Vibration of Rectangular Plates with Circular Holes*

By Shin TAKAHASHI**

The frequencies in vibration of rectangular plates with circular holes are studied. The method is due to Ritz and the displacement is expressed in terms of the products of bar-solutions which satisfy the boundary conditions of plate. The kinetic and potential energies are given in the cases of clamped-clamped, clamped-free and clamped-supported ends. Numerical calculations are carried out in the case of the plate of which all edges are fixed and with a hole in centre. Experimental data are also given in the above mentioned case in which the ratio of edge lengths is 1/2 and the agreement with calculations is good.

D. Young and G.B. Warburton[2] have solved the problem of vibration of plates using the bar-solutions. The author has studied the vibrations of rectangular bars with many holes, and moreover he has studied, as sequel to that paper, the vibration of rectangular plates with circular holes. The method followed is, according to that of Ritz, to apply bar-solutions, which satisfy the boundary conditions of the edges (without considering the conditions round the holes) and gives the frequencies of any mode.

\[ S'(nl, pl) = (1 + \lambda_0 \lambda_2) \cos (g_n - g_p) \cos (\lambda_0 - \lambda_2) \sin (g_n - g_p) \cos (g_n + g_p) + (\lambda_0 - \lambda_2) \sin (g_n + g_p) \]

\[ C'(nl, pl) = (1 - \lambda_0 \lambda_2) \cos (g_n + g_p) \cos (\lambda_0 - \lambda_2) \sin (g_n + g_p) \sin (g_n - g_p) \cos (g_n + g_p) + (\lambda_0 - \lambda_2) \cos (g_n + g_p) \sin (g_n + g_p) \]

A reference to the paper by A. Thum and H. Ochs: Korrosion und Dauerfestigkeit, VDI-Verlag, Hr. 9 (1937), S. 4.


\[ H.F. Gough : Jour. Inst. of Metal. Vol. 49 (1932), p. 117. \]

\[ H. Schenck and E. Schmidtmann : Archiv. fur Eisenhuettenw., Bd. 25, Hr. 11/12 (1954), S. 579. \]


---

*N Received 17th October, 1957. ** Assistant Professor, Engineering Faculty, Yamagata University, Yonezawa.
These quantities are the combinations of \( \phi, \psi, \Phi, \) and \( \Psi, \) reported in the author's preceding paper (2). \( \Sigma \) signifies the summation as many as the number of holes.

### Kinetic and potential energies

The coordinates axes are on the edges of rectangular plate: \( x \) and \( y \) are non-dimensional coordinates, then the maximum kinetic and potential energies are as follows,

\[
T = T_1 - T_2 = \frac{R h}{2 g} \rho ab \left[ \int_0^L \int_0^1 w^2 dx dy - \sum_i \int_{x_i}^{x_i + r} \int_{y_m}^{y_m + q} w^2 dz dy \right]
\]

\[
V = V_1 - V_2 = \frac{Dab}{2 a^4} \left[ \int_0^1 \int_0^L dz dy - \sum_i \int_{y_m}^{y_m + q} \int_{x_i}^{x_i + r} dy dz \right] \left[ \left( \frac{\partial^4 w}{\partial x^4} \right)^2 + q^2 \left( \frac{\partial^4 w}{\partial y^4} \right)^2 + 2 \nu_1^2 \frac{\partial^4 w}{\partial x^2 \partial y^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\nu) q^2 \left( \frac{\partial^4 w}{\partial x^2 \partial y^2} \right)^2 \right]
\]

where the first integrations are the energies of the holeless plate and the second are those which must be subtracted from the first on account of holes. When the boundary conditions are clamped-clamped, clamped-free or clamped-supported, \( w \) is as follows,

\[
w = \sum_n \sum_m (\sin g_x - \sinh g_x) + \lambda_n (\cos g_y - \cosh g_y) \sum_n (\sin g_y - \sinh g_y) + \lambda_n (\cos g_y - \cosh g_y)
\]

where \( a_{n,m} \) is constant and in the case of free-free, the negative signs in the above equation must be changed to the positive and in the case of simply supported ends, the sine terms only must be used, \( g_x \) and \( \lambda_n \) are as follows.

In the cases clamped-clamped or free-free, \( g_x \) is \( n \)-th root of \( \cos g_x \cosh g_x = 1 \) and \( \lambda_n = \frac{\cos g_x - (-1)^n}{\sin g_x} \).

In the case clamped-free, \( g_x \) is \( n \)-th root of \( \cos g_x \cosh g_x = -1 \) and \( \lambda_n = \frac{\cos g_x + (-1)^n}{\sin g_x} \).

