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On the Transverse Flexure of a Semi-Infinite Thin Plate with a Semi-Circular Notch*

By Osamu TAMATE** and Shunsuke SHIOYA***

In this paper, the effects of a semi-circular notch in a semi-infinite thin plate subjected to plain bending are investigated by the use of the Poisson-Kirchhoff theory of thin plates.

The analysis is developed with the help of the complex variable method and the unknown coefficients included in the solution are determined by the method of perturbation. Numerical results are presented in the form of graphs.

In order to clarify the effects of making a semi-circular notch, the distribution of bending couples and stress concentration factor k_t are calculated and presented in the form of graphs.

1. Introduction

As is well known, the presence of a hole in a bent plate has a great influence on the maximum stress and a large number of investigations were directed towards determining the exact extent of this influence (1). Notches of various shapes may also have a similar effect. But no problem of determining the effect of the notch appears to have been solved except for two symmetrically disposed hyperbolic notches (2), (3).

In the present paper, a theoretical solution will be given for a semi-infinite plate, notched by a single semi-circular notch and subjected to plain bending. The analysis will be developed on the basis of the Poisson-Kirchhoff theory (4) and by means of Muskhelishvili's complex variable method (5). The unknown coefficients included in the solution are determined by a method of perturbation. In order to clarify the effect of a semi-circular notch, the distributions of bending couples are calculated numerically. The stress concentration factor k_t also is computed and plotted versus Poisson's ratio of the plate material.

2. Preliminary results

When the middle plane of an isotropic homogeneous thin plate free from lateral load lies originally in the complex plane of $z=x+iy$ and the plate is assumed subject to the limitation of the Poisson-Kirchhoff theory of thin plates, its deflection w normal to the z -plane is governed by

the biharmonic equation. As is well known, the solution of the biharmonic equation may be represented in terms of two analytic functions $\varphi_1(z)$ and $\psi_1(z)$ of the complex variable z

$$\left. \begin{aligned} w &= R_e[\bar{z}\varphi_1(z) + \chi_1(z)] \\ \chi_1(z) &= \int \psi_1(z) dz \end{aligned} \right\} \dots\dots\dots (1)$$

Overbars are used to indicate conjugate complex quantities and " R_e " denotes "the real part of".

With the aid of mapping function

$$z = \omega(\zeta), \quad \zeta = \rho e^{i\theta} \dots\dots\dots (2)$$

curvilinear co-ordinates (ρ, θ) are introduced into the z -plane and we now have the deflection of the plate given by

$$w = R_e[\bar{\omega}(\bar{\zeta})\varphi(\zeta) + \chi(\zeta)] \dots\dots\dots (3)$$

where

$$\begin{aligned} \varphi(\zeta) &= \varphi_1[\omega(\zeta)] = \varphi_1(z), \\ \chi(\zeta) &= \int \psi(\zeta) \omega'(\zeta) d\zeta \end{aligned}$$

and

$$\psi(\zeta) = \psi_1[\omega(\zeta)] = \psi_1(z)$$

The moments and shearing forces per unit length with reference to the new co-ordinates (ρ, θ) are expressible by the formulas:

$$\left. \begin{aligned} M_\rho + M_\theta &= -4D(1+\nu)R_e[\Phi(\zeta)], \\ M_\theta - M_\rho - 2iM_{\rho\theta} &= 2D(1-\nu)\zeta^2[\Phi'(\zeta)\bar{\omega}(\bar{\zeta}) + \Psi(\zeta)\omega'(\zeta)] / \{\rho^2\bar{\omega}'(\bar{\zeta})\}, \\ Q_\rho - iQ_\theta &= -4D\zeta\Phi'(\zeta)/\{\rho|\omega'(\zeta)\} \end{aligned} \right\} \dots\dots\dots (4)$$

in which

$$\begin{aligned} \Phi(\zeta) &= \varphi'(\zeta)/\omega'(\zeta), \\ \Psi(\zeta) &= \psi'(\zeta)/\omega'(\zeta) \end{aligned}$$

and $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of plate; E , ν , and h being, respectively, Young's

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modulus, Poisson's ratio and the thickness of the plate.

