STABILITY AND ACCURACY ANALYSES OF IMPROVED
EXPLICIT FINITE-DIFFERENCE SCHEME IN
OPEN CHANNEL FLOW

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Abstract Stability and accuracy analyses were performed for the linearized system of the explicit
finite-difference scheme in an open channel flow which was previously developed. Sensitivity of the
numerical solutions to special treatment of the boundaries was also analysed empirically by carrying
out various computations. One interesting result was that the stability criterion was expressed by
\[ \frac{c \Delta t}{\Delta x} \leq 2 \] (\(c\) = wave celerity, \(\Delta t\) = time increment, \(\Delta x\) = distance increment), showing that the scheme
provided higher stability than the other widely used explicit scheme subject to the condition
\[ \frac{c \Delta t}{\Delta x} \leq 1. \] Another interesting result was that the overspecification problem envisaged at the boundary
points could successfully be solved by applying either the continuity method or the expedient method
rather than the characteristic method.

I. INTRODUCTION

This paper describes the fundamental aspects of
the improved explicit finite-difference method pre-
ceding developed for open channel flow computation.
In the corresponding paper\(^1\), in fact, no persuasive
discussions of primarily important problems such as
stability and accuracy were made although the method
was practically applied to analysis of secondary
undulations in tidal channel.

In this paper, using the simplified shallow water
equations, the computational scheme of the method
is first examined from the standpoint of stability
and accuracy to obtain the stability criterion and to
estimate difference in celerity and amplitude of wave
between numerical and exact solutions. Then, possible
three ways are considered to cope with the
overspecification problem caused in treatment of
boundary conditions. To determine which one of these
ways is the best at approximating the solution, a
number of actual computations under various hy-
draulic and computational conditions are carried out
and the deviations between numerical and exact
solutions are measured.

II. BASIC EQUATIONS AND SCHEME
OUTLINE

Aiming at investigation of basic features of the
explicit finite-difference method under consideration,
the following linearized shallow water equations are
considered throughout the paper.

\[ \frac{\partial V}{\partial t} + g \frac{\partial h}{\partial x} = 0 \]  \hspace{1cm} (1)
\[ \frac{\partial h}{\partial t} + h \frac{\partial V}{\partial x} = 0 \]  \hspace{1cm} (2)

where \(V\) = cross-sectional mean velocity, \(h\) = water
depth, \(h_0\) = mean water depth, and \(g\) = gravitational
acceleration. The method is now briefly reviewed.

Based upon a one-level scheme of first order ac-
curacy, the method uses a net of points as indicated
in Fig. 1. Each gridpoint has two quantities \(V\) and
\(h\) as variable. The unknowns \(V\) and \(h\) at the ad-
vanced row \(j+1\) are explicitly determined by use of

\[ \begin{array}{lll}
\text{Fig. 1 Net in } (x, t) \text{ plane}
\end{array} \]
Eqs. (1) and (2), respectively. Differing from the other explicit schemes, however, the V-values estimated by Eq. (1) are introduced into Eq. (2) to compute h-values at the same time stage. Mathematical description of the computational procedure is, for intermediate cross-sections, made as follows. The finite-difference approximation of the momentum equation Eq. (1)

\[ V_{i,j+1} = V_{i,j} - \frac{1}{2} g \frac{d^2 t}{dx} (h_{i,j} - h_{i,j}) \]  

is used to compute velocity values for \( i = 2, N-1 \) at the advanced row \( j+1 \), and then the version of the continuity equation Eq. (2)

\[ h_{i,j+1} = h_{i,j} - \frac{1}{2} h \frac{d^2 t}{dx} (V_{i,j+1} - V_{i-1,j+1}) \]  

is applied to compute water depth values at the same row \( j+1 \) using estimated values of \( V_{i,j+1} \) and \( V_{i-1,j+1} \). It is easily known that when replacing \( j+1 \) in the right hand side of Eq. (4) by \( j \) the scheme is quite identified with the so-called unstable scheme having basic instability.

