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On Vibration of an Ideal Fluid contained in an Annular Region bounded by two Eccentric Circular Cylinders (Case of Finite Amplitude)*

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An infinitely long cylindrical wall d_2 of circular cross-section is assumed to exist. Inside which, another cylindrical wall d_1 , also of infinite length, is assumed to exist, their axes being parallel to each other. The axes of these two cylindrical walls need not necessarily coincide. The annular space lying between these two walls d_1 and d_2 is assumed to be filled up with an ideal fluid (non-viscous, incompressible).

When the inner cylindrical wall d_1 makes a transverse vibration, while the outer cylindrical wall d_2 remains fixed, particles of the fluid contained between these walls will also make a vibratory motion. In this report, theoretical calculation is made of the vibratory hydraulic pressure thus set up in the fluid, for a case in which the amplitude of vibration is of finite value, not necessarily confining to the case of very small amplitude.

1. Fundamental relations for vibration of an ideal fluid contained between two eccentric circular cylinders

When we consider a two-dimensional potential flow of an ideal fluid, the velocity of flow is expressed by $V_x = \partial\phi/\partial x$, $V_y = \partial\phi/\partial y$. When the motion of the fluid is not stationary, the velocity potential ϕ is a function of time t , as well as the rectangular coordinates x , y . Denoting the stream function by ψ , and putting $w = \phi + i\psi$, the complex velocity potential w is a function of the complex variable $z = x + iy$, as well as the time t .

The hydraulic pressure p at any point is given

$$-p = \rho \frac{\partial\phi}{\partial t} + \frac{1}{2} \rho \left| \frac{dw}{dz} \right|^2 + p_0 \dots \dots \dots (1)$$

by where p_0 is a constant.

Fig. 1 shows the z -plane, in which an inner circle of radius a is placed eccentrically inside an outer circle of radius b . These two circles represent the walls of inner and outer circular cylinders d_1 , d_2 respectively. These two eccentric circles in the z -plane are transformed, by a conformal transformation expressed by

$$z = c \frac{c + \zeta}{c - \zeta} \dots \dots \dots (2)$$

into two concentric circles of radii R_1 and R_2 in the ζ -plane (Fig. 2). c is a parameter which has real positive values. The region of the z -plane lying between two eccentric circles d_1 and d_2 is thus transformed into the region in ζ -plane lying between two concentric circles d_1 and d_2 . When

Fig. 1 z -plane Fig. 2 ζ -plane

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the inner circle in the z -plane moves, while the outer circle remains fixed, the parameter c and the position of the origin of z -plane changes with time. The radii a and b are to be kept constant irrespective of time t . With these ideas we wish to calculate the amount of vibratory hydraulic pressure p when the inner cylinder d_1 is making an oscillation of finite amplitude while the outer cylindrical wall d_2 is kept fixed.

Let the transverse displacement of the inner cylinder be A , which is a function of time t . The boundary condition to be satisfied by the potential function ϕ is (in the z -plane):

$$\begin{aligned} \text{on circle } d_2 \quad & \partial\phi/\partial n = 0 \\ \text{on circle } d_1 \quad & \partial\phi/\partial n = \dot{A} \cos \theta \end{aligned}$$

Where $\partial/\partial n$ represent the derivative in direction normal to the wall, and $\dot{A}$ the derivative with respect to time t , of A . The value of the potential function ϕ , which satisfy the two dimensional potential equation $\nabla^2\phi=0$, together with the above-mentioned boundary condition, is to be found by taking the real part of the expression

$$w = \phi + i\psi = \dot{A}W + i\frac{\Gamma}{2\pi} \log \zeta \dots \dots \dots (3)$$

where

$$W = \Phi + i\Psi = A_0 + \sum_{n=1}^{\infty} \frac{a_n}{\zeta^n} + \sum_{n=1}^{\infty} b_n \zeta^n \dots \dots \dots (4)$$

Γ is the amount of circulation around the inner cylinder d_1 , when there is any rotational motion of the fluid. When there is no rotational motion

$$\cos \theta = \frac{4R_1}{d_1} \left(\frac{\cos \varphi - \mu}{1 + \mu^2 - 2\mu \cos \varphi} \right) - \frac{2(\beta_1 - c)}{d_1} \dots \dots \dots (5)$$