For a free boundary of a plate, the boundary condition is given by

$$-(1/\epsilon)\varphi(\zeta) + \zeta\bar{\varphi}'(\bar{\zeta}) + \bar{\psi}(\bar{\zeta}) = iC_1\omega(\zeta) + C_2 \quad \dots\dots\dots(5)$$

where

$$\epsilon = (1-\nu)/(3+\nu)$$

and C_1 is a real constant and C_2 a complex one. When the region occupied by the middle plane of the plate is simply connected, these constants may be taken to be zero.

With these formulas we shall seek the solution for the case where the semi-infinite plate, having a semi-circular notch, is subjected to the plain bending in the direction of the straight boundary. In this case, the straight boundary and the rim of the notch are assumed to be stress free.

3. Solution of problem

Let the middle plane of the plate occupy the right half of the x, y -plane, with the exception of the interior of a semi-circle whose center is at

Fig. 1

$$\left. \begin{aligned} \mathbb{I}\varphi(\zeta) &= -\frac{M_0 a}{D(1-\nu)} \sum_{n=0}^{\infty} \left\{ \frac{a_{2n}}{2n+2} \zeta^{-(2n+2)} + \frac{b_{2n}}{2n+1} \zeta^{-(2n+1)} \right\} \\ \mathbb{I}\psi(\zeta) &= -\frac{M_0 a}{D(1-\nu)} \sum_{n=0}^{\infty} \left\{ \frac{c_{2n}}{2n+2} \zeta^{-(2n+2)} + \frac{d_{2n}}{2n+1} \zeta^{-(2n+1)} \right\} \end{aligned} \right\} \dots\dots\dots(11)$$

where a_{2n} , b_{2n} , c_{2n} , and d_{2n} are real constants.

Substituting Eqs. (10) and (11) into Eq. (6) and equating to zero the coefficients of various powers of ρ , we obtain

$$\left. \begin{aligned} c_{2n} &= -\{(2n+2) - (1/\epsilon)\} a_{2n} \\ d_{2n} &= -\{(2n+1) + (1/\epsilon)\} b_{2n} \end{aligned} \right\} \dots\dots\dots(12)$$

In a similar way, Eq. (7) gives

$$\begin{aligned} &\frac{1}{2}(e^{i\theta} + e^{-i\theta}) + \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n+2)\epsilon} e^{-i(2n+2)\theta} + e^{i(2n+4)\theta} + \left(1 - \frac{1}{(2n+2)\epsilon}\right) e^{i(2n+2)\theta} \right\} a_{2n} \\ &+ \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n+1)\epsilon} e^{-i(2n+1)\theta} + e^{i(2n+3)\theta} + \left(1 + \frac{1}{(2n+1)\epsilon}\right) e^{i(2n+1)\theta} \right\} b_{2n} = 0 \dots\dots\dots(13) \end{aligned}$$

from which it follows that

$$\left. \begin{aligned} \cos\theta + \sum_{n=0}^{\infty} \{\cos(2n+2)\theta + \cos(2n+4)\theta\} a_{2n} + \sum_{n=0}^{\infty} \left\{ \left(1 + \frac{2}{(2n+1)\epsilon}\right) \cos(2n+1)\theta + \cos(2n+3)\theta \right\} b_{2n} &= 0 \\ \sum_{n=0}^{\infty} \left\{ \left(1 - \frac{2}{(2n+2)\epsilon}\right) \sin(2n+2)\theta + \sin(2n+4)\theta \right\} a_{2n} + \sum_{n=0}^{\infty} \{\sin(2n+1)\theta + \sin(2n+3)\theta\} b_{2n} &= 0 \end{aligned} \right\} \dots\dots\dots(14)$$

the origin (Fig. 1). The radius of this semicircle may be taken as a .