III. STABILITY AND ACCURACY ANALYSES

The solutions, \( V(x,t) \) and \( h(x,t) \), to the linearized equations Eqs. (1) and (2) are assumed to be differentiable for all \( x \) and \( t \), and thus represented as a Fourier series

\[ V(m \Delta x, j \Delta t) = \sum_{n=1}^{\infty} V_n \exp(i(m \sigma_n \Delta x + j \beta_n \Delta t)) \]  

and

\[ h(m \Delta x, j \Delta t) = \sum_{n=1}^{\infty} H_n \exp(i(m \sigma_n \Delta x + j \beta_n \Delta t)) \]  

where \( V_n \) and \( H_n \) are Fourier coefficients, \( \sigma_n \) and \( \beta_n \) are wave numbers in space and time, and the spacewise indication \( i \) is replaced by \( m \) in order to avoid confusing with \( i = 1, N-1 \). Since in general the exact solutions cannot be reproduced in numerical computations, \( \beta_n \) is complex taking the form of \( \beta_n = \beta_n^R + i \beta_n^I \) (7)

Then two factors

\[ \frac{e^{-\beta_n^I \Delta t}}{e^{-\beta_n^I \Delta t}} = \frac{e^{\beta_n^R \Delta t}}{e^{\beta_n^R \Delta t}} \]  

(\( \beta_n^I = 0 \)) (8)

and

\[ R = \frac{c}{\varepsilon} = \frac{\beta_n^R}{\varepsilon \sigma_n} \]  

(\( \varepsilon = \sqrt{g h_n} \)) (9)

\[ \frac{1}{2} \left( \beta_n^R \right)^2 \leq 1 \]  

can be defined to measure the stability and accuracy of numerical solutions, the circumflexed values implying those of the exact solution. The former factor \( R_1 \) is associated with damping or amplification of the finite-difference solution, while the latter one \( R_2 \) with difference in propagation celerity.

Considering only the \( n \)-th component of the Fourier series because of the linear system, introduction of Eqs. (5) and (6) into Eqs. (3) and (4) yields

\[ V_n(e^{i \beta_n^R \Delta t} - 1) + \frac{1}{2} g H_n \frac{d^2 t}{dx} (e^{i \sigma_n \Delta x} - e^{-i \sigma_n \Delta x}) = 0 \]  

and

\[ H_n(e^{i \beta_n^R \Delta t} - 1) + \frac{1}{2} h_0 H_n \frac{d^2 t}{dx} (e^{i \sigma_n \Delta x} - e^{-i \sigma_n \Delta x}) = 0 \]  

which are homogeneous in \( V_n \) and \( H_n \). Requiring that because \( V_n = 0 \) and \( H_n = 0 \) the coefficient determinant of these quantities should be zero results in the quadratic equation

\[ \left( \frac{d^2 t}{dx} \right)^2 \sin^2 \sigma_n \Delta x - 2 \right) e^{i \beta_n^R \Delta t} + 1 = 0 \]  

Putting \( P^2 = \frac{d^2 t}{dx} \sin \sigma_n \Delta x \), the solution

\[ e^{i \beta_n^R \Delta t} = \frac{1}{2} \left[ -P^2 \pm 2 \pm \sqrt{P^2(P^2 - 4)} \right] \]  

is obtained. Two possibilities now arise depending on the value of \( P \).

(A) \( P^2 > 4 \)

Clearly the right hand side of Eq. (13) is real and its absolute value is by no means equal to unity. This indicates \( \beta_n \) is complex. Applying Eq. (7) to Eq. (13) and solving for real and imaginary parts

\[ R_1 \sin \beta_n^R \Delta t = 0 \]  

and

\[ R_1 \cos \beta_n^R \Delta t = \frac{1}{2} \left[ -P^2 + 2 \pm \sqrt{P^2(P^2 - 4)} \right] \]  

Immediately from Eq. (14)

\[ \beta_n^R \Delta t = \pi (s-1) \]  

(\( s \): integer) (16)

and thus from Eq. (15)

\[ R_1 = \frac{1}{2} \left[ P^2 - 2 \pm \sqrt{P^2(P^2 - 4)} \right] \]  

Retaining the positive sign in the second term, the stability condition is

\[ \frac{1}{2} \left( \beta_n^R \right)^2 \leq 1 \]  

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which means the amplification factor $R_1$ should be equal to or less than unity. This inequality, however, cannot be satisfied under the condition $P^2 > 4$. Thus the scheme is unconditionally unstable.

(B) $P^2 \leq 4$

This condition is rewritten as

$$\varepsilon \frac{dt}{dx} \leq 2$$

(19)

since $|\sin \sigma_n \Delta x| \leq 1$. Then the solution Eq. (13) takes the form

$$e^{i\beta_n t} = \frac{1}{2} [ -P^2 + 2 \pm i \sqrt{P^2 (4 - P^2)} ]$$

(20)

This clearly indicates $\beta_n$ is real and therefore

$$R_1 = 1$$

(21)

Namely if the condition Eq. (19) is satisfied, the scheme is stable and there is neither damping nor amplification of the perturbations.

