Bead shape optimization in frequency response problem

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Abstract
The present paper describes a method finding bead shapes in shell structure to decrease the absolute value of mean compliance under periodic loading by using a solution to shape optimization method. Variation of the shell structure in out-of-plane direction is chosen as a design variable. To create beads, the out-of-plane variation is restricted by using the sigmoid function. The integrated absolute value of mean compliance in target frequency range is used as objective function. An iterative algorithm based on the $H^1$ gradient method is used to solve the problem. The effectiveness of the method is confirmed by numerical example.

Keywords bead shape optimization, shell structure, shape derivative, $H^1$ gradient method

1. Introduction
Shell structure is widely used in mechanical structures such as vehicles and architecture. An outstanding property of shell structure is that stiffness is increased by making beads, which mean convexo-concave shapes with a small height, without increasing its thickness. Until now, optimization methods of beads based on basis vector method have been presented and available in commercial software (for example OptiStruct, Genesis). However, it is pointed out that the solution based on basis vector method causes serious deficiencies in mesh quality [1].

On the other hand, a non-parametric shape optimization problem of beads under static loading was formulated and solved by the authors [2]. To create beads, the out-of-plane variation was restricted by using the sigmoid function. The mean compliance was used as objective function.

The present paper shows an application of the method to a frequency response problem. In this study, the main problem is formulated as a frequency response problem of shell structure. An integrated absolute value of mean compliance in target frequency range is used as objective function.

2. Admissible set of design variable
First, let us define an initial shell structure and admissible set of design variable for creating beads. Fig. 1 shows an initial shell model and boundary conditions. In the present paper, we call a median curved surface of a shell structure the mid-surface. We assume that a thickness of a shell structure and a height of a bead are given as positive constants $t$ and $h$, respectively. We define the mid-surface of an initial shell structure as

$$M_0 = \{ \mu_0 (\xi) \in \mathbb{R}^3 \mid \xi \in D \} .$$

(1)

Here, $D \subset \mathbb{R}^2$ ($\mathbb{R}$ denotes the set of all of real numbers) is a two-dimensional bounded domain with a Lipschitz and piecewise $C^2$ (continuous until second derivative) class boundary denoting a reference domain previously given. $\mu_0 \in C^2 (D; \mathbb{R}^3)$ is a given function for defining the mid-surface of shell structure without beads satisfying $1/|D| |\mu_0|_{C^2(D;\mathbb{R}^3)} \leq (t + h)/2$ in order not to overlap any point in the shell structure after creating beads.

A domain of initial shell structure is defined by using $M_0$. Let $\nu_0$ be the outer unit normal vector on $M_0$ on one side. With $\nu_0$, we define a domain of initial shell structure as

$$\Omega_0 = \{ x + \xi \nu_0 (x) \in \mathbb{R}^3 \mid x \in M_0, \xi \in (-t/2, t/2) \} .$$

(2)

Moreover, the boundary of $\Omega_0$, which we denote as $\partial \Omega_0$, is divided into

$$\partial \Omega_0 = M_{T0} \cup M_{B0} \cup \partial M_0 ,$$

(3)

where

$$M_{T0} = \{ x + (0, 0, -t/2) \nu_0 (x) \} ,$$

$$M_{B0} = \{ x + (0, 0, -2t/2) \nu_0 (x) \} ,$$

$$\partial M_0 = \partial M_0 \times (-t/2, t/2) .$$

Domain variation of $\Omega_0$ is defined in the following way. We introduce a function $\theta : D \rightarrow \mathbb{R}$ as a design variable. We assume that a displacement of domain variation to
create beads is given by a sigmoid function of $\theta$. In the present paper, we use
\begin{equation}
\phi(\theta) = \frac{h}{\pi} \tan^{-1} \theta + \frac{h}{2}.
\end{equation}

By the use of $\phi(\theta)$, we define the varied mid-surface as
\[ M(\theta) = \{ (\mu_0 + \phi(\theta) \nu_0 \circ \mu_0) (\xi) | \xi \in D \}, \]
where $\nu$ denotes the composite function. The varied domain of shell structure is defined as
\[ \Omega(\theta) = \{ x + \xi \nu(\xi) | x \in \Omega(\theta), \xi \in (-t/2, t/2) \}, \]
where $\nu(x)$ denotes the outer unit normal vector on $M(\theta)$. $\partial \Omega(\theta)$ is divided into
\begin{equation}
\partial \Omega(\theta) = M_T(\theta) \cup M_B(\theta) \cup \partial M(\theta)
\end{equation}
corresponding to (3).

