Remarks on optimal strategies to utility maximizations in continuous time incomplete markets

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Abstract

In continuous time diffusion models, the optimal strategies to utility maximizations can be obtained by solving a certain partial differential equation. In this paper, we give another proof of this fact in an incomplete market without using the well-known fictitious security arguments. Since we avoid using the fictitious security arguments, we can apply our method to the situations when the markets cannot be completed. We provide an example of such cases where the asset price follows a simple jump process with unpredictable jump sizes and see that we can derive the equation which determines the optimal strategy as usual.

Keywords utility maximization, incomplete markets, martingale methodologies

Research Activity Group Mathematical Finance

1. Introduction

Theorem 7 in [1] derives the partial differential equation (PDE) which provides the optimal strategies to utility maximization problems in incomplete markets by the well-known fictitious security arguments which are originally developed by [1, 2]. In this paper, the same result as Theorem 7 in [1] will be obtained without fictitious security arguments when maximizations of utility from terminal wealth are concerned. Moreover, we will provide an example in which our derivation of optimal strategies works in a simple jump model where the fictitious security method cannot be applied.

The martingale approaches for utility maximization problems are originally studied by [3–6] in complete markets and by [1, 2, 7] in incomplete markets. In particular, Theorem 2.1 in [4] gives the PDE which yields the optimal strategies in complete markets in which price processes are supposed to be Markovian. The PDE which provides the optimal strategies to utility maximization problems in incomplete diffusion models is provided initially by Theorem 7 in [1] under the assumption that price processes are Markov processes and the result is a direct extension of Theorem 2.1 in [4].

However, incompleteness of markets makes the derivation of the PDE somewhat complicated. The author believes that the completion of the market with fictitious securities and the duality arguments are not crucial to get the result. In this paper, the optimal strategy will be derived as a simple necessary condition of optimality without using fictitious security arguments. We will also provide an example to derive a similar result in a simple jump model. In [8], it is shown that when a market model admits jumps with unpredictable jump sizes, the model is not able to be completed by any number of additional securities, so the fictitious security arguments

cannot be applied. Nevertheless, our method still works in such a case since the fictitious completion is not used.

The rest of the paper is organized as follows. In Section 2, a security market model by Brownian motions will be specified, all the equivalent martingale measures in the model will be characterized and a utility maximization problem will be recalled. Under the assumption that the utility maximization problem admits an optimal strategy, Section 3 will derive the PDE which provides the optimal strategy as a necessary condition of optimality. An example of a simple jump model is exhibited in Section 4. Section 5 concludes the paper.

2. Model settings and a utility maximization problem

In this section, a utility maximization problem is recalled.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T > 0$ be the finite terminal time. Let $W(t) = (W^1(t), W^2(t), \dots, W^N(t))^T, 0 \leq t \leq T$ denote an N -dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ and we suppose that we have a filtration $\mathcal{F}_t, 0 \leq t \leq T$ which is the augmented filtration generated by W . Note that, for a matrix A , A^T denotes the transpose of A .

We consider a single investor's utility maximization problem. We assume that there are $M (\leq N)$ risky assets that the investor can trade in a market. The discounted price process of those risky assets, denoted by $S(t) = (S^1(t), S^2(t), \dots, S^M(t))^T, 0 \leq t \leq T$, is supposed to be defined by

$$\begin{aligned} dS(t) &= a(t)dt + b(t)dW(t), \quad 0 \leq t \leq T, \\ S(0) &= S_0 \end{aligned}$$

where a is an $M \times 1$ vector valued progressively measurable process, b is an $M \times N$ matrix valued progressively measurable process and $S_0 > 0$ is a constant. We suppose further that

$$\int_0^T |a(t)| dt < \infty \text{ a.s.}, \quad \int_0^T |b(t)|^2 dt < \infty \text{ a.s.}$$

and $\text{rank}(b(t)) = M$ for all t where, for a matrix A , we assume $|A| = \text{tr}(AA^\top)^{\frac{1}{2}}$.