On the other hand, setting the left hand side against the right hand side of Eq. (20) yields

$$\cos \beta_n \Delta t = \frac{1}{2} [ -P^2 + 2 ]$$

(22)

From this, it follows that the celerity of waves computed is expressed by

$$c = \frac{\Delta t}{\sigma_n} \frac{\cos \left( 1 - \frac{1}{2} P^2 \right)}{\sigma_n \Delta t}$$

(23)

whence the difference in propagation celerity is given by

$$R_2 = \frac{\varepsilon \frac{dt}{dx} \sin^2 \sigma_n \Delta x}{\sigma_n \Delta x}$$

(24)

Obviously if the condition Eq. (19) is satisfied exactly, that is, if $\varepsilon \frac{dt}{dx} = 2$, the factor $R_2$ comes to unity and then the scheme is non-dispersive. Fig. 2 shows the relations between the factor $R_2$ and the celerity ratio $\varepsilon \frac{dt}{dx}$ for some selected values of $\sigma_n \Delta x$. It is clear from the figure that in the stable region of $\varepsilon \frac{dt}{dx} < 2$ the factor $R_2$ is less than unity implying the scheme is dispersive and therefore computed celerity always becomes less than exact one. Considering $\sigma_n \Delta x$ is, according to $\sigma_n = \frac{2\pi}{L_n}$ with wave length $L_n$, related to $\frac{\Delta x}{\Delta x}$ designating the number of spacewise difference increments in one wave length, the figure also describes that finer division of a wave length leads to approach of computed celerity to exact one.

IV. SENSITIVITY OF SOLUTION TO BOUNDARY TREATMENT

1. Methods of boundary treatment

As an inviolable rule, the number of conditions specified at the boundary must be equal to that of characteristics originating on the boundary. In case of subcritical flow, therefore, only one condition, either velocity or water depth, is given at each end of a channel. The values at the unknown boundary point, for instance, at $S$ in Fig. 1, are then over-specified having the two shallow water equations and one boundary condition. This is an inherent property of the present scheme. In order to overcome this overspecification, there are three possibilities applicable to boundary treatment without too much difficulties.

(A) Continuity method (1st possibility) Assuming the conservation of momentum is of little importance compared with that of water mass, the momentum equation is discarded. This method for boundary treatment was in fact used in the previ
ous work\(^1\). If at the downstream end \(i=1\) velocity is given as a function of time, the remaining unknown is determined by the aid of
\[
h_{i+1} = h_i - h_0 \frac{dx}{dt} (V_{i+1} - V_i) \quad \ldots \ldots \ldots (25)
\]
which stems from applying the continuity equation Eq. (2) to the rectangular grid adjacent to the boundary. In the other case where water depth is given, Eq. (25) is rearranged as
\[
V_{i+1} = V_i + \frac{1}{h_0} \frac{dx}{dt} (h_{i+1} - h_i) \quad \ldots \ldots \ldots (26)
\]
to compute the unknown velocity. The expressions for the upstream end \(i=N\), analogous to Eqs. (25) and (26), are also obtainable with ease.

(B) Expedient method (2nd possibility) This is more reasonable than the 1st possibility, in point that estimation of the unknown is made in the same way as in the intermediate cross-sections. Then one of the shallow water equations in expeditiously employed for determination of the unknown, depending on the manner of boundary value specification.

Again considering the downstream end of a channel, the continuity equation Eq. (25) can be used without any changes if time-varying velocity values are imposed there. On the other hand, the case of water depth specification requires use of the momentum equation represented in a form of
\[
V_{i+1} = V_i - g \frac{dx}{dt} (h_{i+1} - h_i) \quad \ldots \ldots \ldots (27)
\]
discarding the continuity equation. Also for the upstream end, choice of the shallow water equation used is performed in the same way.

