |W_t|^2}{2^3} \operatorname{sech}^2 \left(\frac{1}{2} \left(\frac{c}{2} T - \frac{c^3}{2^3} t \right) \right) \\ =: v(t, T) + |W_t|^2 u(t, T).$$

By this scale change, the functions u and v satisfy $4 \frac{\partial v}{\partial T} = u$ and

$$\frac{\partial u}{\partial t} = -6u \frac{\partial u}{\partial T} - \frac{\partial^3 u}{\partial T^3}. \quad (16)$$

Eq. (16) is known as the Korteweg-de Vries equation (KdV equation for short), which describes waves on shallow water surfaces. The KdV equation is mathematically as well as physically quite important in that there are many infinite dimensional symmetries which allow it to have great many explicit solutions including elliptic ones, rational ones, and most importantly in our context, *soliton* ones.

The relation has been extensively studied, especially by N. Ikeda and S. Taniguchi [6–10]. An extended relation to KP solitons using stochastic areas is given in [11].

5. Concluding remark

We have pointed out that the forward rates of some (but actually almost all) affine term structures are multi-solitons. This observation may give new insights to fitting or calibrating of affine term structures.

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