By Proposition 2.2 in [7] or Proposition 1 in [1], the Radon-Nikodym derivatives of equivalent martingale measures in the current model are characterized as follows: If Q is an equivalent martingale measure with respect to P , then $dQ/dP = \xi_\nu(T)$ where

$$\begin{aligned} \xi_\nu(t) = \exp \left\{ \int_0^t (\kappa(s) + \nu(s))^\top dW(s) \right. \\ \left. - \frac{1}{2} \int_0^t (|\kappa(s)|^2 + |\nu(s)|^2) ds \right\}, \\ 0 \leq t \leq T \end{aligned} \quad (1)$$

with $\kappa(t) = -b(t)^\top (b(t)b(t)^\top)^{-1}a(t)$, $0 \leq t \leq T$ and $\nu \in \text{Ker}(b)$ where

$\text{Ker}(b) = \{\nu | \nu \text{ is progressively measurable,}$

$$\nu(t) \in \mathbb{R}^N \text{ and } b(t)\nu(t) = 0, 0 \leq t \leq T\}$$

and $\int_0^T |\nu(t)|^2 dt < \infty$ a.s.

The whole set Θ of the investor's strategies is defined by

$$\begin{aligned} \Theta := \left\{ z(t) = (z^1(t), z^2(t), \dots, z^M(t))^\top, 0 \leq t \leq T \right. \\ \left. \begin{aligned} & \left| z \text{ is adapted, } \int_0^T |z(t)^\top a(t)| dt < \infty \text{ a.s.} \right. \\ & \left. \text{and } \int_0^T |z(t)^\top b(t)|^2 dt < \infty \text{ a.s.} \right\} \end{aligned} \right\} \end{aligned}$$

where $z^i(t), i = 1, 2, \dots, M$ is the number of the i th asset that the investor holds at time $t, 0 \leq t \leq T$. Obviously, Θ forms a vector space on $\mathbb{R}$. The investor's gain processes are defined by

$$G_t(z) := \int_0^t z(s)^\top dS(s), \quad z \in \Theta, \quad 0 \leq t \leq T.$$

Let $x > 0$ be the investor's initial endowment. Then the value of a self-financed portfolio corresponding to a strategy $z \in \Theta$ at time t can be obtained by $x + G_t(z)$. Suppose that the investor has a utility $u(y)$ defined on $(k, \infty), -\infty < k < \infty$. Suppose further that u is a continuously differentiable, strictly increasing and strictly concave function and $I(y) := (u')^{-1}(y), \lim_{l \rightarrow k+0} u'(l) < y < \infty$ is continuously differentiable.

We will consider the following utility maximization problem:

$$\sup_{z \in \Theta} E[u(x + G_T(z))]. \quad (2)$$

3. Optimal strategies in Brownian models

In this section, the PDE which yields the optimal strategy to the problem (2) is obtained.

Before providing the result, we need some preparations. We assume that the problem (2) admits an optimal strategy which is denoted by $z_0(t), 0 \leq t \leq T$. We define a functional f on Θ by

$$f(z) := E[u(x + G_T(z))], \quad z \in \Theta.$$

Theorem 1 of Section 7.4 in [9] suggests that if z_0 is an optimal strategy, the Gateaux differential of f at z_0 equals zero, i.e.,

$$\begin{aligned} \frac{\partial}{\partial \alpha} f(z_0 + \alpha z)|_{\alpha=0} = E[u'(x + G_T(z_0))G_T(z)] = 0, \\ \forall z \in \Theta \end{aligned}$$

where $\alpha \in \mathbb{R}$. Then, since $u' > 0$, we can claim that $u'(x + G_T(z_0))/E[u'(x + G_T(z_0))]$ is the Radon-Nikodym density of an equivalent martingale measure. Therefore, there exists a process $\nu_0 \in \text{Ker}(b)$ such that

$$\frac{u'(x + G_T(z_0))}{E[u'(x + G_T(z_0))]} = \xi_{\nu_0}(T) \quad (3)$$

where $\xi_{\nu_0}(t), 0 \leq t \leq T$ is the process defined by (1) corresponding to ν_0 .