(C) Characteristics method (3rd possibility)
The method is to use the characteristic equations arising from coupling the two shallow water equations. Replacing \(h_0\) by \(h\) in Eq. (2), the characteristic equations are
\[
\frac{\partial}{\partial t}(V \pm 2C) \pm C \frac{\partial}{\partial x}(V \pm 2C) = 0 \quad \ldots \ldots \ldots (28)
\]
with \(C = \sqrt{gh}\). The simplest way of finite-difference representation of the characteristic equations is to assume that the characteristic \(SP_1\) (or \(P'_1\)) is identical with the diagonal \(SP\) (or \(P'\)). Then, in case of the downstream boundary where the backward characteristic with negative sign is valid, one of the explicit equations expressed in terms of boundary values and previously estimated values
\[
C_i V_{i+1} = C_i V_i + \frac{1}{2} (V_{i+1}^{j+1} - V_i) \nonumber
\]
\[
- C_i \frac{dx}{dt} (V_{i+1} - V_i) - 2(C_i - C_i) \nonumber
\]
and
\[
V_{i+1} = V_i + 2(C_i^{j+1} - C_i) \nonumber
\]
\[
- C_i \frac{dx}{dt} (V_{i+1} - V_i) - 2(C_i - C_i) \nonumber
\]
is used for determination of the unknown variable which is not specified as boundary condition. Similarly the forward characteristic equation for the upstream end can be changed to explicit finite-difference form.

2. Sensitivity and accuracy analyses

Sample computations for a single reach channel are made to analyse sensitivity of numerical solution to boundary treatment, and to investigate applicability and validity of the computational scheme.

The fictitious channel has a unit width, overall length of \(l=2000\) m and mean depth of \(h_0=3.0\) m, and is then divided into 20 segments having uniform incremental distance \(dx=100\) m. From the stability criterion Eq. (19), the critical time increment \(dt_{crit} = 36.886\) sec is calculated.

To examine numerical solutions under the different set of boundary conditions, two different flow cases of fundamental interest in which exact solutions\(^3\) are available are considered; (a) tidal flow in a reservoir end channel and (b) tidal flow in a closed end channel. For both cases, the vertical tide \(\eta=\eta_0 \cos \omega t\) (\(\eta_0=0.1\) m, \(\omega=2\pi/T\), \(T=600\) sec) is given at the mouth of the channel. As for the upstream end, \(\eta=0\) and \(V=0\) are imposed for the cases (a) and (b) respectively. Initial conditions in numerical computation can reasonably be set up by using the exact solutions at \(t=0\).

For each of the abovementioned three possibilities for boundary treatment, a number of time-varying solutions are computed for various values of the time increment \(dt\), that is, of the celerity ratio \(\epsilon = \frac{dt}{dx}\). Discrepancy between computed and exact solutions for a fixed value of the celerity ratio is measured by spatially averaging the standard deviations each
of which is calculated for the first five periods of the fluctuation at the corresponding cross-section. Obviously such a discrepancy is the result of rounding off fractions, truncating higher order terms, discretizing space and time, introducing special treatment for the boundaries, and violating the assumption $\gamma_0 < h_0$ under which any exact solutions are derived, and thus the spatially averaged standard deviation (hereinafter called SD) indicates the resultant error during the entire computation.

Fig. 3 illustrates the relations between $\frac{\partial u}{\partial x}$ and normalized SD in the reservoir end case, while Fig. 4 in the closed end case. Typical results of velocity and vertical tide fluctuations, for the reservoir end case treated by the 2nd possibility, are shown in Fig. 5, including numerical solutions for two different increments $\Delta t = 10$ sec and $\Delta t = 36.886$ sec and the corresponding exact solutions. Similarly Fig. 6 illustrates those for the closed end case by the 1st possibility.

What these figures substantially describe may be summarized as follows.

1) For both flow cases, the 3rd possibility produces relatively inaccurate solutions although it has the same order of approximation as the other possibilities. In the worst case, as shown in Fig. 4, the computations are destroyed by introduction of the 3rd possibility. This fact implies that unsuitable method of boundary treatment often causes instability even in the region of basic stability.
It is of interest that in stable computations the resultant error nearly linearly increases with increasing wave celerity ratio.

(iii) Accuracy of water depth is higher than that of velocity. As the celerity ratio increases, the difference in error between water depth and velocity solutions tends to be large.

(iv) There is a close resemblance in error feature between the 1st and 2nd possibilities. Then, even if $\frac{\partial t}{\partial x} = 2$, it is possible to obtain accuracy more than 99% since the maximum error, appearing in Fig. 3, is less than 0.8%.

(v) A saw-toothed solution, especially of velocity highly sensitive to the computational environment, is often produced by computation with the time increment equal or close to its critical value, as shown in Fig. 5. Such a solution should not be, however, confused with basic instability and may be possible to be smoothed by introduction of the dispersive resistance term $R_1$.

V. CONCLUSIONS

The following conclusions can be drawn from this study.