Using these values, we have, on the surface of inner cylinder;

$$\left| \frac{dw}{dz} \right|^2 = \left| \frac{dw}{d\zeta} \frac{d\zeta}{dz} \right|^2 = \left(\frac{R_1^2 + c^2 - 2R_1c \cos \varphi}{2c^2} \right)^2 \left[\dot{A}^2 \cos^2 \theta + \left(F\dot{A} + \frac{\Gamma}{2\pi R_1} \right)^2 \right]$$

where we put for shortness,

$$F = \frac{1}{R_1} \sum_{n=1}^{\infty} (-n) A_n (1 + \epsilon^{2n}) \sin n\varphi \dots \dots \dots (6)$$

Thus the value of hydraulic pressure p on inner cylinder is found to be

$$-\frac{1}{\rho} p = \ddot{A}\Phi + \dot{A} \frac{\partial\Phi}{\partial t} + \frac{p_0}{\rho} + \frac{1}{2} \left(\frac{R_1^2 + c^2 - 2R_1c \cos \varphi}{2c^2} \right)^2 \left[\dot{A}^2 \cos^2 \theta + \left(F\dot{A} + \frac{\Gamma}{2\pi R_1} \right)^2 \right] \dots \dots \dots (7)$$

In the above equation, $\partial\Phi/\partial t$ is to be understood to mean $[\partial\Phi/\partial c][\partial c/\partial t]$. Also we have, on inner cylinder

$$\Phi = \sum_{n=1}^{\infty} A_n \cos n\varphi + \sum_{n=1}^{\infty} B_n \cos n\varphi = \sum_{n=1}^{\infty} A_n [1 + \epsilon^{2n}] \cos n\varphi$$

The relation between the angles φ and θ are given by (5), as mentioned above, but this relation is not the same for every instant t , because the expression in (5) contains parameters R_1 , β_1 , etc. The curves showing the relation between φ and θ for various values of the ratio c/a is given in Fig. 3

Expressing the parameters μ , R_1 , etc., in terms of a single parameter c , we have

of fluid around inner cylinder d_1 , we must have $\Gamma=0$. Γ is a constant and is independent of time t . If Γ is a function of t , the pressure p as given by Eq. (1) will not be a single valued function.

The values of coefficients A_n and B_n are determined by the above mentioned boundary conditions, and are found to be:

$$\begin{aligned} a_n &= A_n R_1^n, \quad b_n = B_n / R_1^n, \quad A_0 = 0, \\ A_n &= -\frac{4R_1 \mu^n}{n\lambda_n} \left[\left(\frac{2R_1}{d_1} \right) \left(\frac{1}{1-\mu^2} \right)^2 \left\{ n \left(\frac{1}{\mu} - \mu \right) + 2\mu \right\} \right. \\ &\quad \left. - \frac{2(\beta_1 - c)}{d_1} \frac{1}{1-\mu^2} \right], \end{aligned}$$

$$B_n = (R_1/R_2)^{2n} A_n$$

where we put

$$\mu = R_1/c \quad \epsilon = R_1/R_2 \quad \lambda_n = 1 - \epsilon^{2n}$$

β_1 and β_2 are abscissa of the centers of circles d_1 , d_2 in the z -plane (Fig. 1).

When the inner cylinder is vibrating with a finite amplitude, the parameters μ , β_1 , c , ϵ , ... are functions of t .

For a practical purpose, it is convenient to regard that μ , β_1 , ϵ , ... are functions of a single parameter c , while c is a function of time t . The value of pressure p is to be found by Eq. (1). Especially, the value of p on the surface of inner cylinder can be found by putting

$$z = \beta_1 + ae^{i\theta}, \quad \zeta = R_1 e^{i\varphi}$$

into the expression (1) for p . Relation between the angles θ and φ as deduced from the relation (2) is;

$$(2) \text{ is;}$$

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Fig. 3 Curves showing relations between the angles θ and φ

Fig. 4 Variation with time of central distance $\beta_2 - \beta_1$

Fig. 5 Relation between c and $(\beta_2 - \beta_1)$

a and b are to be kept at constant values. When the inner cylinder is vibrating, the parameter c is a function of t . When $c/a = 0$ the two cylinders d_1 and d_2 touch each other. While when $c/a \rightarrow \infty$, two cylinders d_1 , d_2 become concentric. Using these notations, the Eq. (5) may also be written;

$$\varepsilon = \frac{\xi_1}{\xi_2} = 1 - \left(\frac{s}{a} \right) \frac{(c/a)}{\sqrt{(c/a)^2 + 1}}$$

$$A_n = -\frac{2a}{n} \left(\frac{a}{s} \right) \sqrt{\left(\frac{c}{a} \right)^2 + 1} \xi_1^n$$