As a linear space for the design variable $\theta$, we use
\[ X(\theta) = \{ \theta \in H^1(D; D') | \theta = 0 \text{ on } \partial D \}, \]
where $D_{C0}$ is a subset of $D$ (denotes the closure) of $\partial D$, which corresponds to a subset of $\mu_0(D)$ or $\mu_0(D)$ in $\Omega_0$, previously given from design demands. Here, $H^1(D; D')$ denotes $W^{1,2}(D; D')$, where $W^{s,p}(D; D')$ denotes the Sobolev space for the set of functions defined in $D$ and having values in $\mathbb{R}$, which are $s \in [0, \infty)$ times differentiable and are $p \in [1, \infty]$-th order Lebesgue integrable. In the present study, we assume that a Dirichlet boundary $\Gamma_{D0} \subset \partial \Omega_0$ and a non-homogeneous Neumann boundary $\Gamma_{N0} \subset \partial \Omega_0 \setminus \Gamma_{D0}$ are included in $\mu_0(D_{C0})$.

Moreover, to keep the continuity of $\phi(\theta)$ in $D$, we define an admissible set of $\theta$ as
\[ D = X \cup W^{1,\infty}(D; D'). \]
As shown above, although the Fréchet derivative of cost function with respect to arbitrary variation of $\theta$ is not included in $D$, we can show that the variation of $\theta$ obtained by the $H^1$ gradient method [3, 4] remains in $D$ under appropriate conditions.

3. Main problem

For $\theta \in D$, we define a frequency response problem in $\Omega(\theta)$ as a main problem in a shape optimization problem of beams.

Let $\hat{x}_M = (\hat{x}_1, \hat{x}_2, 0)^T$ be an in-plane coordinate of $M(\theta)$, $\hat{x}_3$ be normal coordinate, and $\tau \in \mathbb{R}$ be time. $\hat{u}_M : M(\theta) \times \mathbb{R} \rightarrow \mathbb{R}^2$, $\hat{w} : M(\theta) \times \mathbb{R} \rightarrow \mathbb{R}$ and $\hat{q} : M(\theta) \times \mathbb{R} \rightarrow \mathbb{R}^2$ denote the in-plane displacement of the mid-surface $M(\theta)$ of the undeformed shell structure, the out-of-plane displacement of $M(\theta)$ and the rotational angles of the cross sections which the normals agree with $x_1$-axis and $x_2$-axis, respectively. Here, we assume the Mindlin hypothesis in which the displacement in the global coordinate system $u : \Omega(\theta) \times \mathbb{R} \rightarrow \mathbb{R}^3$ is given in the local coordinate system as
\[ u(x, \tau) = R(\hat{x}_3) \hat{u}(\hat{x}_M, \tau) \]
where
\[ R = \begin{pmatrix} 1 & 0 & 0 & -\hat{x}_3 & 0 \\ 0 & 1 & 0 & 0 & -\hat{x}_3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \]
\[ \hat{u} = (\hat{u}_M, \hat{w}, \hat{q})^T = (\hat{u}_M, \hat{w}, \hat{q}_1, \hat{q}_2)^T. \]