(i) The stability criterion is represented by $\frac{\partial t}{\partial x} \leq 2$. The scheme not only dismisses basic instability encountered in use of the unstable method but also provides higher stability (allowable time increment $\Delta t_{\text{crit.}}$ becomes twice for a fixed space increment $\Delta x$) than the usual explicit scheme restricted by the Current condition $\frac{\partial t}{\partial x} \leq 1$.

(ii) If the stability criterion is satisfied, there is neither damping nor amplification of the perturbations because of the amplification factor $R_1 = 1$. For all wave lengths, the features of the wave amplitudes are reproduced accurately and any discontinuity would propagate without any changes of its form.

(iii) The scheme is non-dispersive when $\frac{\partial t}{\partial x} = 2$, but dispersive when $\frac{\partial t}{\partial x} < 2$. Therefore there are no wave celerity differences only when $\frac{\partial t}{\partial x} = 2$ is exactly satisfied.

(iv) Accuracy is linearly reduced as $\frac{\partial t}{\partial x}$ increases.

(v) Even if special technique of boundary treatment is introduced to get over the overspecification problem, the solution can be reproduced with considerably high accuracy.

(vi) In both respects of stability and accuracy, use of either the 1st or 2nd possibilities is recommended for boundary treatment.

[All the computations in this paper have been carried out on FACOM M-190 in the Electric Data Processing Center of Kyoto University]

REFERENCES


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傾斜地の日射量分布計算法

三浦 健志・三野 徹
丸山 利輔・四方田 穆

複斜な地形をなす傾斜地の温度環境を明らかにするため、合理的な土地利用を図る上に不可欠な事項である。本論文では、その基本となる、日射量の面的分布の計算法を検討したので、傾斜面をほぼ二つの三角面に分割し、その面にたいする日射を日の出から日没まで積分することによって求められている。従来、点の日射量を求めることは明らかにされているが、複斜な地形のところで、システム的に、しかも円的に求める工夫をしてい

水収支法による錦江（韓国）流域の季節蒸発散量の推定

厳柄 鉄・丸山 利輔

韓国の代表的な河川である錦江流域については、水収支法による際別蒸発散量を推定した。その結果、本流域の年平均蒸発散量は 470mm であった。際別蒸発散量 E_v は月別に計器蒸発散 E_d および蒸発散散（Penman 式による）E_p との比で整理した。その結果は日本および英国で用いられている値よりかなり小さいことがわかった。その理由は、降水の雨期乾期の差が激しいこと、ヘグ山が多いことによるのではないかと推定した。

傾斜開田地における排水路内の滞砂現象とその
実験的考察

松本 康夫・五十崎 恒

開田地の排水機能は滞砂により低下する。本報では、排水路の滞砂状態を現地調査し、その滞砂過程を検討するために水路実験を行った。現地調査の結果、排水路場配約 2°以下で滞砂が進み、水路実験の結果、排水路の滞砂が進むのは水路力の 10 dyne/cm² 以下、場所から流玄と場配は約 20 dyne/cm² 以上であると思われた。これを本報の対象場配の範囲で検討すると、排水路場配約 2°以下で満足する水理条件がわかった。

自由水面をもつ Densimetric Exchange Flow
のシミュレーション

武内 智行

従来からの理論的・実験的研究を補う手段としてコンピュータシミュレーションが用いられている。MAC 法（Marker-and-Cell method）を密度不定流解析用に展開改良したものを用いて、二次元二層密度不流定の Densimetric Exchange Flow を解析した。その結果は実験値をよく説明するものであり、本手法の有効性が実証された。なお、本論文では拡散等を考慮していない。

都市水路定常流解析における降雨の安定性と
精度について（本文＝英文）

河地 利彦・A.H. ガイラン・雨 哲

筆者らが先に提案した開水路非定常流れに対する流形式の差分解法に基づく安定性の精度の観点から解法の特性を調べる。安定規範 c (dt/dx) ≤ 2 (c = 波速、dt = 時間差、dx = 距離差) であることが示され、Courant 規範に拘束される変の一般的な流形法に比べて高安定であることが理論的に証明される。また、境界点での拘束問題の処理は特性曲線法によるものも連続法あるいは適宜法によるべきことが示される。（農土論集 第88号）

モデルコンクリートのひびわれと破壊に関する
基礎的研究

北條 弘生

本論文は二相モデル供試体を利用してコンクリートのひびわれや破壊に対する材料の特性やモルタルと粗骨材の付着強度の影響について研究したものである。ひびの進行や破壊の過程は粗骨材の強度、大きさおよび付着

= 88 号 論文紹介