By means of the mapping function $z = \omega(\zeta) = a\zeta$, the straight boundary of the plate and the rim of the notch will be mapped onto the lines $\theta = \pm\pi/2$ and $\rho = 1$, respectively. Then the boundary conditions of the present problem are expressed as follows:

$$(i) \quad \theta = \pm\pi/2$$

$$-(1/\epsilon)\varphi(\zeta) + \zeta\bar{\varphi}'(\bar{\zeta}) + \bar{\psi}(\bar{\zeta}) = 0 \dots\dots\dots(6)$$

$$(ii) \quad \rho = 1$$

$$-(1/\epsilon)\varphi(\zeta) + \zeta\bar{\varphi}'(\bar{\zeta}) + \bar{\psi}(\bar{\zeta}) = 0 \dots\dots\dots(7)$$

The load applied at the infinite part of the plate is $M_y = \text{constant} = M_0$, with $M_x = M_{xy} = 0$, which may be easily transformed into

$$(iii) \quad \rho \rightarrow \infty$$

$$\left. \begin{aligned} M_\rho + M_\theta &= M_0, \\ M_\theta - M_\rho - 2iM_{\rho\theta} &= M_0 e^{2i\theta} \end{aligned} \right\} \dots\dots\dots(8)$$

To satisfy these conditions, let us consider the functions

$$\left. \begin{aligned} \varphi(\zeta) &= \mathbb{I}\varphi(\zeta) + \mathbb{II}\varphi(\zeta) \\ \psi(\zeta) &= \mathbb{I}\psi(\zeta) + \mathbb{II}\psi(\zeta) \end{aligned} \right\} \dots\dots\dots(9)$$

where $\mathbb{I}\varphi(\zeta)$ and $\mathbb{I}\psi(\zeta)$ correspond to the deflection of the plate, without the notch, subjected to plain bending and can be written as

$$\left. \begin{aligned} \mathbb{I}\varphi(\zeta) &= -M_0 a \zeta / \{4D(1+\nu)\} \\ \mathbb{I}\psi(\zeta) &= M_0 a \zeta / \{2D(1-\nu)\} \end{aligned} \right\} \dots\dots\dots(10)$$

$\mathbb{II}\varphi(\zeta)$ and $\mathbb{II}\psi(\zeta)$ are the respective parts of the functions which are analytic in the entire region of the semi-infinite plate surrounding the notch and therefore include only terms of negative powers of ζ . Taking into account the requirement of the symmetry of the problem we may write

$\mathbb{II}\varphi(\zeta)$ and $\mathbb{II}\psi(\zeta)$ in the form of series:

Meanwhile, sines and cosines of even multiples of θ can be expanded in Fourier series of sines or cosines, respectively, of odd multiples of θ valid between $\theta = -\pi/2$ and $\theta = \pi/2$

$$\left. \begin{aligned} \cos 2n\theta &= \frac{4}{\pi} \sum_{s=0}^{\infty} \frac{(-1)^{n+s}(2s+1)}{(2s+1)^2 - (2n)^2} \cos(2s+1)\theta \\ \sin 2n\theta &= \frac{4}{\pi} \sum_{s=0}^{\infty} \frac{(-1)^{n+s}2n}{(2s+1)^2 - (2n)^2} \sin(2s+1)\theta \end{aligned} \right\} \dots\dots\dots (15)$$

