A model of referendum

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Abstract

This study provides a theoretical framework to understand how campaign advertising works in a referendum. Our model analyzes a referendum with a straight choice between two alternatives, Yes or No. In this model, after the parties decide their policies, they promise benefits to voters during the campaign, which ultimately results in a fiscal cost and, thus, a burden to the voters themselves. We construct a two-party two-stage game in which parties choose a policy in the first stage and the benefits in the second stage. We show that one party shifts to a more extreme (polarized) position in an equilibrium.

Keywords referendum, municipal mergers, campaign advertising.

Research Activity Group Mathematical Politics

1. Introduction

Recently in Japan, referendums on policy issues with regard to local administration have been held; in other words, direct democracy has become common in Japan (see, e.g., Sorensen 2006) [1]. For example, the Japanese central government occasionally enacts large-scale municipal mergers across the nation to control the size of its municipal jurisdictions (see, e.g., Nakagawa 2016) [2]. Municipal jurisdiction mergers in Japan have a deep impact on the residents in those regions. Therefore, during the recent Heisei consolidation, some municipalities held a referendum via a municipal ordinance to decide on the merger. This was despite the fact that the law governing such mergers gave local representative assemblies the power to decide on the merger (see, e.g., Weese 2015) [3].

When a referendum is held, voters are usually asked to answer either Yes or No to an issue. On the one hand, both proposing and opposing sides are fixed; on the other hand, there are more voters who have not decided which side to vote for. We call those who could switch to either side “switchers.” In such a referendum, one of either side who succeeds in obtaining switchers will win the referendum. To obtain switchers, both a proposing party and an opposing party conduct a campaign either for or against the issue in a referendum in order to entice switchers to move to their side. A proposing party usually conducts a positive campaign and an opposing party usually conducts a negative campaign. In fact, both proposing and opposing campaigns are so polarized that we could consider them as either positive campaigns for the issue or negative campaigns against the issue. (see, e.g., Lemenicier 2005, Hobolt 2009) [4, 5].

We construct a model based on the spatial theory of voting à la Downs (1957) [6] to consider this situation, incorporating the following three assumptions, and examine whether campaigns diverge extremely in an equilibrium. The first assumption is that parties cannot discriminate between their enthusiastic supporters and switchers when they conduct their campaigns. Next, the preferences of each party’s supporters and switchers are not distributed uniformly over the policy space in which parties choose the characteristics of their campaigns, but instead, voters’ ideology is a discrete distribution. Third, if parties’ campaigns focus on switchers, their enthusiastic supporters will not be satisfied by the campaigns.

2. Model

Citizens are exogenously assigned to groups: group 1 citizens vote Yes only; group 2 citizens vote No only; and group 3 citizens vote either Yes or No, depending on which option offers the highest net utility. For ease of reference, we classify these three groups into two types: informed and uninformed. Switchers are called informed voters and citizens who belong to groups 1 and 2 are collectively called uninformed voters. Groups 1 and 2 are the same size. We normalize the number of uninformed voters of each group to 1, while x denotes the number of informed voters. Furthermore, we assume that x ≥ 1, namely the group of group 3 voters, is at least as large as group 1 or 2 voters. Our model considers a spatial model, such as Downs; the explicit location point of each voter represents her ideology. In this case, 0 and 1 represent the ideologies of uninformed voters and 1/2 represents informed voters’ ideology. This is considered to be equivalent to the voters’ location point in Kamada and Kojima (2014) [7].

There are two parties in this model: proposing and opposing. The two parties compete on policy for voters, who are either uninformed or informed. In addition, we assume that each party provides one issue campaign. In other words, the proposing party chooses 0 or 1/2 and the opposing party chooses 1 or 1/2. In this model, those who fit a campaign pay no additional cost. If not, an additional cost 1/2 is incurred.

Let us consider the payoff of each party V_i, (i = 1, 2).
Uninformed voters, $C_i$, always support one party $i$, whereas informed voters, $C_i$, support either party. Let us consider party 1’s payoff: party 2’s payoff can be found analogously. Let $V_i$ denote party 1’s payoff. This is written as $V_i = V_{C_1}^i + V_{C_2}^i$, where $V_{C_1}^i$ is party 1’s payoff from uninformed voters and $V_{C_2}^i$ is party 1’s payoff from informed voters. First, we define each party’s payoff as the number of votes in the election. Thus, we can calculate each component as follows: $V_{C_1}^i = 1$ and $V_{C_2}^i = x$. When both uninformed and informed voters support party 1’s policy, party 1 obtains $1 + x$. This game is described by the following $2 \times 2$ matrix, see Table 1. Clearly, we find a pure-strategy equilibrium. In an equilibrium, both choices are informed, that is, $1/2$. In other words, campaigns conducted by both parties are not polarized.