The following theorem provides an explicit expression of z_0 when (ξ_{ν_0}, S) are Markov processes. Below we use the abbreviation that $F_t = \partial F(t, \xi, S)/\partial t$, $F_\xi = \partial F(t, \xi, S)/\partial \xi$, $F_{\xi\xi} = \partial^2 F(t, \xi, S)/\partial \xi^2$, F_S is a $1 \times M$ vector valued process $\partial F(t, \xi, S)/\partial S$, $F_{S\xi}$ is a $1 \times M$ vector valued process $\partial^2 F(t, \xi, S)/\partial S \partial \xi$ and F_{SS} is an $M \times M$ matrix valued process $\partial^2 F(t, \xi, S)/\partial S^2$.

Theorem 1 Suppose that (ξ_{ν_0}, S) forms a Markov process where ξ_{ν_0} is defined by (3) and z_0 is an optimal strategy to (2). Define the function $F(t, \xi, S)$ by

$$\begin{aligned} F(t, \xi_{\nu_0}(t), S(t)) \\ := \frac{1}{\xi_{\nu_0}(t)} \\ E[I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T)|\xi_{\nu_0}(t), S(t)] \end{aligned}$$

where $I = (u')^{-1}$. Suppose further that F is a $C^{1,2,2}$ function and $F_\xi \neq 0$. Then F satisfies the partial differential equation

$$F_t + \frac{1}{2} |\kappa|^2 \xi_{\nu_0}^2 F_{\xi\xi} + |\kappa|^2 \xi_{\nu_0} F_\xi + \frac{1}{2} \text{tr}(F_{SS} b b^\top) - F_{S\xi} a = 0$$

with the boundary conditions

$$\begin{aligned} F(T, \xi_{\nu_0}(T), S(T)) &= I(E[u'(x + G_T(z_0))\xi_{\nu_0}(T)]), \\ F(0, 1, S_0) &= x \end{aligned}$$

where $\nu_0 = 0$. Moreover, the optimal strategy z_0 is given from F by

$$z_0^\top = \xi_{\nu_0} F_\xi \kappa^\top b^\top (b b^\top)^{-1} + F_S.$$

Proof Since $u'(x + G_T(z_0)) = E[u'(x + G_T(z_0))]\xi_{\nu_0}(T)$, $x + G_T(z_0) = I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))$.

By multiplying both side of this equation by $\xi_{\nu_0}(T)$, we obtain

$$(x + G_T(z_0))\xi_{\nu_0}(T) = I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T).$$

Since the left hand side of this equation is a P -martingale, by taking conditional expectations,

$$\begin{aligned} (x + G_t(z_0))\xi_{\nu_0}(t) \\ = E[I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T)|\mathcal{F}_t]. \end{aligned}$$

Then we get

$$\begin{aligned} x + G_t(z_0) \\ = \frac{1}{\xi_{\nu_0}(t)} E[I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T)|\mathcal{F}_t] \\ = \frac{1}{\xi_{\nu_0}(t)} \\ E[I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T)|\xi_{\nu_0}(t), S(t)] \\ = F(t, \xi_{\nu_0}(t), S(t)). \end{aligned}$$

By Itô's formula,

$$\begin{aligned} z_0^\top adt + z_0^\top bdW \\ = \left(F_t + \frac{1}{2}(|\kappa|^2 + |\nu_0|^2)\xi_{\nu_0}^2 F_{\xi\xi} + F_S a + \frac{1}{2}\text{tr}(F_{SS}bb^\top) \right. \\ \left. + F_{S\xi}b\kappa \right) dt + (\xi_{\nu_0} F_\xi(\kappa + \nu_0)^\top + F_S b) dW. \end{aligned}$$

Then we obtain the following simultaneous equations:

$$\begin{aligned} z_0^\top a = F_t + \frac{1}{2}(|\kappa|^2 + |\nu_0|^2)\xi_{\nu_0}^2 F_{\xi\xi} + F_S a \\ + \frac{1}{2}\text{tr}(F_{SS}bb^\top) + F_{S\xi}b\kappa, \end{aligned} \quad (4)$$

$$z_0^\top b = \xi_{\nu_0} F_\xi(\kappa + \nu_0)^\top + F_S b. \quad (5)$$

By multiplying both sides of (5) by $b^\top(bb^\top)^{-1}$, $z_0^\top$ is determined as

$$z_0^\top = \xi_{\nu_0} F_\xi \kappa^\top b^\top (bb^\top)^{-1} + F_S. \quad (6)$$