So that we have, to the same order of of approximation;

$$\Phi = -4a \left(\frac{a}{s} \right) \sqrt{\left(\frac{c}{a} \right)^2 + 1} \sum_{n=1}^{\infty} \frac{\xi_1^n}{n} \cos n\varphi$$

$$= -2a \left(\frac{a}{s} \right) \sqrt{\left(\frac{c}{a} \right)^2 + 1} \log[1 - 2\xi_1 \cos \varphi + \xi_1^2]$$

$$\frac{\partial \Phi}{\partial c} = -2 \left(\frac{a}{s} \right) \frac{(c/a)}{\sqrt{\left(\frac{c}{a} \right)^2 + 1}} \log[1 - 2\xi_1 \cos \varphi + \xi_1^2] - 4 \left(\frac{a}{s} \right) \frac{\xi_1(\xi_1 - \cos \varphi)}{1 - 2\xi_1 \cos \varphi + \xi_1^2}$$

$$\beta_2 - \beta_1 = \frac{s}{\sqrt{\left(\frac{c}{a} \right)^2 + 1}}$$

$$\cos \theta = 2 \left(\frac{c}{a} \right) \frac{\xi_1(\cos \varphi - \xi_1)}{1 + \xi_1^2 - 2\xi_1 \cos \varphi} - \xi_1$$

2. Case of simple harmonic motion

When the inner cylindrical wall is vibrating transversally in the direction of x -axis, while the outer cylindrical wall is kept at rest, we put

$$A = \alpha \sin \omega t \quad (8)$$

Then the central distance $\beta_2 - \beta_1$ (Fig. 1) of two cylinders becomes

$$\beta_2 - \beta_1 = \beta_0 - \alpha \sin \omega t \quad (9)$$

where β_0 is a positive constant such that $\beta_0 > \alpha$ (Fig. 4).

According to the above notation, we have

$$\beta_2 - \beta_1 = \sqrt{b^2 + c^2} - \sqrt{a^2 + c^2}$$

and the relation between the value of $\beta_2 - \beta_1$ and parameter c will be of the form shown in Fig. 5. Therefore, for any given instant t , the value of $\beta_2 - \beta_1$ can be found by (9), while the corresponding value of c to the value of $\beta_2 - \beta_1$ thus found can be determined by Fig. 5, it may be said that c is given as a function of time t . Thus we have

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial c} \frac{\partial c}{\partial (\beta_2 - \beta_1)} \frac{\partial (\beta_2 - \beta_1)}{\partial t}$$

So that the right hand side of the expression (7) can be evaluated.

3. Case of a narrow clearance

The case most frequently met with in practical application is that in which the breadth of clearance space between two cylindrical walls is very small in comparison to the radius a .

In order to make approximate formula suitable to that case, let us put $b = a + s$, and let us neglect $(s/a)^2$ in comparison to unity. Then we may put approximately

$$\frac{\partial(\beta_2 - \beta_1)}{\partial c} = \left(\frac{s}{a}\right) \frac{\left(\frac{c}{a}\right)}{\left[\left(\frac{c}{a}\right)^2 + 1\right]^{3/2}}$$

$$F = 2\left(\frac{a}{c}\right)\left(\frac{a}{s}\right)\sqrt{\left(\frac{c}{a}\right)^2 + 1} \frac{\sin \varphi}{1 + \xi_1^2 - 2\xi_1 \cos \varphi}$$

Making use of these values, the pressure p may be expressed in the form:

$$p = P_1 \sin \omega t + P_2 [\cos \omega t]^2 + P_3 \cos \omega t \quad \dots\dots\dots (10)$$

where we have

$$\frac{1}{\rho} P_1 = \omega^2 \alpha \Phi,$$

$$\frac{1}{\rho} P_2 = -(\omega \alpha)^2 \left[-\frac{\partial \Phi}{\partial c} \frac{\partial c}{\partial (\beta_2 - \beta_1)} + \frac{1}{2} \left\{ \left(\frac{a}{c}\right) \left(\frac{a}{s}\right) \sqrt{\left(\frac{c}{a}\right)^2 + 1} \sin \varphi \right\}^2 \right]$$

$$\frac{1}{\rho} P_3 = -\frac{1}{4} (1 - 2\xi_1 \cos \varphi + \xi_1^2) \frac{\Gamma}{\pi} \left(\frac{\omega \alpha}{c \xi_1} \right)$$

The expression for pressure p contains linear and quadratic terms in $\sin \omega t$ and $\cos \omega t$. If $\Gamma = 0$, p is proportional to ω^2 . As to the relation of p to vibration amplitude α , we see that p contains a term P_1 which is proportional to α , and a term P_2 which is proportional to α^2 .

However, since Φ , $\partial \Phi / \partial c$ etc. are functions of c , while c is dependent upon t , the pressure p is actually a complicated function of time t .