In the case clamped-supported, \( g_x \) is \( n \)-th root of \( \tan g_x = \tanh g_x \) and \( \lambda_n = - \tan g_x \).

Using these values, \( T_1 \) and \( V_1 \) are easily obtained, e. g., in the case that all edges are clamped,

\[
T_1 = \frac{R h}{2 g} \rho ab \sum_n \sum_m a_{n,m}^2 \lambda_n \lambda_m
\]

\[
V_1 = \frac{Dab}{2 a^4} \sum_n a_{n,m}^2 \left[ (\sinh g_x)^2 \lambda_n \lambda_m + 2 g_x^2 (\sinh g_x)^2 \right] (\sinh g_x)^2 + 2 g_x \lambda_n)
\]

When \( w \) is expressed with the above form (i. e. the cases clamped-clamped, clamped-free or clamped-supported), \( V_1 \) for each hole is as follows,

\[
2 V a^2 abD = \sum_n \sum_m \sum_i \sum_j a_{n,m} a_{n,m} \frac{\pi^2 q}{2} \times \left[ (\sin g_x - \sinh g_x) + C(n, m) \right] \left[ \cos g_x - \cosh g_x \right] \left( \sinh g_x \right)^2 + 2 g_x \lambda_n)
\]

\[
+ \left( \gamma K(m, m) - \gamma G(m, m) \right) \left[ (2 R) + \left( \frac{D}{2} G(m, m) \right) \left( \sinh g_x \right)^2 + 2 g_x \lambda_n)
\]

\[
+ \left( \beta - \gamma \right) \left( S(m, m) - S(m, m) \right) \left[ \cos g_x - \cosh g_x \right] \left( \sinh g_x \right)^2 + 2 g_x \lambda_n)
\]

\[
+ \left( \alpha - \gamma \right) \left( S(m, m) - S(m, m) \right) \left[ \cos g_x - \cosh g_x \right] \left( \sinh g_x \right)^2 + 2 g_x \lambda_n)
\]

\[
+ \left( \beta - \gamma \right) \left( S(m, m) - S(m, m) \right) \left[ \cos g_x - \cosh g_x \right] \left( \sinh g_x \right)^2 + 2 g_x \lambda_n)
\]

\[
+ \left( \alpha - \gamma \right) \left( S(m, m) - S(m, m) \right) \left[ \cos g_x - \cosh g_x \right] \left( \sinh g_x \right)^2 + 2 g_x \lambda_n)
\]

\[
+ \left( \beta - \gamma \right) \left( S(m, m) - S(m, m) \right) \left[ \cos g_x - \cosh g_x \right] \left( \sinh g_x \right)^2 + 2 g_x \lambda_n)
\]

