A random thinning model with a latent factor for improvement of top-down credit risk assessment

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Abstract

We introduce a credit portfolio risk model within the “top-down approach” and then demonstrate applicability of our model to practical credit risk management via some empirical studies using some historical data on down-grades of Japanese firms. Specifically we present a simple random thinning model with some latent factor so as to explain the fact that downgrades are observed in some sub-portfolio much more or much less than expected naively.

Keywords Credit risk, top-down approach, random thinning, credit quality vulnerability factor

Research Activity Group Mathematical Finance

1. Introduction

Credit risk is the risk associated with financial losses caused by credit events such as debtors’ defaults and decreases in creditworthiness of debtors. Financial institutions strive to develop their credit risk quantification models for measuring the total credit risk of their large credit portfolio more accurately since they are required by regulation to adequately manage their total credit risk.

In this article we introduce one of the quantification models for credit portfolio risk based on the “top-down approach” and then demonstrate applicability of our model to practical credit risk management via some empirical studies using some historical data on down-grades of Japanese firms. The top-down approach has an advantage to allow a relatively simple representation of credit risk dependence among the constituents of a large portfolio. In general, credit risk modeling within the top-down approach has two steps. First, specify the model of occurrence intensity for target credit events in the universe portfolio of the model, without reference to constituents of the universe portfolio. Second, obtain individual event intensity models for sub-portfolios which is a subset of the universe portfolio according to the ratio of the number of constituents to that of the universe.

In this article we introduce a new idea to improve the second step of the top-down approach for modeling downgrade risk of sub-portfolios. Specifically we present a simple random thinning model with some latent factor so as to sufficiently capture time-series variation of allotted downgrade intensities for the sub-portfolios.

Several previous studies suggest some intensity models within top-down approach for assessing credit risks of large credit portfolios and/or pricing some credit portfolio derivatives. (Refer to [1–5].)

As for random thinning, the most primitive thinning model, employed in [1] and [5], is specified by the size of sub-portfolios. In short, the most primitive thinning model allocates the universe portfolio intensity to a sub-portfolio according to the ratio of the number of constituents in the sub-portfolio to that of the universe portfolio.

However, we observe that actual credit event can occur in some sub-portfolios and/or in some periods much more or much less frequently than the primitive random thinning model implies. As an example, Fig. 1 displays two time-series plots of the ratio of downgrades happened in the manufacturing sector to ones happened in the universe constituted by all firms rated by Rating and Investment Information, Inc. (R&I), one of the rating agencies in Japan.

One (dotted line) is the time series of the actual downgrade ratio obtained every half year by the ratio of the number of downgrades observed in the manufacturing sector to all downgrades in the universe, and the other (solid line) is the one implied by the most primitive thinning model, that is, given by the ratio of the number of the rated firms in the manufacturing sector to that of the constituents in the universe (all the firms rated by R&I).

As can be seen, Fig. 1 indicates that time-series of the actual downgrade ratio is more volatile than those implied by the most primitive thinning.

In this paper, we propose a new random thinning model with some latent factor in order to explain the time-variation around the naive downgrade ratio given by the size ratio as Fig. 1 implies. Also, we give some empirical results to examine the applicability of our new thinning model.
2. Top-down approach

We focus on modeling portfolio credit risk in the whole economy on a filtered complete probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,H]})\), where \(\{\mathcal{F}_t\}_{t \in [0,H]}\) is a right-continuous and complete filtration, and \(H > 0\) is a finite time horizon. We implicitly suppose that there are sufficiently many debtors which can be faced with some credit events such as defaults and credit rating transitions in our credit portfolio. Denote by \(S^*\) the set of all of such debtors, called the economy-wide portfolio. Let \(T^* = \{T_n\}_{n=1,2,...}\) be a strictly increasing sequence of totally inaccessible \(\{\mathcal{F}_t\}\)-stopping times with \(0 < T_1 < T_2 < \cdots\). We regard the stopping times \(\{T_n\}_{n=1,2,...}\) as the ordered credit event times observed in the economy-wide portfolio \(S^*\). Moreover, we specify that \(N^*_t\) has an intensity process \(\lambda^*_t\), which is indeed an \(\{\mathcal{F}_t\}\)-progressively measurable non-negative process such that the compensated process \(N^*_t - \int_0^t \lambda^*_ysds\) is an \(\{\mathcal{F}_t\}\)-local martingale. We call \(\lambda^*_t\) the total intensity (process). The total intensity is quite important for credit risk management since the total intensity specifies the probability distribution of the credit event times.