As a result, it seems that a model based on spatial theory fails to explain the split of the two parties’ campaigns. However, this is incorrect, because the trade-off from which campaign polarization emerges is lacking in the aforementioned setting. The model above describes that parties choose their policies when a campaign starts. However, this model does not include what happens during a campaign. After a campaign starts, parties promise benefits to move switchers to their side. Voters pay attention to the benefits they will obtain by voting for a party. In particular, for switchers, who have a neutral ideology, the benefits they will obtain are more important than they are for those who strongly support either side. Thus, we need to introduce benefit competition into this spatial voting game. In other words, we can use a two-stage model to analyze the polarization. This benefit competition generates pressure to choose a policy when campaign advertisement begins. This trade-off may result in a polarization of campaign advertisement.

We construct a two-party two-stage game. Political campaigns are set simultaneously in the first stage; in the second stage, the chosen campaign becomes broadly known and the benefits $p_i$, ($i = 1, 2$) of its policy that underlies its campaign are set simultaneously. We solve this game by backward induction.

The two parties compete on policy and its benefit to voters, who are either uninformed or informed. Let us consider the payoff of each party $V_i$. The parties maximize their payoffs. Let us consider party 1’s payoff; party 2’s payoff can be found analogously. Let $V_i(p_1, p_2)$ denote party 1’s payoff. This is written as $V_i(p_1, p_2) = V_{C_1}^i(p_1) + V_{C_2}^i(p_1, p_2)$. Here, unlike the earlier payoffs, the definition of each party’s payoff depends on $p$. We define each party’s payoff as “$p$ times the number of votes obtained by each party.” Now, we can calculate each component as follows: $V_{C_1}^i(p_1) = p_1$ and $V_{C_2}^i(p_1, p_2) = p_1x$. When both uninformed and informed voters support party 1’s policy, party 1 obtains $p_1(1 + x)$.

The reserve values (resp. densities) of the groups are 1 (1) for uninformed voters and $y(x)$ for informed voters. Voter utilities can be formulated in their reservation utility minus their costs, which include $p$ and their foot cost 0 or 1/2. Uninformed voters always support one party, whereas informed voters support either party. Equations (1) and (2) show the utility for uninformed $C_i$ and informed $C_3$ voters, here $i = 1, 2$.

$$u_{C_1} = 1 - p_i - \text{foot cost}, \quad (1)$$

$$u_{C_3} = y - p_i - \text{foot cost}. \quad (2)$$

Voters cannot accept negative utility when $p$ exceeds their reservation utility, 1 or $y$; hence, whenever $u < 0$, $u = 0$ instead.

Although $p$ is a cost, which is the same as the foot cost not actually paid by voters, it also affects the decision making of voters. We introduce this second parameter $p$ into this model in addition to the foot cost and then define the payoff of each party as “$p$ times the number of votes obtained by each party in the election.” This payoff is different from the earlier payoffs defined as the number of votes obtained by each party in the election. As a result of this modification of the model, we obtain a polarization of the policy in the equilibrium. Importantly, $p$ has no convection with securing financial resources. Thus, our model does not consider fiscal expansion after the election. We also assume that a source of revenue to give a benefit to voters could be secured after the election. In other words, citizens will bear the burden of providing the benefit and this burden itself in the future will not affect their behavior.

### 3. Benefit game

In this section, we analyze the subgames given a policy campaign by each party. Under the assumption of a voter’s behavior, subgames are classified into the following three cases: 1) both parties focus on uninformed voters; 2) both parties focus on informed voters; and 3) party 1 focuses on informed voters, whereas party 2 focuses on uninformed voters. In this section, we show that the equilibrium payoff in each subgame is determined uniquely.

#### 3.1 Both parties focus on uninformed voters

Here, both parties focus on uninformed voters. Therefore, we find that both parties can persuade uninformed voters at $p_i = 1$, $i = 1, 2$ and can obtain $V_i = 1$, $i = 1, 2$. When party $i$ charges $p_i > 1$, because it could not persuade uninformed voters, party $i$ obtains $V_i = p_i x$. Furthermore, even if party $i$ persuades informed voters at $p_i \leq 1$, because of $p_i(1 + x) \geq 1$, party $i$ does not charge $p_i \leq 1/(1 + x)$.