Based on the definitions above, we define the linear strain as $E(\hat{u}) = (\epsilon_{ij}(\hat{u}))_{ij} \in \mathbb{R}^{3 \times 3}$ where
\[ (\epsilon_{ij}(\hat{u}))_{ij}^{(\alpha, \beta) \in (1, 2)^2} = \left[ \begin{array}{c} \nabla_M \hat{u}_M + (\nabla_M \hat{u}_M)^T \\ -\hat{x}_3 \frac{\nabla_M \hat{u}_M^T + (\nabla_M \hat{u}_M)^T}{2} \end{array} \right] \]
\[ E_M(\hat{u}) = \epsilon_{i3}(\hat{u}) = x_3 K(\hat{u}), \]
\[ \epsilon_{33}(\hat{u}) = 0, \]
where $E_M$, $K$ and $\gamma$ denote the in-plane strain, the curvature and the shear strain, respectively. Here, $\nabla_M$ denotes $\partial / \partial x_M$. In the Mindlin-Reissner plate theory, the stress $S(\hat{u}) = (\sigma_{ij}(\hat{u}))_{ij} \in \mathbb{R}^{3 \times 3}$ is given as
\[ (\sigma_{ij}(\hat{u}, \hat{x}_3))_{ij}^{(\alpha, \beta) \in (1, 2)^2} = S_M(\hat{u}, \hat{x}_3) = C_M E_M(\hat{u}) - \hat{x}_3 K(\hat{u}), \]
\[ (\sigma_{33}(\hat{u}))_{ij}^{(\alpha, \beta) \in (1, 2)^2} = C_S \gamma(\hat{u}), \]
\[ \sigma_{33}(\hat{u}) = C_N E(\hat{u}), \]
where $C_M \in W^{1,\infty}(\Omega(\theta); \mathbb{R}^{2 \times 2 \times 2 \times 2})$, $C_S \in \mathbb{R}^{3 \times 3}$ and $C_N \in \mathbb{R}^{1,\infty}(\Omega(\theta); \mathbb{R})$ denote the stiffnesses for the in-plane deformation, the shear deformation and the out-of-plane deformation, respectively, having the ellipticity and symmetric property. In the Mindlin-Reissner plate theory, $c_S$ is called the shear correction factor for which $5/6$ is used for isotropic elastic plates. However, $c_{33}(\hat{u})$ does not appear in the equilibrium equations.

External forces are defined in the following manner. We assume that a volume force and a traction force are given as $b : \Omega(\theta) \times \mathbb{R} \rightarrow \mathbb{R}^3$ and $p_N : \Gamma_{N0} \times \mathbb{R} \rightarrow \mathbb{R}^3$, respectively. In the local coordinate system of $M(\theta)$, the external forces are rearranged as
\[ \hat{b}_N = t b_{\alpha} + \chi_{\Gamma_{N0}}(\bar{\theta}) \circ \bar{M}(\theta) \bar{M}(\theta) p_{\alpha} \]
\[ \hat{p}_{N3} = t b_{3} + \chi_{\Gamma_{N0}}(\bar{\theta}) \circ \bar{M}(\theta) \bar{M}(\theta) p_{3}. \]
\[ \hat{m} = \left( \chi_{\Gamma_{N0}}(\bar{\theta}) \circ \bar{M}(\theta) \right)^T \frac{t}{2} \hat{p}_{N3} - \chi_{\Gamma_{N0}}(\bar{\theta}) \circ \bar{M}(\theta) \frac{t}{2} \hat{p}_{N3} \]
where $\chi(\cdot)$ denote characteristic function which is defined as 1 on $(\cdot)$ otherwise 0 on other domains or boundaries. Moreover, the external forces on $\Gamma_{N0} \cap \partial M_0$ are integrated in thickness as
\[ \hat{p}_N = \int \hat{b}_N d\bar{x}_3, \]
\[ \hat{p}_{N3} = \int \hat{p}_{N3} d\bar{x}_3, \]
\[ \hat{m} = \int \hat{m} d\bar{x}_3, \]
\[
\hat{m} = \int_{-t/2}^{t/2} \bar{x}_3 \rho_{N,0} d\bar{x}_3.
\]

With respect to the external forces, the stress matrices for the Mindlin-Reissner plate theory are defined by integrating the stress \( T (\bar{u}) = (\sigma_{ij} (\bar{u}))_{ij} \in \mathbb{R}^{3 \times 3} \) in thickness as

\[
T_M (\bar{u}) = \left( \int_{-t/2}^{t/2} \sigma_{ij} (\bar{u}, \bar{x}_3) d\bar{x}_3 \right)_{(i,j) \in \{1,2\}^2},
\]

\[
m (\bar{u}) = c_S \left( \int_{-t/2}^{t/2} \sigma_{ij} (\bar{u}, \bar{x}_3) d\bar{x}_3 \right)_{(i,j) \in \{1,2\}^2},
\]

\[
M (\bar{u}) = \left( \int_{-t/2}^{t/2} \bar{x}_3 \sigma_{ij} (\bar{u}, \bar{x}_3) d\bar{x}_3 \right)_{(i,j) \in \{1,2\}^2}.
\]