Substituting this into (5), we have

$$\xi_{\nu_0} F_\xi \kappa^\top = \xi_{\nu_0} F_\xi(\kappa + \nu_0)^\top$$

by the definition of κ and we get $F_\xi \nu_0^\top = 0$. This implies $\nu_0 = 0$ by the assumption $F_\xi \neq 0$. From (6) and $\nu_0 = 0$, (4) becomes

$$\begin{aligned} F_t + \frac{1}{2}|\kappa|^2 \xi_{\nu_0}^2 F_{\xi\xi} + |\kappa|^2 \xi_{\nu_0} F_\xi + \frac{1}{2}\text{tr}(F_{SS}bb^\top) - F_{S\xi}a \\ = 0. \end{aligned}$$

(QED)

Remark 2 Theorem 1 is a special case of Theorem 7 in [1]. In our model, there is no state variable process which is denoted by Y in [1]. Our result can be obtained by assuming that $Y \equiv 0$ in Theorem 7 of [1].

4. An optimal strategy in a jump model

In this section, we provide an example which shows that the derivation of the equation which provides the optimal strategy given in the previous section may work in models where the fictitious security method cannot be applied.

Let $N(t), 0 \leq t \leq T$ be a Poisson process defined on $(\Omega, \mathcal{F}, P)$ with the intensity $\lambda > 0$ and $T_i, i = 1, 2, \dots$ denote times of N 's i -th jump. It is assumed that we have the filtration $\mathcal{F}_t^N, 0 \leq t \leq T$ that is generated by N . Let $C_i, i = 1, 2, \dots$ be identically and independently distributed (i.i.d.) random variables which are supposed to have common distribution μ . Suppose further that each C_i is $\mathcal{F}_{T_i}$ -measurable and C_i and N are independent for $i = 1, 2, \dots$. Moreover, we define an adapted process $c(t)$ by

$$c(t) := \sum_{i=1}^{\infty} C_i 1_{\{T_i \leq t < T_{i+1}\}}(t).$$

Suppose that there are one risky asset and one risk-free asset in the market. We suppose further that the discounted asset price process $Y(t)$ is given by

$$\begin{aligned} dY(t) &= Y(t-)(\alpha dt + c(t)dN(t)), \quad 0 \leq t \leq T, \\ Y(0) &= Y_0 \end{aligned} \quad (7)$$

where α and Y_0 are constants and $\alpha \neq 0$. By [8], we realize that since the jump sizes of this asset price process are unpredictable, this market cannot be completed with any number of additional securities. Therefore the fictitious completion method cannot be applied. However, as we will see in the following example, our method provided in the previous section may work in such a situation.

Example 3 We consider a problem which is similar to the problem (2) in the model (7). We define the whole set of strategies by

$$\Theta^Y := \left\{ z(t), 0 \leq t \leq T \mid z \text{ is predictable,} \right.$$

$$\left. \int_0^T |\alpha z(t)Y(t-)|dt < \infty \text{ a.s.} \right.$$

$$\left. \text{and } \int_0^T |c(t)z(t)Y(t-)|dt < \infty, \text{ a.s.} \right\}$$

and the gain processes by

$$H_t(z) := \int_0^t z(t)dY(t), \quad z \in \Theta^Y, \quad 0 \leq t \leq T$$

and we consider the following utility maximization problem:

$$\sup_{z \in \Theta^Y} E[u(x + H_T(z))]. \quad (8)$$

We assume that the problem (8) admits an optimal strategy and the strategy is denoted by $z^Y(t), 0 \leq t \leq T$. In the subsequent argument, we see that the similar result to Theorem 1 may hold true in this situation.

First, the whole of the equivalent martingale measures for the model (7) is characterized. Let Q be an equivalent

martingale measure with respect to P . Then, according to Theorem 4.2 in [8], there exists a predictable process $\zeta(t, y)$ which satisfies

$$\zeta > 0 \text{ and } -\frac{\alpha}{\int_{\mathbb{R}} y\zeta(t, y)d\mu(y)} > 0, \quad \forall t$$

and dQ/dP is written by $dQ/dP = R_{\zeta}(T)$ where

$$\begin{aligned} R_{\zeta}(t) &:= \exp \left\{ \lambda t + \int_0^t \frac{\alpha}{\int_{\mathbb{R}} y\zeta(s, y)d\mu(y)} ds \right. \\ &\quad \left. + \int_0^t \log \left(\frac{-\alpha\zeta(s, c(s))}{\lambda \int_{\mathbb{R}} y\zeta(s, y)d\mu(y)} \right) dN(s) \right\}, \end{aligned}$$

$$0 \leq t \leq T.$$

Suppose ζ_0 is the process corresponding to z^Y , i.e.,

$$\frac{u'(x + H_T(z^Y))}{E[u'(x + H_T(z^Y))]} = R_{\zeta_0}(T)$$

and we assume that R_{ζ_0} is a Markov process.