Let us consider an equilibrium at which either party $i$ persuades informed voters, without loss of generality. In this equilibrium, party $j, j \neq i$ always charges $p_j = 1$. Now, we obtain $p_i < 1$ when $p_i = 1$ is given. This is because if party $i$ obtains $V_i = p_i x$ with $p_i > 1$, from $x \geq 1$, the opposition party $i \neq i$ always chooses $1 < p_j < p_i$. Thus, no equilibrium exists such that $p_i > 1, p_j = 1$. Furthermore, we find that $p_i = 1, p_j = 1$ is not an equilibrium benefit pair when $p_j = 1$.

Moreover, if a party cannot obtain $p_i x > 1$, it chooses...
\(p_i = 1\) and decides to persuade only uninformed voters with its density 1 because it can always obtain \(V_i = 1\). It follows that when a party wants to obtain informed voters, its lower bound benefit is \(p_i = 1/x\). In addition, it follows that \(1/x \leq 1\) from \(x \geq 1\). Therefore, we also find that party \(i\) can obtain \(V_i > 1\), which is obtained only from informed voters when it charges \(p_i\) such that \(1/x < p_i \leq 1\), and persuades only informed voters. Here, we also obtain \(1/(1+x) \leq 1/x\) from \(x \geq 1\). Thus, we find that the lower bound of undercut benefit competition is always \(1/(1+x)\) in an equilibrium at which \(p_i \leq 1\) is charged.

As mentioned earlier, \(p_i = p_j = 1\) is not an equilibrium benefit. Furthermore, we find that \(p_i = 1/(1+x)\) is not the best response when opponent \(j \neq i\) charges \(p_j = 1\). This is because we obtain \(1/(1+x) \leq 1/2\) from \(x \geq 1\); thus, party \(i\) increases its payoff by charging benefit \(p_i\) such that \(1/(1+x) < p_i \leq 1\). On the contrary, we find that opponent \(j\)'s \(p_j = 1\) is not the best response when \(p_i\) is given such that \(1/(1+x) < p_i \leq 1\). This is simply because \(j\) has an incentive to deviate to \(p_j = 1/(1+x)\).

Finally, we show that \(p_i = p_j = 1/(1+x)\) is not an equilibrium benefit. We obtain \(V_i = (1+x)/(2/(1+x) = (x+2)/(2[x+1])\) if \(p_i = p_j = 1/(1+x)\). This equation is monotonically decreasing with regard to \(x\). Then, we obtain \(\lim_{x \to \infty} 1/(1+x) = 1/2\). According to \(x \geq 1\), it follows that the maximum of this equation is 3/4 because it has a maximum value at \(x = 1\). For this reason, opponent \(j\) deviates to \(p_j = 1\).

Similar arguments apply to the case of replacing \(i\) and \(j\) because \(i\) and \(j\) are symmetric. Thus, we find that a pure-strategy equilibrium in this subgame does not exist. Let us consider a mixed-strategy equilibrium \(F^*(p)\) with the following support, where both parties \(i, 2\) obtain \(V_i^* = 1/(1+x) \leq p_i \leq 1\). By solving \((1-F^*(p))p_i(1+x) + F^*(p)p_i = V_i^*\), we obtain \(F^*(p) = 1 - (V_i^* - p_i)/(p_i x)\) (see, e.g., Varian 1980) [8].

3.2 Both parties focus on informed voters

With regard to informed voters, both parties are symmetric in that neither has locational advantages, and this competition leads to simple Bertrand competition. Now, we consider whether uninformed voters support their supporting party. We obtain \(p_i \leq 1/2, i = 1, 2\) by solving the following equation: \(1 - p_i - 1/2 \geq 0\).

Now, each party is guaranteed to earn a payoff \(V_i = 1/2, i = 1, 2\), because both can be supported by their uninformed voters at benefit \(p_i = 1/2\). Thus, from \(p_i(1+x) \geq 1/2\), we find that neither party charges a benefit such that \(p_i \leq 1/2(1+x)\). Here, \(x \geq 1, 1/2(1+x) \leq 1/4\) holds.

Furthermore, if a party charges \(p_i > 1/2\), it obtains a payoff \(V_i = p_i x\) because uninformed voters abandon it. Now, we consider the condition such that \(p_i x > p_i(1+x)\). We obtain \(p_i > (1+x)/(2x)\) from \(p_i x > (1+x)/2\). Thus, we find that when a party focuses only on informed voters and charges benefit \(p_i > (1+x)/(2x)\), it obtains a higher payoff than \((1+x)/2\). In addition, we obtain \(1/2 < (1+x)/(2x) \leq 1\). According to \(\lim_{x \to \infty} (1+x)/(2x) = 1/2\), we obtain the lower value of this equation. The upper value is obtained by substituting \(x = 1\) into this equation. Thus, we obtain \(V_i,\ i, j = 1, 2, i \neq j\) as follows: \(p_i(1+x), 1/[2(1+x)] \leq p_i \leq 1/2, p_i < p_j; p_i x, p_i > (1+x)/(2x), p_i < p_j\).