Let \(M( \geq 2)\) be a given integer. Assume that the economy-wide portfolio \(S^*\) is decomposed into \(M\) non-empty sub-portfolios \(\{S_i\}_{i=1,...,M}\) so that \(S_1 \cup \cdots \cup S_M = S^*\) and \(S_i \cap S_j = \emptyset\) (\(i \neq j\)).

When we focus on credit events occurred in some sub-portfolio \(S_i\), we need the intensity process \(\lambda^*_i\) for sub-portfolio \(S_i\). For the purpose, the procedure called a random thinning, a method of turning the total intensity \(\lambda^*_i\) into the sub-portfolio intensity \(\lambda^*_i\), is usually used in the top-down approach.

The random thinning is specified by \(\{\mathcal{F}_t\}\)-adapted processes \(\{Z^*_i\}_{i=1,...,M}\) such that

\[0 \leq Z^*_i \leq 1, \text{ for any } i = 1, \ldots, M,\] (1)

\[\sum_{i=1}^{M} Z^*_i = 1 \text{ a.s. for any } t \in [0,H].\] (2)

The process \(\{Z^*_i\}\) is called a thinning process associated with sub-portfolio \(S_i\).

In fact, we can notice that, for any sub-portfolio \(S_i\), the variable \(Z^*_i\) at time \(t\) can be regarded as the conditional probability that one credit event is observed in sub-portfolio \(S_i\) if it occurs in \(S^*\) at time \(t\). As a result, it follows that the intensity process \(\lambda^*_i\) for sub-portfolio \(S_i\) can be given by

\[\lambda^*_i = Z^*_i \lambda^*_i.\] (3)

We remark that \(\lambda^*_i\) is naturally associated with the counting process \(N^*_t\) of credit events observed in sub-portfolio \(S_i\), which is specified by \(N^*_t = \sum_{n \geq 1} 1_{\{T_n \leq t\} \cap \{T_n \in T^*_i\}}\), where \(T^*_i\) stands for the set of event times contained in sub-portfolio \(S_i\).

3. Random thinning model with some latent factor

As seen in the last section, the way of random thinning completely depends upon how to specify the thinning processes \(\{Z^*_i\}\).

We introduce our random thinning model \(\{Z^*_i\}\) for sub-portfolio \(S_i\) with some latent factor as follows:

\[Z^*_i = \theta^*_i \tilde{Z}^*_i.\] (4)

Here we suppose that \(\theta^*_i\) is a positive \(\{\mathcal{F}_t\}\)-adapted process and that \(\tilde{Z}^*_i\) is the most primitive thinning process, namely \(\tilde{Z}^*_i = X^*_i / X^*_i\) where \(X^*_i\) (resp. \(X^*_t\)) denotes the number of debtors contained in sub-portfolio \(S_i\) (resp. the economy-wide portfolio \(S^*\)) at time \(t\).

The most primitive thinning case of \(\theta^*_i = 1\), or \(Z^*_i = \tilde{Z}^*_i\), has been employed in the previous studies (for example in [1] and [5]) under the assumption of homogeneous sub-portfolios.

However it seems necessary to adjust the primitive thinning process with some random factor like (4) so as to explain what Fig. 1 indicates.