Next, by the same argument as the proof of Theorem 1 in the previous section, when we define a function $J(t, R)$ by

$$\begin{aligned} J(t, R_{\zeta_0}(t)) &:= \frac{1}{R_{\zeta_0}(t)} \\ &\quad E[I(E[u'(x + H_T(z^Y))]|R_{\zeta_0}(T))R_{\zeta_0}(T)|R_{\zeta_0}(t)], \end{aligned}$$

we can assert that by Itô's formula,

$$\alpha z^Y(t)Y(t-)dt + c(t)z^Y(t)Y(t-)dN(t) = dJ(t, R_{\zeta_0}(t))$$

$$\begin{aligned} \therefore \quad &\alpha z^Y(t)Y(t-)dt + c(t)z^Y(t)Y(t-)dN(t) \\ &= \left\{ J_t(t, R_{\zeta_0}(t-)) + \left(\lambda + \frac{\alpha}{\int_{\mathbb{R}} y\zeta(t-, y)d\mu(y)} \right) \right. \\ &\quad \left. \cdot R_{\zeta_0}(t-)J_R(t, R_{\zeta_0}(t-)) \right\} dt \\ &\quad + \left[J \left(t, -\frac{\alpha\zeta(t, c(t))}{\lambda \int_{\mathbb{R}} y\zeta(t, y)d\mu(y)} R_{\zeta_0}(t-) \right) \right. \\ &\quad \left. - J(t, R_{\zeta_0}(t-)) \right] dN(t), \end{aligned}$$

where the subscripts of the function J denote the partial differential of J with respect to each argument. Then we obtain the following simultaneous equations:

$$\begin{aligned} \alpha z^Y(t)Y(t-) &= J_t(t, R_{\zeta_0}(t-)) + \left(\lambda + \frac{\alpha}{\int_{\mathbb{R}} y\zeta(t-, y)d\mu(y)} \right) \\ &\quad \cdot R_{\zeta_0}(t-)J_R(t, R_{\zeta_0}(t-)), \\ c(t)z^Y(t)Y(t-) &= J \left(t, -\frac{\alpha\zeta(t, c(t))}{\lambda \int_{\mathbb{R}} y\zeta(t, y)d\mu(y)} R_{\zeta_0}(t-) \right) \\ &\quad - J(t, R_{\zeta_0}(t-)). \end{aligned}$$

From these, we conclude that J satisfies the following

equation:

$$\begin{aligned} &\frac{c(t)}{\alpha} \left[J_t(t, R_{\zeta_0}(t-)) + \left(\lambda + \frac{\alpha}{\int_{\mathbb{R}} y\zeta(t-, y)d\mu(y)} \right) \right. \\ &\quad \left. \cdot R_{\zeta_0}(t-)J_R(t, R_{\zeta_0}(t-)) \right] \\ &= J \left(t, -\frac{\alpha\zeta(t, c(t))}{\lambda \int_{\mathbb{R}} y\zeta(t, y)d\mu(y)} R_{\zeta_0}(t-) \right) - J(t, R_{\zeta_0}(t-)). \end{aligned}$$

5. Conclusion

We have provided another proof of a special case of Theorem 7 in [1] which provides the optimal strategies to problems of maximizing utility from terminal wealth in incomplete market models without fictitious security arguments. Moreover, we have provided an example of an optimization problem in a simple jump model in which although the fictitious security method cannot be applied, our derivation of the equation to determine the optimal strategies works.

However, in the example, a strong assumption that the Radon-Nikodym derivative process corresponding to the optimal solution is a Markov process is imposed. The justification of this assumption is left for a future subject.

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