Either party always has an incentive to undercut its opponent’s benefit when both parties charge benefits such that \(p_i x > (1+x)/2\). This is simple Bertrand competition, which results in \(p_i = (1+x)/(2x), i = 1, 2\). Now, from \(x \geq 1\), we find that \((1+x)/4 - 1/2 = (x-1)/4 \leq 0\). This equation means that when this benefit competition results in the lower bound benefit, \(p_i = (1+x)/(2x), i = 1, 2\), both parties are guaranteed to earn a higher payoff than \(1/2\). Moreover, we obtain \(1/2 < (1+x)/(2x)\) above; thus, we find that the lower bound benefit in this competition is higher than \(1/2\), which is also the benefit at which the payoff is guaranteed from uninformed voters. However, we obtain \(1/2(2x)[1+x(x/2) = (1+x)/4 < (1+x)/2\). Thus, this competition, both parties finally undercut their benefits at \(1/2\), because they persuade both uninformed and informed voters and then earn more payoffs. Thus, we conclude that \(p_i > 1/2\) is not charged.

It follows that the upper and lower bounds of benefit are as follows: \(1/[2(1+x)] \leq p_i \leq 1/2\). No pure-strategy equilibrium exists in this subgame, and we also consider a mixed-strategy equilibrium \(F^*(p)\) to obtain \(V_i^*\) as in the case analyzed before.

3.3 Either party focuses on informed voters

Because of symmetry, without loss of generality, we assume that party 1 focuses on promotion to informed voters through intensive advertising. Party 1 always persuades its loyal supporters when it charges \(p_1 = 1/2\). It follows that party 1 is guaranteed to bring \(V_1 = 1/2\). Thus, party 1 does not charge its benefit such that \(p_1 < 1/[2(1+x)]\). Informed voters are indifferent between party 1 and party 2 if \(y - p_1 - 0 = y - p_2 - 1/2\) holds. We obtain \(p_1 = p_2 + 1/2\). Thus, party 1’s payoff \(V_2\) is obtained as follows: \(p_1(1+x), 1/[2(1+x)] \leq p_1 \leq 1/2, p_1 < p_2 + 1/2; p_1 x, p_1 > 1/2, p_1 < p_2 + 1/2\).

For a given \(x\), let us consider a benefit at which an equal payoff is earned for party 1 obtaining both informed and uninformed voters at \(p_1 = 1/2\) or party 1 obtaining only informed voters. In this case, \(p_1 x = (1+x)/2\) holds. We obtain \(p_1 = (1+x)/(2x)\). Here, \(1/2 < (1+x)/(2x) \leq y\) holds.

Now, for any \(x\) such that \(x \geq 1\) holds, \(1/2 < (1+x)/(2x) \leq 1\) holds. Thus, when \(p_2 = 1\) is given, for some \(x\), party 1 can find a benefit \(p_1\) such that its payoff increases and \(p_1 x \geq (1+x)/2\) holds. The left-hand side of this equation denotes payoff \(p_1 x\), which is earned only from informed voters. The right-hand side of this equation denotes the maximum payoff \((1+x)/2\) gained from both informed and uninformed voters. In other words, party 1 could abandon 1-loyal voters. This deviation improves party 1’s payoff to an even larger extent because \(p_1 x > (1+x)/2\) holds when party 1 chooses a benefit \(p_1\) such that \(p_1 > (1+x)/(2x)\) holds. This finding indicates that \(p_1 = (1+x)/(2x)\) is not the best response to \(p_2 = 1\).

On the contrary, when party 1 chooses its benefit such that \(p_1 > (1+x)/(2x)\) holds, \(p_2 = 1\) is not the best
response. This is because party 2 has an incentive to change its benefit from \( p_2 = 1 \) to \( p_2 = 1/(1+x) \). Let us now consider the incentive of party 2. When \( p_1 = (1+x)/(2x) \) is given, party 2 can obtain informed voters at \( p_2 < 1 \) and earn a payoff such that \( V_2 = p_2 (1+x) > 1 \).