Substituting Eq. (15) into Eq. (14) and then equating to zero the coefficients of $\cos(2s+1)\theta$, $\sin(2s+1)\theta$, we get

$$\left. \begin{aligned} 1 + \left(1 + \frac{2}{\varepsilon}\right)b_0 + \frac{8}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{2a_{2n}}{(2n+1)(2n+3)(2n+5)} &= 0 \\ b_0 + \frac{8}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+3) - (2n+5)/\varepsilon}{(2n+1)(2n+3)(2n+5)} a_{2n} &= 0 \end{aligned} \right\} \dots\dots\dots (16)$$

$$\left. \begin{aligned} \left(1 + \frac{2}{(2s+1)\varepsilon}\right)b_{2s} + b_{2s-2} + \frac{8}{\pi} \sum_{n=0}^{\infty} (-1)^{n+s} \frac{2(2n+3)(2s+1)a_{2n}}{(2n-2s+1)(2n-2s+3)(2n+2s+3)(2n+2s+5)} &= 0 \\ b_{2s} + b_{2s-2} + \frac{8}{\pi} \sum_{n=0}^{\infty} (-1)^{n+s} \frac{(2n+2)(2n+4) + (2s+1)^2 - \frac{1}{\varepsilon}(2n-2s+3)(2n+2s+5)}{(2n-2s+1)(2n-2s+3)(2n+2s+3)(2n+2s+5)} a_{2n} &= 0 \end{aligned} \right\} \dots\dots\dots (17)$$

($s \geq 1$)

The set of Eqs. (16) and (17) are easily transformed into

$$\left. \begin{aligned} b_0 &= -\frac{8}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+3) - (2n+5)/\varepsilon}{(2n+1)(2n+3)(2n+5)} a_{2n}, \\ b_{2s} &= \frac{4}{\pi} (2s+1) \sum_{n=0}^{\infty} (-1)^{n+s+1} \frac{(2n+2s+5) - (2n-2s+1)\varepsilon}{(2n-2s+1)(2n+2s+3)(2n+2s+5)} a_{2n} \quad (s \geq 1) \end{aligned} \right\} \dots\dots\dots (18)$$

and

$$\left. \begin{aligned} \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{2(2n+5) - (2n+1)\varepsilon - (2n+1)\varepsilon^2}{(2n+1)(2n+3)(2n+5)} a_{2n} &= -\frac{\varepsilon^2}{2} \\ \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^{n+s+1} \frac{(2n+2s+1)(2n+2s+5) + 2(2n-2s+1)\varepsilon - (2n-2s+1)(2n-2s+3)\varepsilon^2}{(2n-2s+1)(2n+2s+3)(2n+2s+5)} a_{2n} &= \varepsilon^2 \delta_{1,s}/4 \end{aligned} \right\} \dots\dots\dots (19)$$

($s \geq 1$)

where $\delta_{1,s}$ is the Kronecker delta.

If the system of Eqs. (19) can be solved for a_{2n} , b_{2s} are determined at once from Eqs. (18).

To solve these equations, we now apply a method of perturbation in which ε is the perturbation parameter. Assuming the expansions

$$a_{2n} = \sum_{r=2}^{\infty} a_{2n}^{(r)} \varepsilon^r, \quad b_{2n} = \sum_{r=1}^{\infty} b_{2n}^{(r)} \varepsilon^r \dots\dots\dots (20)$$

where $a_{2n}^{(r)}$ and $b_{2n}^{(r)}$ are independent of ε , and substituting these into Eqs. (18) and (19) and finally comparing the coefficients of the same power of ε on both sides which result, we get

$$\left. \begin{aligned} \sum_{n=0}^{\infty} (-1)^n \frac{a_{2n}^{(2)}}{(2n+1)(2n+3)} &= -\frac{\pi}{16} \\ \sum_{n=0}^{\infty} (-1)^{n+s+1} \frac{a_{2n}^{(2)}}{(2n-2s+1)(2n+2s+3)} &= \frac{\pi}{16} \delta_{1,s} \quad (s \geq 1), \end{aligned} \right\} \dots\dots\dots (21)$$