We regard the multiplier \(\theta^*_i\) for \(\tilde{Z}^*_i\) as some latent factor that can affect credit event frequency independent of the portfolio size. If \(\theta^*_i\) is more than one, the credit event frequency for sub-portfolio \(S_i\) is higher than is expected by the primitive thinning model. Thus the credit quality of sub-portfolio \(S_i\) can be seen as more vulnerable. On the other hand, if \(\theta^*_i\) is less than one, the credit events occur less in portfolio \(S_i\) than is expected by the primitive thinning model, so the credit quality of \(S_i\) would be less vulnerable.

As such, we tentatively call \(\theta^*_i\) “Credit Quality Vulnerability Factor (CQVF)".

For empirical illustrations below, we take up a piece-wise constant CQVF defined by

\[\theta^*_i = c^i + \epsilon^i,\] (5)

where \(c^i\) is some positive constant and \(\epsilon^i\) is a process with piece-wise constant paths.

Specifically, we assume that for a sequence
\( \{H_j\}_{j=0,1,2,...} \) of increasing times, we have
\[
\epsilon_i^t = \beta_j^t \quad \text{if} \quad H_j \leq t < H_{j+1},
\]
where \( \{\beta_j^t\}_{j=0,1,2,...} \) are independent random variables defined by \( \beta_j^t(a^t) = \eta_j^t(a^t) - E[\eta_j^t(a^t)] \) where \( \eta_j^t(a^t) \) is a random variable following a symmetric beta distribution with density \( f(x) = x^{a^t-1}(1-x)^{a^t-1}/B(a^t,a^t) \) and \( B(a,b) \) is the beta function, that is, \( B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \).

In addition, we should remark that the constant \( c_i^t \) should satisfy the inequality \( 1/2 \leq c_i^t \leq \delta_i^t - 1/2 \) with \( \delta_i^t = \min_{t \in [0,H]} X_i^t / \max_{t \in [0,H]} X_i^t \) in order to be consistent with condition (1). Since \( c_i \) must be given at time zero, we should set \( \max_{t \in [0,H]} X_i^t \) and \( \min_{t \in [0,H]} X_i^t \) a priori.

4. Empirical analysis

In this section, we show some empirical results obtained by analyses of historical data on downgrades of Japanese firms.

4.1 The data

As the data for our empirical study we use the historical records on downgrade from April 1, 2000 to September 30, 2013 of Japanese firms, which were rated by R&I.

We suppose that the economy-wide portfolio \( S^t \) is constituted by all the firms where R&I provides some credit rating. We focus on the manufacturing sector as the target sub-portfolio \( S_1 \). Sub-portfolio \( S_2 \) is given as the complement of \( S_1 \), namely non-manufacturing sector portfolio.

In fact, we pay attention to only the pair of what time a downgrade occurred and to which sub-portfolio the downgraded firm belonged. Hence we suppose that the data available at time \( t \), denoted by \( \mathcal{H}_t \), can be represented as follows:
\[
\mathcal{H}_t = \left( \bigcup_{j=1}^M T_j^t \right) \cap [0,t].
\]

4.2 Estimation procedure

In order to estimate our random thinning model specified by the piece-wise constant CQVF in (5) and after that, we employ the so-called generalized method of moments (GMM).

First we estimate the values \( \beta_j^t \) in (6) for each period \( \{[H_j, H_{j+1}]\}_{j=1,2,...,J} \) for some \( J > 0 \). In detail, we specify the estimate \( \hat{\beta}_j^t \) of the piece-wise constant CQVF by the following. For a subset \( A \), we denote by \( 1_A \) an indicator function of \( A \).
\[
\hat{\theta}_j^t = E \left[ \left\{ \frac{1_{(T_j^t \in T \cap [H_j,H_{j+1}])}}{Z_{T_j^t}} \right\} \right] = \frac{\sum_{n=n_j=1} N_{H_{j+1}} 1_{(T_j^t \in T \cap [H_j,H_{j+1}])}}{N_{H_{j+1}} - N_{H_j}},
\]
where \( Z_{T_j^t} \) stands for the observed ratio at \( n \)-th downgrade time of the number of constituents in sub-portfolio \( S_j \) to that of the universe \( S^t \).

