Experiment on the Lateral Buckling of a Cantilever Beam with Narrow Rectangular Cross Sections*

By Seiji Kondo**

This is a report of an experimental study for the lateral buckling of a narrow rectangular cantilever beam with an end load. These experiments were carried out systematically in a wide range of the dimension of beam. The test pieces used for the experiment were of thin plates of mild steel end aluminum with varying heights and spans. The results obtained have enabled us to decide the range of beam dimension applicable to the theoretical formula of the buckling load, and also enabled us to establish an experimental formula for a range in which the old formula does not hold.

1. Introduction

An experiment of the lateral buckling of a narrow rectangular beam has been made by Carrington(1), Dumont and Hill(2) etc., but these experiments have been carried out respectively in a small range of beam dimensions. In case of lateral buckling of beam as in Euler's theoretical formula for column, there must be some limit in using these formulae for the practical design, and the empirical formulae are necessary for the beam, of which the range of the dimension does not conform to the theoretical formulae.

The results of the present experiment have made it clear that the ratio of the height to the width of the cross section is more than 30 in a range where the old formula holds, while the empirical formulae must be used for a beam of which the ratio mentioned above is below 30.

2. Experiment

Apparatus The apparatus for fixing the test piece is mounted on the over-hanging part of [-] beam fastened to the concrete foundation with bolts. In this apparatus the test piece is settled both laterally and vertically by the two plates. At the free end of the test piece, a pail is hung with a pin passing through a hole on the horizontal centre line of the test piece. The load is supplied by water jetting from a small nozzle and estimated by the water quantity pouring into the pail for a given test time. The capacity of the nozzle varies from 15 to 500 gr. per minute and its selection depends upon the strength of the beam to be tested. Fig. 1 shows the photograph of the apparatus of our experiment.

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and the lateral rotation $\varphi$ at the free end of the test piece have been measured. The deflections are measured with dial indicators with an accuracy of $10^{-4}$ mm. The twisted and lateral rotation angles $\theta$ and $\varphi$ are measured by optical-lever method of a small mirror mounted at the free end of the test piece and its accuracy is $2 \times 10^{-4}$ radians. As for the displacement determining the buckling load, the error of twisted angle caused by the defect of the measuring apparatus is less than 2% and therefore, these errors have not been rectified.

**Materials** The test pieces are cut from the super rolled thin plate of mild steel and pure aluminum. The thickness of the plate is 0.6 mm for mild steel and 1.0 mm for aluminum, respectively. The tensile properties are given in Table 1. The width of the cross section of the test piece is constant, being same as the thickness of the plate, and the height of cross section and span of beam vary as the case may be. The pieces are carefully machined in order to guarantee the accuracy in both dimension and geometrical size. After being machined out, they are precisely measured, and the best ones are selected. Heat treatment is not given at all.

**Scope of experiment** The height and width of cross section, and span of beam are denoted respectively as $h$, $b$ and $l$. And let us call $h/b$ the section ratio and $l/h$ the slenderness ratio of beam. The section ratio in the present experiment consists of five kinds, namely, 5, 10, 20, 30 and 50 for mild steel and six kinds, namely, 3, 6, 10, 20, 30 and 50 for aluminum, for all of which the slenderness ratio is prepared as far as possible for testing and its limits are about 2 at minimum and 50 at maximum. The test was carried out about ten times for one testing point such as $h/b=20$ and $l/h=10$, and as a rule it was our aim to obtain some data of buckling load that can be adopted.

### Table 1 Tensile Properties of Materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Thickness of Plate $b$ mm</th>
<th>Yield Point $\sigma_y$ kg/mm²</th>
<th>Tensile Strength $\sigma_b$ kg/mm²</th>
<th>Elongation $\delta$ %</th>
<th>Youngs Modulus $E$ kg/mm²</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mild Steel</td>
<td>0.606</td>
<td>22.3</td>
<td>36.4</td>
<td>25.7</td>
<td>$2.12 \times 10^4$</td>
<td>JIS SPK 2</td>
</tr>
<tr>
<td>Aluminum</td>
<td>0.996</td>
<td>13.0</td>
<td>14.4</td>
<td>3.2</td>
<td>7100</td>
<td>Pure Aluminum Plate</td>
</tr>
</tbody>
</table>