$$\left. \begin{aligned} \sum_{n=0}^{\infty} (-1)^n \frac{a_{2n}^{(3)}}{(2n+1)(2n+3)} &= \sum_{k=0}^{\infty} (-1)^k \frac{a_{2k}^{(2)}}{2(2k+3)(2k+5)} \\ \sum_{n=0}^{\infty} (-1)^{n+s+1} \frac{a_{2n}^{(3)}}{(2n-2s+1)(2n+2s+3)} &= -\sum_{k=0}^{\infty} (-1)^{k+s+1} \frac{2a_{2k}^{(2)}}{(2k+2s+1)(2k+2s+3)(2k+2s+5)}, \quad (s \geq 1) \end{aligned} \right\} \dots\dots\dots (22)$$

$$\left. \begin{aligned} \sum_{n=0}^{\infty} (-1)^n \frac{a_{2n}^{(r)}}{(2n+1)(2n+3)} &= \sum_{k=0}^{\infty} (-1)^k \frac{a_{2k}^{(r-1)} + a_{2k}^{(r-2)}}{2(2k+3)(2k+5)} \\ \sum_{n=0}^{\infty} (-1)^{n+s+1} \frac{a_{2n}^{(r)}}{(2n-2s+1)(2n+2s+3)} &= -\sum_{k=0}^{\infty} (-1)^{k+s+1} \frac{2a_{2k}^{(r-1)} - (2k-2s+3)a_{2k}^{(r-2)}}{(2k+2s+1)(2k+2s+3)(2k+2s+5)}, \quad (s \geq 1) \end{aligned} \right\} \dots\dots\dots (23)$$

($r \geq 4$)

$$b_0^{(1)} = -1/2, \quad b_{2s}^{(1)} = 0 \quad (s \geq 1), \dots\dots\dots (24)$$

$$b_0^{(2)} = 1/4, \quad b_{2s}^{(2)} = 3\delta_{1,s}/4 \quad (s \geq 1), \dots\dots\dots (25)$$

$$b_0^{(r)} = -\frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{a_{2n}^{(r-1)}}{(2n+1)(2n+3)} - \frac{a_{2n}^{(r-2)}}{(2n+3)(2n+5)} \right] \quad (26)$$

$$b_{2s}^{(r)} = \frac{4}{\pi} (2s+1) \sum_{n=0}^{\infty} (-1)^{n+s+1} \left[\frac{a_{2n}^{(r-1)}}{(2n-2s+1)(2n+2s+3)} - \frac{a_{2n}^{(r-2)}}{(2n+2s+3)(2n+2s+5)} \right] \quad (s \geq 1) \quad (r \geq 3)$$

With these relations, the coefficients $a_{2n}^{(r)}$ and $b_{2n}^{(r)}$ in the Series (20) can be determined successively for $r=1, 2, \dots$.

It may be noted that the coefficients of $a_{2n}^{(r)}$ are invariant for all values of r in the infinite sets of Eqs. (21), (22), and (23).

Hence, if we solve these equations once, the solutions in successive cycles are obtained only by the calculations about the constant terms.

The foregoing solution is valid in so far as the Series (20) is convergent. Though the proof of convergence is necessary, no further investigation will be pursued here for brevity. However, numerical computations carried out indicate that this method gives results to any required degree of approximation.

4. Numerical examples

The foregoing solution will be worked out in some detail. The calculation of $a_{2n}^{(r)}$, $b_{2n}^{(r)}$ is carried out until $r=6$, but their values are not shown here for brevity. It is now a simple matter to calculate the deflection and stress resultants in the neighbourhood of the notch.

The values of (1) M_θ on the edge of the semi-circular notch, (2) M_r on the straight edge, (3) M_r and M_θ along the line of symmetry, ox , for the case of $\nu=0.3$ are calculated and shown graphically in Fig. 2. The distribution of stress resultants shows a general likeness to that around a circular hole in an infinite plate under plain bending. The principal difference seems to be in the value of M_θ near the corners of the notch. This might have been expected, as the nature of the problem makes M_θ vanish when θ is $\pm\pi/2$.