Using the definitions above and introducing \( \rho \) as the density and \( \zeta \) as the structural damping, the frequency response problem of shell structure is written as below. Here, the Fourier transform of \( \bar{u} \) is defined by

\[
\hat{u} (x, \omega) = \int_{\mathbb{R}} \bar{u} (x, \tau) e^{-i\omega \tau} d\tau,
\]

where \( i \) and \( \omega \in \mathbb{R} \) are the imaginary unit and angular frequency, respectively. In the same manner, " denotes the Fourier transform of ".

**Problem 1 (Frequency response of shell)** For \( \phi \in \mathcal{D} \) and \( \omega \in [\omega_1, \omega_2] \), find \( u : M (\theta) \times \mathbb{R} \rightarrow \mathbb{R}^3 \) such that

\[
\begin{align*}
&-\rho \omega^2 \hat{M} - \left( 1 + i \zeta \right) \nabla_x T_M (\bar{u}) = \hat{b}^T \\
&-\rho \omega^2 \hat{w} - \left( 1 + i \zeta \right) \nabla_x m (\bar{u}) = \hat{p}_{N3}^T \\
&-\rho \omega^2 \hat{q} - \left( 1 + i \zeta \right) \left( \nabla_x M (\bar{u}) + m^T (\bar{u}) \right) = \hat{m}^T \\
&\quad \text{in } M (\theta), \\
&\quad \left( 1 + i \zeta \right) T_M (\bar{u}) \nu = \hat{p}_N \\
&\quad \left( 1 + i \zeta \right) m (\bar{u}) \cdot \nu = \hat{p}_{N3} \\
&\quad \left( 1 + i \zeta \right) M (\bar{u}) \nu \cdot \nu = \hat{0}_R \\
&\quad \left( 1 + i \zeta \right) m (\bar{u}) \cdot \nu = 0 \\
&\quad \left( 1 + i \zeta \right) M (\bar{u}) \nu = \hat{0}_R \\
&\quad \text{on } \Gamma_{M0} \cap \partial M_0, \\
&\quad \text{on } \Gamma_{\nu} \cap \partial M (\theta), \\
&\quad \hat{u} = \hat{0}_R \text{ on } \Gamma_{D0}.
\end{align*}
\]

For use later, we here define the Lagrange function for Problem 1 as

\[
\mathcal{L}_M (\theta, \hat{u}, \hat{v}) = \int_{\omega_1}^{\omega_2} \text{Re} \left[ a (\theta, \hat{u}, \hat{v}) - \omega^2 b (\theta, \hat{u}, \hat{v}) - l (\theta, \hat{v}) \right] d\omega,
\]

where \( \ast \) denotes the conjugate, and

\[
a (\theta, \hat{u}, \hat{v}) = \int_{M(\theta)} \left( 1 + i \zeta \right) (T_M (\bar{u}) \cdot E_M (\bar{v})) c_S m (\hat{u}) \cdot \gamma (\hat{v}) + M (\hat{u}) \cdot K (\hat{v}) d\bar{x},
\]

\[
b (\theta, \hat{u}, \hat{v}) = \int_{M(\theta)} \rho \hat{u} \cdot \hat{v} d\bar{x},
\]

\[
l (\theta, \hat{v}) = \int_{M(\theta)} \left( \hat{b} \cdot \hat{v}_M + \hat{p}_{N3} \hat{z} - \hat{m} \cdot \hat{r} \right) d\bar{x}
\]

\[
+ \int_{\Gamma_{M0} \cap \partial M_0} \left( \hat{p}_N \cdot \hat{v}_M + \hat{p}_{N3} \hat{z} - \hat{m} \cdot \hat{r} \right) d\gamma.
\]

In the present paper, \( \cdot \) denotes the scalar product for vectors and matrices. Here, we assume that \( \hat{u} \) and \( \hat{v} \) belong to

\[
\hat{U} = \{ \hat{u} \in H^1 (M (\theta) \times \mathbb{R}; \mathbb{C}^3) | \hat{u} = 0_{3 \times 3} \text{ on } \Gamma_{D0} \}.
\]

By the use of \( \mathcal{L}_M (\theta, \hat{u}, \hat{v}) \), if \( \hat{u} \) is the weak solution of Problem 1,

\[
\mathcal{L}_M (\theta, \hat{u}, \hat{v}) = 0
\]

holds for all \( \hat{v} \in U \).