* 0.2% Permanent Deformation

\[ \sigma_{cr} = 2.006E \frac{h}{l} \sqrt{\frac{B_1C}{B_2b^2}} \quad \cdots \cdots \cdots (2) \]

where $B_1$ and $B_2$ are the principal flexural rigidities and $C$ is the torsional rigidity of the cross section, $E$ is the modulus of longitudinal elasticity of the material. By using $B_1 = b^4E/12$, $B_2 = bh^3E/12$ and $C = k\delta^2hG$ for a rectangular section and taking Poisson's ratio equal to 0.3, the formula (2) is converted into the form

\[ \sigma_{cr} = 4.31E \frac{\kappa}{n^2} \frac{h}{l} \quad \cdots \cdots \cdots (3) \]

In the formula, $n$ stands for the section ratio. For a narrow rectangular cross section, we can take $\kappa \equiv (1-0.65/n)/3$ and we obtain

\[ \sigma_{cr} = 2.487E \frac{1}{n^2} \frac{h}{l} \quad \cdots \cdots \cdots (4) \]

These formulae give the correct value of the critical stress only within the elastic region. Beyond the elastic limit, buckling occurs at a smaller stress than that given by the formulae.

When the beam yields solely owing to the bending stress, the yield load is represented by

\[ P_{y} = \frac{bh^2}{6l} \sigma_y \quad \cdots \cdots \cdots (5) \]

where $\sigma_y'$ denotes the bending yield stress of the material. In case of $(P_{cr} - P_{y}) > 0$, the beam yields by common bending, while in case of $(P_{cr} - P_{y}) < 0$, the beam buckles at a smaller load than $P_{y}$ by lateral buckling. Denoting the slenderness ratio as $(l/h)_{cr}$ in the case of $P_{cr}$ just being equal to $P_{y}$, we obtain $(l/h)_{cr} = 4.31E \kappa/(n^2 \sigma_y')$. The line PQS in Fig. 2 shows the relation between $\sigma_{cr}$ and $l/h$.

In a short beam with a smaller value than $(l/h)_{cr}$ which is represented by the point $P$ in the Fig. 2, the beam yields at a constant yield stress $\sigma_y'$ by common bending, while in a long beam having larger value than $(l/h)_{cr}$, the beam

![Fig. 2 Characteristic of critical stress](image-url)
collapses by lateral buckling and its buckling stress becomes smaller according to hyperbolic line PQS as the beam becomes longer.

If the beam is very long, the influence of self weight of the beam on the critical load must be considered. But actually this effect is so small even in the longest beam in the present experiment, that it can be altogether ignored.

**Method of determining the critical load** In the experiment of lateral buckling of a beam, we generally experience some difficulties in determining the buckling load, since the deformations gradually increase even in the vicinity of the critical load. To overcome these difficulties a method has been proposed which can be induced from the relation between the load and the deformation of the beam end.

Taking the origin at the fixed end of beam, the following equilibrium equations between the load \( P \) and the deformations are given:

\[ B_1 u'' = -\theta P(1-z); \quad C \cdot \theta'' = P(1-z)u'' - P(u_1 - u) \]

Let us assume the next equation for the twist angle

\[ \theta = \theta_1 \left( \frac{z}{l} \right)^2 - 2\left( \frac{z}{l} \right) \]

Substituting this equation into the first of the equilibrium equations, we have the lateral deflection \( u_1 \) at the free end as

\[ u_1 = \frac{2}{15} \frac{P l^3}{B_1 \theta_1} \]

And also substituting it into the next energy equilibrium equation

\[ \frac{B_1}{2} \int_0^l u''^2 dz + \frac{C}{2} \int_0^l \theta''^2 dz = P \int_0^l u'' \theta (1-z) dz \]

We have the following result:

\[ \frac{30}{7} \frac{B_1}{l^3} u_1^2 + \frac{4}{3} \frac{C}{l} \theta_1^2 = \frac{8}{7} P u_1 \theta_1 \]

From the equations (b) and (c), we have \( P = 4.181 \sqrt{B_1 C / l^3} \). The last equation corresponds to the theoretical formula (1), and the error is about 4%.