Curve I: M_θ at the rim of the notch
Curve II: M_y on the minimum section
Curve III: M_x on the minimum section
Curve IV: M_y on the straight edge

Fig. 2 Stress resultants in the neighbourhood of the notch.

The maximum bending couple occurs at the apex of the notch. Since $M_\theta=0$ on the same edge, the maximum couple is given by

$$M_{\max}/M_0 = (M_\theta + M_r)_{\theta=1} / M_0$$

$$= 1 - 2(1-\epsilon) \sum_{r=1}^{\infty} M_r' \epsilon^{r-1} \quad (27)$$

where

$$M_1' = b_0^{(1)}$$

$$M_r' = \sum_{n=0}^{\infty} a_{2n}^{(r)} + \sum_{n=0}^{\infty} b_{2n}^{(r)} \quad (r \geq 2) \quad (28)$$

The calculated values of M_r' are given in Table 1. With these values, the stress concentration factor $k_b = M_{\max}/M_0$ is calculated and plotted against ϵ in Fig. 3. It is interesting that the value of k_b changes almost linearly with ϵ .

5. Conclusion

The problem of determining the stress resultants in a semi-infinite plate with a semi-circular notch under plain bending is solved on the basis of the Poisson-Kirchhoff theory of thin plates and with the aid of Muskhelishvili's complex variable method. A method of perturbation is adopted for the determination of parametric coefficients involved in the solution.

Numerical calculations are carried out in some

Table 1

r	M_r'
1	-5.0×10^{-1}
2	-3.34×10^{-3}
3	-1.156×10^{-2}
4	5.64×10^{-3}
5	2.60×10^{-3}
6	4.34×10^{-3}

Fig. 3 Stress concentration factor

detail, and the distribution of stress resultants in the neighbourhood of the notch for $\nu=0.3$ is clarified. The values of the stress concentration factor also are plotted against ν .

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On Shear Lag in Thin-Walled Cantilever Pipes

(General Theory of Cylindrical Shells, Example 4)*

By Koki MIZOGUCHI**

In this paper, the shear-lag in a thin-walled cantilever pipe is discussed, utilizing the author's fundamental differential equation in a single displacement for a thin cylindrical shell.

The results are compared with those obtained by the simple beam theory or modified beam theory.

The deflection at the loaded side is very close to that estimated by the modified beam theory, while the configuration of the elastic line and the stress distribution near the fixed side are fairly different from those presumed by such a beam theory.

1. Solution of problem

Referring to Fig. 1, for the sake of simplicity, the following assumptions are made, i.e., one end of the pipe is fixed (not built-in or clamped) and the other end is fitted with a thin plate, the extensional and the flexural rigidities of which are sufficiently large and small respectively.

The concentrated force P is applied to this plate, and the origin O , which coincides with the centroid of the section of the loaded side, remains fixed during deformation.

Under these assumptions, the boundary conditions take the following forms

- [1] for $x=-l$; $w=\delta \cos \theta$, $v=-\delta \sin \theta$,
 $u=0$, $\partial w / \partial x=0$
- [2] for $x=0$; $w=0$, $v=0$, $T_x=0$, $M_x=0$

$$2 \int_0^\pi \left[\left(N_x + \frac{1}{a} \frac{\partial \mathcal{M}}{\partial \theta} \right) \cos \theta + \left(S' - \frac{\mathcal{M}}{a} \right) \sin \theta \right] a d\theta = P$$

where δ is a quantity equivalent to the deflection of the loaded end, and w, v, u , are the displacements for radial, tangential, and axial directions, and $T_x, N_x, S', \dots$ or $M_x, \mathcal{M}, \dots$ are the stress resultants and couples, the meanings and the positive directions of which are defined in the author's previous paper⁽¹⁾ as shown in Fig. 2.

By means of the strain energy equation, we can prove that the solution under the boundary

Fig. 1

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