In addition, party 2 must choose its benefit such that \( p_2 > 1/(1+x) \) holds in order that it gains a payoff such that \( V_2 = p_2 (1+x) > 1 \). Further, party 2 does not lose its locational advantage for its uninformed voters, which is equal to the transportation cost margin 1/2. Because of this margin, party 2 undercuts its opponent by charging \( p_2 = 1/(1+x) \) when \( p_1 - p_2 = (1+x)/(2x) - 1/(1+x) \geq 1/2 \) holds. \((x^2 + 1)/[x(x+1)] \geq 1\). However, when \( p_2 = 1/(1+x) \) is given, \( p_1 > (1+x)/(2x) \) is not the best response of party 1 because it has an incentive to change its benefit to its payoff is 0.

Let us consider the condition that \( p_1 \) is lower than \( p_2 \), including the transportation cost margin 1/2. According to \( p_1 \leq p_2 + 1/2 \) and \( p_2 = 1/(1+x) \), we obtain \( p_1 \leq 1/2 + 1/(1+x) \). By taking \( 1/2 < 1/2 + 1/(1+x) \) into consideration, party 1 can obtain both informed and uninformed voters when it charges \( p_1 = 1/2 \). However, \( p_2 = 1 \) is the best response to this undercutting. In addition, we find that when \( p_2 = 1 \) is given, \( p_1 = 1/2 \) is not the best response, according to the discussion above.

It follows that no pure-strategy equilibrium exists in this subgame. At the same time, we find that the support of \( p_2 \), now that party 2 focuses on its uninformed voters, is as follows: \( 1/(1+x) \leq p_2 \leq 1 \). On the contrary, party 1 charges benefit \( p_1 \) such that \( 1/2 \leq p_1 \leq y \) holds. We now consider a mixed-strategy equilibrium \( F^*(p) \) to obtain \( V^*_1 \). By solving \((1 - F^*(p)/p_1)(1+x) + F^*(p)/p_1 = V^*_1 \), we obtain \( F^*(p) = 1 - (V^*_1 - p_1)/(p_1x) \). In an equilibrium, both parties obtain \( V^*_1 = (1+x)/2 \). Here, \((1 - F^*(p)/p_1)(1+x) + F^*(p)/p_1 = V^*_1 \) holds.

Finally, we consider the assumption with regard to \( y \). In the discussion above, we consider the case in which party 1 charges \( p_1 = 1 \). From \( xy > (1+x)/2 \), we obtain \( y > (1+x)/(2x) \). In addition, we find \( 1/2 \leq y \leq 1 \) from \( x \geq 1 \). Thus, \( x, y \) are well defined.

4. Location game

In this section, we solve the first stage of the so-called “location game.” Each party can expect to receive at least those voters loyal to it, and, furthermore, it earns a payoff at least equal to charging the maximum benefit that those voters are willing to pay. Thus, when either party that is farther from informed voters obtains 1, the opposition party that focuses on informed voters obtains \((1+x)/2\). When both focus on informed voters, they obtain 1/2.

We consider a subgame perfect equilibrium of this game. This location game is described by the following 2 x 2 matrix, see Table2. Clearly, we find two pure-strategy equilibria. In both equilibria, one party chooses informed voters, while the other chooses uninformed voters. Furthermore, we find a symmetric mixed-strategy equilibrium in which both parties obtain \( V^*_1 = 1 \). Let \( \lambda \) be the probability of a party choosing a campaign to uninformed voters. By

\[
1\lambda + 1(1-\lambda) = \frac{x^2 + 1}{2} \lambda + \frac{1}{2}(1-\lambda),
\]

we obtain \( \lambda^* = 1/x \).

5. Concluding remarks

In this study, we presented a model that analyzes a referendum with a straight choice between two alternatives, Yes or No. In a pure-strategy equilibrium, one party focuses on informed voters, while the other focuses on uninformed voters. In other words, the parties shift to more extreme positions (i.e., they become polarized). In this model, we also obtained a mixed-strategy equilibrium in addition to the pure-strategy equilibria.

We introduce the second parameter \( p \) into a spatial model like Downs. However, in our model, \( p \) is the theoretical hypothesis. We thus have to consider this second parameter empirically. This task remains for future research. We now consider an extreme case in which each party chooses only either informed voters or uninformed voters. However, it is more natural to consider the case in which the continuum interval between the location points of the informed and uninformed voters is the point at which a party determines the characteristics of its campaign. For example, a party chooses a midway point between uninformed and informed voters: in other words, it chooses its campaign to have a non-committal/gray attitude. This theoretical extension to a continuum interval also remains for future research.

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