Now we transform the equations (b) and (c) into the next forms, respectively

\[ P = 7.5 \frac{B_1}{l^3} \left( \frac{u_1}{\theta_1} \right) \]

(6)

\[ P \left( \frac{u_1}{\theta_1} \right) = 3.75 \frac{B_1}{l^3} \left( \frac{u_1}{\theta_1} \right)^2 + \frac{C}{6} \frac{1}{l} \]

(7)

In these equations, \( B_1, C, l \) are constants for a given beam and a straight line relation exists between \( P \) and \( u_1 / \theta_1 \) for the equation (6), and also between \( P \left( u_1 / \theta_1 \right) \) and \( \left( u_1 / \theta_1 \right)^2 \) for the equation (7) respectively. And if we multiply both sides of the equation (6) by \( u_1 / \theta_1 \) we have the next equation

\[ P \left( \frac{u_1}{\theta_1} \right) = 7.5 \frac{B_1}{l^3} \left( \frac{u_1}{\theta_1} \right)^3 \]

(8)

The last equation also represents a straight line passing through the origin, if we take the variable as similar to equation (7). All these equations (6)~(8) prescribe the rule between the load and the end deformations \( u_1, \theta_1 \). Equations (6) and (8) represent the relation of the elastic deformation and equation (7) represents the relation of the buckling deformation. During the load equilibriums in elastic deformation, the relation of (6) or (8) holds true. When the load reaches the buckling load, owing to the abnormal deformation resulting from the buckling, the relation suddenly transforms into the equation (7). On the basis of this theory we can graphically determine the critical load from the experimental data.

**4. Results of experiment**

Fig. 3 shows an example of the load deformation curves of mild steel and aluminum, and it represents the ordinary deformation curve of the test piece which is of a suitable proportion. The displacements \( u, u, \theta \) and \( \varphi \) represent the values at the free end of test piece. From the data shown in Fig. 3 we can construct a diagram which determines the buckling load by the equations (6)~(8). Fig. 4 shows the diagram for aluminum. In the diagram, the line connecting the plotted points can be seen as a straight line broken in two stages. Although the loads at the breaking points of the lines differ in some measure according to the adopted equation, the values at the second breaking points are nearly equal to the theoretical buckling load of this beam in the light of \( P_r = 10.8 \text{ kg} \) which is computed by the theoretical formula (1). As is seen from this example, we can decide the breaking point more clearly by the equation (7) than by (6), and so...
all data are treated by the equation (7). Fig. 5 is constructed from the data of mild steel shown in Fig. 3 (B) according to the equation (7). In this case, only one breaking point can be obtained and the load 7.0 kg at the point is nearly equal to the theoretical buckling load whose value is calculated at 7.2 kg for this test piece.

Considering the results of our test, it can be recognized that the first breaking point occurring in aluminum comes out from the discontinuity of the deformations owing to the defects mainly of the loading piece in the testing apparatus and the second breaking point is caused by the discontinuity which is due to the buckling. And for such comparatively smaller deformations as appears on a short and deep beam of mild steel, the point corresponding to the first breaking point is not found. Therefore, in case two breaking points are obtained we take the value of the second one as the buckling load, but in case of only one breaking point, the value of that point is taken.

The deformation which can determine the buckling load according to the method mentioned above usually appears in the beam whose size is suitable for the lateral buckling. When the dimension of the beam runs to an extremity, different types of deformation appear very often. These are classified into three types, namely, common, uncommon and bending yield types. The relation between each type of deformations and beam dimension and the methods of determining the buckling load for each type are not mentioned here.