Second, we use GMM to estimate the parameters \((c^t,a^t)\) of the piece-wise constant CQVF. The moment condition of GMM is given as below.
\[
\begin{align*}
E \left[ \hat{\theta}_j^t - c^t \right] &= 0, \\
E \left[ (\hat{\theta}_j^t)^2 \right] &= \frac{1}{3(20+\epsilon)} - (c^t)^2, \\
E \left[ (\hat{\theta}_j^t)^3 \right] &= \frac{3c^t}{4(20+\epsilon)} - (c^t)^3.
\end{align*}
\]

Finally we calculate an estimated transition of the thinning process \( Z_t^i \) via (4).

4.3 Results


We assume that each period \( [H_j, H_{j+1}] \) corresponds to \( j \)-th half-year from April 1, 2000 as the starting point.

Fig. 2 shows the time variation of half-year average estimated CQVF (solid line) for downgrades in the manufacturing sector during the in-sample period (Jan 2000 and Mar. 2010) as well as that of the range for one standard error around the half-year average for each period (dotted line). In Fig. 2, the estimated CQVF values are significantly above or below 1 in some periods. This result is consistent with the fact that downgrades occurred much more or less frequently than the primitive random thinning model expects.

Using the time-series of estimated CQVF \( \{\hat{\theta}_j^t\}_{j=1,2,...,J} \) in Fig. 2, we use GMM to obtain the estimates \( a^t = 0.90 \) (0.42) and \( c^t = 0.89 \) (0.06), where the standard error is reported in each parenthesis. Kolmogorov-Smirnov test, one of the goodness of fit tests, implies that the estimated CQVF model is not rejected under 10% significant level for in-sample (P-valued: 0.579) as well as for out-of-sample (P-valued: 0.364) and thus we cannot say CQVF model is inadequate.

Finally, we obtain every half year the distribution of the number of downgrades under the estimated CQVF model. In addition, we compare it with the number of actual downgrades for each half year.

The distribution at each period of the number of...
downgrades is achieved by Monte Carlo simulation with 10,000 scenarios generated from the estimated CQVF model.

Each scenario consists of the sub-portfolio downgrade times $T^1$ (manufacturing sector) and $T^2$ (non-manufacturing sector), which is generated by allocating each observed downgrade time $T_n$ to either $T^1$ or $T^2$ according to the estimated conditional thinning probability $Z_t^{1}$. The primitive thinning model $Z_t^i$ is calculated by the observed number of constituent firms for each portfolio.

In Fig. 3, the upper figure shows the observed number per half year of downgrades in our data as well as the boxplot for the distribution of the number of downgrades obtained from our random thinning model (4) with piecewise constant CQVF based on the beta distribution. In comparison to this, the lower figure shows the downgrade distribution obtained from the primitive thinning model with $Z_t^i$.

As one can see in Fig. 3, the observed numbers of downgrades in most half-year periods are located between 10% and 90% of the distribution obtained from our model, while in some periods the observations are out of the 10%-90% range of the distribution obtained from the primitive model. Thus, our random thinning model (4) seems to be more conformable to the variation of actual downgrades than the primitive one.

5. Concluding remarks

In this paper, we propose a random thinning model with some latent factor in order to improve portfolio credit risk, especially portfolio downgrade risk within the top-down approach. Specifically, we suppose that our random thinning model incorporates the latent factor called piecewise constant CQVF characterized by the beta distribution.

The model parameters are easily estimated with GMM from the historical downgrade samples of the Japanese firms. As a result of some empirical analysis, we notice that our model is adequate in the sense that our model seems to capture how downgrade frequency in some sector change over time more adequately than the primitive model.

Our model is quite tractable since we need only the historical event data for the universe portfolio and some information on the constituents of the target sub-portfolios. Naively we can also consider another way to model random thinning as a function of some observable variables. This would be studied in our future work.

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