For practical use it may be more convenient to express amplitudes P_1 , P_2 , P_3 in terms of water columns thus;

$$\left. \begin{aligned} \frac{P_1}{\gamma} &= H_1 = \frac{(\omega \alpha)^2}{g} \left(\frac{\alpha}{s}\right) K_1 \\ \frac{P_2}{\gamma} &= H_2 = \frac{(\omega \alpha)^2}{g} \left(\frac{\alpha}{s}\right) K_2 \\ \frac{P_3}{\gamma} &= H_3 = \frac{(\omega \alpha)(\omega a)}{g} \left(\frac{\alpha}{s}\right) K_3 \end{aligned} \right\} \quad \dots\dots\dots (11)$$

where we put for shortness

$$K_1 = -2\sqrt{\left(\frac{c}{a}\right)^2 + 1} \log(1 - 2\xi_1 \cos \varphi + \xi_1^2)$$

$$K_2 = -\left(\frac{\alpha}{s}\right) \left[\frac{(c/a)}{\sqrt{\left(\frac{c}{a}\right)^2 + 1}} \log(1 - 2\xi_1 \cos \varphi + \xi_1^2) - 4 \frac{\xi_1(\xi_1 - \cos \varphi)}{1 - 2\xi_1 \cos \varphi + \xi_1^2} \right] \frac{\left\{ \left(\frac{c}{a}\right)^2 + 1 \right\}^{3/2}}{(c/a)}$$

$$- \frac{1}{4} \left(\frac{\alpha}{s}\right) \left[\left(\frac{a}{c}\right) \sqrt{\left(\frac{c}{a}\right)^2 + 1} \sin \varphi \right]^2$$

$$K_3 = -\left(\frac{a}{c}\right)^2 \sqrt{\left(\frac{c}{a}\right)^2 + 1} \sin \varphi$$

4. Numerical example

When the value of s/a is small in comparison to unity, the relation between $(\beta_2 - \beta_1)/s$ and c/a becomes as shown in Fig. 6. For $c/a = 0$ the two circles just touch each other, while for $c/a \rightarrow \infty$, the two circles become concentric. So that, when the inner cylinder is vibrating in the way as expressed by (9), c/a will take the values A, B, C (Fig. 6) in succession. The Fig. 6 shows a case in which the vibration amplitude α has the value 0.20 s, and is vibrating around the steady configuration expressed by $\beta_0/s = 0.40$.

At an instant when the configuration of inner

Fig. 6 Relation between $(\beta_2 - \beta_1)/s$ and (c/a)

Fig. 7 Mutual configuration of inner and outer cylinders

Fig. 8 Graphs for values of coefficients K_1 , K_2 , K_3

cylinder is at "A", we have $\sin \omega t = -1$, $\cos \omega t = 0$, while when the configuration is at "B" we have $\sin \omega t = 0$, $\cos \omega t = 1$, and also at configuration "C" $\sin \omega t = 1$, $\cos \omega t = 0$.

Thus we see, by Eqs. (10) and (11), that at the configurations "A" and "C" the factor K_1 only matters, while at configuration "B", the factors K_2 and K_3 matter. The numerical values of the factor K_1 (for "A" and "C") and of the factors K_2 and K_3 (for "B") have been calculated (for the case

in which $\alpha/s = 0.20$, $\beta_0/s = 0.40$, as mentioned above) and are shown in graphs in Fig. 8.

The author has previously reported a result of his calculation for the same problem, in which the amplitude of vibration is very small in comparison with the clearance $s^{(1)(2)}$. Comparing it with the present result we see a marked increase in the factor K_1 . The factor K_2 , being the coef. of $\cos^2 \omega t$, is a factor newly appeared because of non-linearity of the vibration.

5. Concluding remarks

The amount of variation of hydraulic pressure p , for the case in which the vibration amplitude α of inner cylinder cannot be regarded to be small as compared to the clearance s , has been calculated by a method which may be called a conformal transformation changing with the time t .

The result, if expressed explicitly in terms of time t , would be much complicated. But for a practical application, it suffices when the result is expressed in terms of the parameter c , which in turn is a function of time t . From the result of numerical evaluation, it was found that the amount of pressure variation may become considerably higher than that amount obtainable by applying the formula for the case of small vibration.

When the vibration amplitude is of finite value, there appear, in the expression for pressure p , a nonlinear term, which makes the mode of variation of pressure much more complicated than the case of small vibration. But, for a practical purpose, the numerical evaluation for three special configurations "A", "B", and "C" of the vibration, may give a good picture of the pressure variation.

References

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