Fig. 6 showing a part of experimental results of mild steel is obtained from the test pieces whose section ratios are 10, 20, and 30. The experimental buckling stress \( \sigma_{2} \) is computed by the equation \( \sigma_{2} = \frac{6P_{cr}l}{(bh)^{2}} \), in which \( P_{cr} \) represents the buckling load determined by the experiment. In the figure, the thin lines indicate the theoretical buckling load by the formula (3) and the thick lines indicate the tendency of \( \sigma_{2} \). \( \sigma_{1} \) is a stress corresponding to the first breaking point of the straight line in the buckling load determining diagram. \( \sigma_{2} \) is the bending yield point determined from the break point of vertical deflection curve of the beam having the section ratio up to 10. \( \sigma_{2} \) is a stress corresponding to the collapsing or breaking point. The equal values of the estimated stresses are arranged on both sides of the same line of \( l/h \). Distribution of each kind of stresses has a tendency parallel to the theoretical value of buckling and for the beam whose section ratio is
less than 20, a certain constant stress exists at the small slenderness ratio.

The points where the stress is constant are found near 42 kg/mm² for buckling stress and 55 kg/mm² for collapsing stresses respectively. The collapsing stress \( \sigma_{c} \) tends to become somewhat larger according to the smaller section ratio, but the buckling stress \( \sigma_{c} \) is constant despite the variation of the section ratio and its variety is confined within the experimental errors. The value of \( \sigma_{c} \) for aluminum is much smaller than \( \sigma_{c} \) owing to the properties of the material. Fig. 7 and 8 show only the critical stresses \( \sigma_{c} \) of mild steel and aluminum for the total range of experiment. The inclined thin lines show the theoretical buckling stresses and the thick lines show the experimental ones for the beams whose section ratios are below 20. Two horizontal broken lines indicate respectively the bending yield stress \( \sigma_{y} \) and the tensile yield point \( \sigma_{t} \) shown in Table 1. Considering that the buckling stress of a short beam is, at the same time, equal to the bending yield stress \( \sigma_{y} \), these values are decided at 42 kg/mm² and 22 kg/mm² for the two materials respectively.

5. Experimental Formulae

The theoretical formula (1) for elastic buckling is used only within the proportional limit of the material and beyond the limit, the buckling is usually treated as plastic one. In this case we decided the separate points of elastic and plastic bucklings by means of the tensile yield point \( \sigma_{t} \), as the proportional limit is not clearly obtained. Beyond the section ratio 30, the buckling stress \( \sigma_{c} \) quite agrees with the old formula as far as our experiment goes, and so some experimental formulae must be sought for the beam with the section ratio less than 30.

**Buckling below the tensile yield point** In the beam with the section ratio less than 30, let us examine the region which corresponds to the elastic buckling and in which the stress is below the tensile yield point. The buckling load in this range, as shown in Figs. 7 and 8 begins to deviate from the theory at the section ratio of 20 and the difference of both values increases as the section ratio becomes smaller. But on the logarithmic diagram it, as it is seen, is exactly a straight line for the slenderness ratio \( \frac{l}{h} \) and the line parallel to the theoretical one. From this fact, the experimental value \( P_{c} \) or \( \sigma_{c} \) can be expressed by the following formulae which are similar to the theoretical formula:

\[
P_{c} = m \sqrt{\frac{E}{2}} \frac{h}{l^{3}} \quad (9)
\]

\[
\sigma_{c} = 1.074m \sqrt{\frac{E}{2}} \frac{h}{l^{3}} \quad (10)
\]

In these formulae \( m \) is the factor corresponding to 4.013 in the theoretical formula (1), and is determined by as follows:

\[
m = 4.013 \frac{m}{P_{c}} = 4.013 \frac{\sigma_{c}}{\sigma_{t}} \quad (11)
\]

Fig. 9 shows the values of \( m \) with respect to the section ratio. It can be seen as a straight line with the range 3 to 7 of section ratio, and for both material, mild steel and aluminum, the value is represented as follows:
\[ m = 0.2 + 0.36n, \quad (3 < n < 7) \]  \hspace{1cm} (12)

When the ratio is over 8 it differs somewhat according to the material, and it can be expressed, for mild steel
\[ m = 0.013(0.84 + 4.0/n), \quad (8 < n < 25) \]  \hspace{1cm} (13)

and for aluminum
\[ m = 0.013(0.84 + 4.2/n), \quad (8 < n < 21) \]  \hspace{1cm} (14)

For the beam whose section ratio is over the upper limit indicated in the parentheses in these formulae, the old theoretical formula should be used. Dinnik\(^{30}\) has given a factor for buckling load for long beam with small section ratio which is shown by a broken line in Fig. 9, but the value he gave runs contrary to that of our experiment.