**4. Bead shape optimization problem**

Let us define a cost function for the shell structure to decrease the magnitude of response \( \hat{u} \) in \([\omega_1, \omega_2]\) by the magnitude of response \( \hat{u} \) in \([\omega_1, \omega_2]\) for \( \theta \in \mathcal{D} \)

\[
f (\theta, \hat{u}) = \int_{\omega_1}^{\omega_2} |\text{Re} \left[ f (\theta, \hat{u}) \right]| d\omega,
\]

where \( l (\cdot, \cdot) \) is defined by (17). We call \( f \) the mean compliance with respect to frequency response. In the present paper, we consider the following problem.

**Problem 2 (Frequency response minimization)**

Let \( f (\theta, \hat{u}) \) be defined in (19). Find \( \theta \) such that

\[
\min_{\theta \in \mathcal{D}} \left\{ f (\theta, \hat{u}) | \hat{u} \in U, \text{ Problem 1} \right\}.
\]

**5. \( \theta \)-derivative of cost function**

To solve Problem 2 by using the \( H^1 \) gradient method, the Fréchet derivative of \( f \) with respect to arbitrary variation of the design variable \( \theta \), which we call the \( \theta \)-derivative, is required. Employing the adjoint variable method, we obtain the \( \theta \)-derivative of \( f \). Introducing the \( \hat{v} \in \hat{U} \) as the adjoint variable (Lagrange multiplier) for Problem 1, and we define the Lagrange function for \( f \) as

\[
\mathcal{L}' (\theta, \hat{u}, \hat{v}) = f (\theta, \hat{u}) + \mathcal{L}_M (\theta, \hat{u}, \hat{v}).
\]

Denoting the arbitrary variation of \( \theta \) by \( \vartheta \in X \), the \( \theta \)-derivative of \( \mathcal{L} \) is written as

\[
\mathcal{L}' (\theta, \hat{u}, \hat{v}) | \vartheta = \mathcal{L}_b (\theta, \hat{u}, \hat{v}) + \mathcal{L}_b (\theta, \hat{u}, \hat{v}) | \vartheta + \mathcal{L}_a (\theta, \hat{u}, \hat{v}) | \vartheta + \mathcal{L}_a (\theta, \hat{u}, \hat{v}) | \vartheta
\]

where \( \hat{u}', \hat{v}' \in \hat{U} \) and \( \hat{v}' \in \hat{U} \) denote the arbitrary variations of \( \hat{u} \) and \( \hat{v} \), respectively. Here, if \( \hat{u} \) is the solution of Problem 1, then the second term in the right-hand side of (20) become 0. The third term in the right-hand side of (20) is calculated as

\[
\mathcal{L}_a (\theta, \hat{u}, \hat{v}) | \vartheta = f (\theta, \hat{u}) + \mathcal{L}_M (\theta, \hat{u}', \hat{v}).
\]
This term become 0 if \( \dot{v} \) satisfies
\[
\dot{v} = \begin{cases} 
\dot{u} & (\text{Re} \left[ i (\theta, \dot{u}) \right] \geq 0) \\
-\dot{u} & (\text{Re} \left[ i (\theta, \dot{u}) \right] < 0).
\end{cases}
\tag{21}
\]

When \( \dot{u} \) is the solution of Problem 1 and \( \dot{v} \) is determined as (21), the first term in the right-hand side of (20) becomes the \( \theta \)-derivative of \( f(\theta) = f(\theta, \dot{u}(\theta)) \) and obtained as
\[
\begin{aligned}
\dot{f}(\theta)[\dot{v}] &= \mathcal{L}_g(\theta, \dot{u}, \dot{v})[\dot{v}] \\
&= \int_{\partial D_C^0 \setminus D_C} g_M \dot{v} d\gamma + \int_{\partial D_M(\theta) \setminus D_C^0} g_{\partial M} \dot{v} d\gamma \\
&= (g, \dot{\theta}),
\end{aligned}
\tag{22}
\]

where
\[
\begin{aligned}
g_M &= \frac{d\phi}{d\theta} \int_{\omega_1}^{\omega_2} \text{Re} \left[ \psi(\dot{u}, \dot{v}, \frac{t}{2}) - \psi(\dot{u}, \dot{v}, -\frac{t}{2}) \right] d\omega, \\
\psi(\dot{u}, \dot{v}, \bar{x}_3) &= -(1 + i\zeta) \left[ T_M(\dot{u}) \cdot (E_M(\dot{v}) + \bar{x}_3 K(\dot{v})) \right] \\
&\quad + \rho \omega^2 \dot{u} \cdot \dot{v}^c + 2b \cdot \dot{u}, \\
g_{\partial M} &= \frac{d\phi}{d\theta} \int_{\omega_1}^{\omega_2} \text{Re} \left[ \left(1 + i\zeta\right) \left( T_M(\dot{u}) \cdot E_M(\dot{v}) \right) \\
&\quad - c_s \left[ m(\dot{u}) \cdot \gamma(\dot{v}) - M(\dot{u}) \cdot K(\dot{v}) \right] \right] d\omega.
\end{aligned}
\]

6. Solution

To solve Problem 2, for iteration number \( k \in \{0, 1, 2, \ldots\} \), we iterate to update \( \theta_k \) to \( \theta_{k+1} = \theta_k + \theta_g \) using the solution \( \theta_g \) of the following problem [3].

**Problem 3 (H^1 gradient method of \( \theta \) type)** Let \( a_X : X \times X \to \mathbb{R} \) be a given coercive bilinear form on \( X \). For \( \vartheta(\theta_k) \in X' \) of (22), find \( \vartheta_g \in X \) such that

\[ a_X(\vartheta_g, \psi) = - \langle g(\theta_k), \psi \rangle \tag{23} \]

for all \( \psi \in X \).

In the present study, we use
\[ a_X(\theta, \psi) = \int_D \nabla \theta \cdot \nabla \psi \, dx \tag{24} \]
under the condition of means \( (\bar{D}_{C^0}) > 0 \) in (6) for the coercive bilinear form.

7. Numerical example

A computer program to solve Problem 2 is developed, in which a program of finite element method, Abaqus 6.14 (Dassault Systèmes), is used for solving Problem 1.

Fig. 2 (a) shows the finite element model used as an initial shell structure. The boundary conditions in Problem 1 and Problem 3 are shown in this figure. We use \( D = (0, 100) \times (0, 200) \) [mm\(^2\)] and \( \mu_0 = (x_1, x_2, \mu_3(x_1, x_2))^T \) in (1), where \( \mu_3(x_1, x_2) \) denotes the shape of shell in \( x_3 \) direction. The thickness \( t = 1 \) [mm] and \( 5 \) [mm] of the width of flange in the long sides are assumed. The bead height is limited within \( h = 5 \) [mm]. The values 210 [GPa] and 0.3 are used for Young’s modulus and Poisson’s ratio, respectively.

\[ \omega_1/(2\pi) = \omega_2/(2\pi) = 250 \, \text{[Hz]} \]

is used for the target frequency defined in (19).

Fig. 2 (b) shows the amplitude of displacement excited with periodic external load \( \bar{p}_N \) of 250 [Hz] in the initial shell structure.

Fig. 3 (a) shows the iteration history of the cost function \( f \) with respect to the number of reshaping. From the result, it is observed that \( f \) decreases monotonically and reaches 0.0%. Fig. 3 (b) shows frequency response functions with respect to the initial shape (dot line) and the optimum shape (solid line). We can see in this figure that 250 [Hz] become anti-resonance frequency in the optimum shape.

Fig. 4 (a) and (b) shows the optimum shape and the amplitude of displacement excited by 250 [Hz], respectively. We can see that the smooth beads are formed as reinforcing the clamped and loaded boundary. From the mechanical point of view, it is clear that the displacement at loaded point is reduced by forming these beads.

**References**