**Buckling beyond the yield point** The state of buckling stress of the beam with section ratio of less than 20 can be shown \( P \), \( P_1 \), \( Q \) in Fig. 2. And the hyperbolic line \( PQR \) in this case represents the stresses in obedience to the experimental formula (10). In the figure, \( a \) and \( b \) denote the slenderness ratio at \( P_1 \) and \( Q \) respectively. The special values of slenderness ratio taken from Figs. 7 and 8 are shown in Fig. 10. But the slenderness ratio \( b \) can also be calculated by the next equation, which is reduced by taking \( \sigma_{\epsilon} \) into \( \sigma_y \) in equation (10) and is represented by
\[ b = 1.074 \frac{\sqrt{\kappa}}{n^2} \frac{E}{\sigma_y} \]  \hspace{1cm} (15)

In Fig. 10, the values of \( a/b \) are nearly equal to 0.4 for mild steel and 0.5 for aluminum respectively, and the approximate values of \( a \) can be computed by finding the value of \( b \).

In the beam whose section ratio is less than 20, further experimental formulae should be sought for the stresses between \( \sigma_{\epsilon} \) and \( \sigma_y \), in other words, for the slenderness ratios from \( a \) to \( b \). For the experimental formulae of these stresses, variety of types can be considered, as in the case with a column. But in this case let us simply adopt a straight line formula and take the following form for the critical stress
\[ \sigma_{\epsilon} = \alpha - \beta \frac{1}{h} \]  \hspace{1cm} (16)

and the critical load can be expressed
\[ P_{\epsilon} = \frac{bh^2}{2\beta} \left( \alpha - \beta \frac{1}{h} \right), \quad (a < l/h < b) \]  \hspace{1cm} (17)

These formulae can be used only for the ranges with special values of \( l/h \) falling between \( a \) and \( b \). Since the formula is shown as the line passing through the two points \( P_{\epsilon}(a, \sigma_{\epsilon}) \) and \( Q(b, \sigma_y) \) in Fig. 2, the values of constants \( \alpha \) and \( \beta \) are given.

![Fig. 11](image-url) Representation of experimental formulae for mild steel
by the next equation
\[ \alpha = \frac{ab}{a-b} \left( \frac{\sigma_s}{a} - \frac{\sigma_s'}{a} \right), \quad \beta = \frac{\sigma_s - \sigma_s'}{a-b} \quad \cdots (18) \]

The inclined straight lines in Figs. 11 and 12 for the beam whose section ratio is less than 20 are drawn by this step. The tested points gradually decrease along this straight lines according as the slenderness ratios increase.

6. Conclusions

The results of our experiment are summarized as follows:

(1) The range of the sizes of beam which can be applied to the theoretical formula for buckling load is greater than 25 for mild steel and 21 for aluminum in terms of the section ratio, provided that in both cases the slenderness ratio is a suitable one.

(2) The short beam whose section ratio is less than 20, yields in such conditions as are seen in simple bending, and its buckling stress is constant regardless the slenderness ratio.

(3) In the beam whose \( h/b \) is less than 20, when the slenderness ratio is of the middle class, the beam yields by bending accompanied by lateral buckling, but these buckling loads gradually decrease as slenderness ratio increases.

(4) Even the beam whose \( h/b \) is less than 20, when the slenderness ratio becomes greater, the beam yields laterally by buckling. But these critical stresses are smaller than the theoretical ones. And the experimental formulae are established for the range of beams whose dimensions do not conform to the old formula.

In closing, the writer wishes to express his thanks to Dr. Y. Nakagawa, Prof. of Kyoto University for his kind encouragement. His thanks are also due to Messrs. T. Furukawa and M. Tutida for their useful help in this experiment. Last but not least, the writer has to acknowledge his obligation to the Ministry of Education for the donation of the Grants for Aid in Scientific Research which has facilitated this modest research.

Reference

(1) H. Carrington, Phil., 36, 1920, p. 311.
(4) do p. 384.
(5) do p. 249.