\[
+ \left( \alpha - \gamma \right) \left( S(m, m) - S(m, m) \right) \left[ \cos g_x - \cosh g_x \right] \left( \sinh g_x \right)^2 + 2 g_x \lambda_n)
\]
\[ S'(nl, pl) \left[ \delta K (lm, sm) - e G (lm, sm) \right] (2R) + \left[ \delta K (lm, sm) - e G (lm, sm) \right] (2B) \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + C'(nl, pl) \left[ \delta K (lm, sm) - e G (lm, sm) \right] (2R) + \left[ \delta K (lm, sm) - e G (lm, sm) \right] (2B) \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + C(sm, tm) \left[ \delta K (nl, pl) - e G (nl, pl) \right] (2R) + \left[ \delta K (nl, pl) - e G (nl, pl) \right] (2B) \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + S(sm, tm) \left[ \delta K (nl, pl) - e G (nl, pl) \right] (2R) + \left[ \delta K (nl, pl) - e G (nl, pl) \right] (2B) \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + C(sm, tm) \left[ \delta K (nl, pl) - e G (nl, pl) \right] (2K) + \left[ \delta K (nl, pl) - e G (nl, pl) \right] (2B) \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + S(sm, tm) \left[ \delta K (nl, pl) - e G (nl, pl) \right] (2R) + \left[ \delta K (nl, pl) - e G (nl, pl) \right] (2B) \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + \left[ (-\alpha) + \epsilon \right] \left[ (G(nl, pl) G(sm, tm) + K(nl, pl) K(sm, tm)) (2R) + \left[ -G(nl, pl) K(sm, tm) + K(nl, pl) G(sm, tm) \right] (2B) \right] \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ - \left[ (-\alpha) - \epsilon \right] \left[ (G(nl, pl) G(sm, tm) - K(nl, pl) K(sm, tm)) (2R) + \left[ G(nl, pl) K(sm, tm) + K(nl, pl) G(sm, tm) \right] (2B) \right] \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + \left[ (-\beta) + \epsilon \right] \left[ (G(nl, pl) G(tm, sm) + K(nl, pl) K(tm, sm)) (2R) + \left[ -G(nl, pl) K(tm, sm) + K(nl, pl) G(tm, sm) \right] (2B) \right] \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ - \left[ (-\beta) - \epsilon \right] \left[ (G(nl, pl) G(tm, sm) - K(nl, pl) K(tm, sm)) (2R) + \left[ G(nl, pl) K(tm, sm) + K(nl, pl) G(tm, sm) \right] (2B) \right] \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + \left[ (-\gamma) \right] \left[ (G(pl, nl) G(sm, tm) + K(pl, nl) K(sm, tm)) (2R) + \left[ G(pl, nl) K(sm, tm) + K(pl, nl) G(sm, tm) \right] (2B) \right] \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ - \left[ (-\beta) - \epsilon \right] \left[ (G(pl, nl) G(sm, tm) - K(pl, nl) K(sm, tm)) (2R) + \left[ G(pl, nl) K(sm, tm) + K(pl, nl) G(sm, tm) \right] (2B) \right] \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + \left[ (-\alpha) - \epsilon \right] \left[ (G(pl, nl) G(tm, sm) + K(pl, nl) K(tm, sm)) (2R) + \left[ -G(pl, nl) K(tm, sm) + K(pl, nl) G(tm, sm) \right] (2B) \right] \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ + \left[ (-\gamma) \right] \left[ (G(pl, nl) G(tm, sm) + K(pl, nl) K(tm, sm)) (2R) + \left[ -G(pl, nl) K(tm, sm) + K(pl, nl) G(tm, sm) \right] (2B) \right] \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ - \left[ (-\beta) + \epsilon \right] \left[ (G(pl, nl) G(sm, tm) - K(pl, nl) K(sm, tm)) (2R) + \left[ G(pl, nl) K(sm, tm) + K(pl, nl) G(sm, tm) \right] (2B) \right] \] \[ Q \sqrt{g_3^2 + g_9^2 + (g_1 + ig_2)^2} q' \]
\[ \text{where} \]
\[ \alpha = \frac{g_3^2 + g_9^2 + (g_1 + ig_2)^2}{g_3^2 + g_9^2 + (g_1 + ig_2)^2} \]
\[ \beta = \frac{g_3^2 + g_9^2 + (g_1 + ig_2)^2}{g_3^2 + g_9^2 + (g_1 + ig_2)^2} \]
\[ \gamma = \frac{g_3^2 + g_9^2 + (g_1 + ig_2)^2}{g_3^2 + g_9^2 + (g_1 + ig_2)^2} \]
\[ \delta = \frac{g_3^2 + g_9^2 + (g_1 + ig_2)^2}{g_3^2 + g_9^2 + (g_1 + ig_2)^2} \]
\[ \epsilon = \frac{2(1 - \nu)}{g_3^2 + g_9^2 + (g_1 + ig_2)^2} \]

In the above expression of \( V_2 \), putting \( \epsilon = 0 \), and \( \alpha = \beta = \gamma = \delta = 1 \) in the first half and \( (-\alpha) = (-\beta) = (-\gamma) \) in the second half, \( 2 T_2 \sqrt{\frac{J_l}{g}} \) is obtained. To obtain \( T_2 \) and \( V_2 \), the following integrations are used and in the case \( n = p \), the limits are used. Putting \( X_s(y) = dx X_s(y) dy \)
\[ \int X_s(y) X_s(y) dy = \frac{1}{g_3^2 + g_9^2} \left[ X_s''(y) X_s(y) - X_s'(y) X_s'(y) - X_s''(y) X_s'(y) + X_s(y) X_s''(y) \right] \]
\[ \int X_s''(y) X_s'(y) dy = \frac{1}{g_3^2 + g_9^2} \left[ g_3^2 X_s''(y) X_s(y) - g_9^2 X_s'(y) X_s'(y) - g_5^2 X_s''(y) X_s(y) + g_9^2 X_s(y) X_s''(y) \right] \]
\[ \int X_s'(y) X_s'(y) dy = \frac{1}{g_3^2 + g_9^2} \left[ g_3^2 X_s'(y) X_s(y) - g_9^2 X_s'(y) X_s'(y) - g_5^2 X_s''(y) X_s(y) + g_9^2 X_s(y) X_s''(y) \right] \]
\[ \text{and} \]
\[ \int_{y=m}^{y=m+q} \cos(\mu y - \nu) \int_{y=m}^{y=m-q} \sin(\mu' y - \nu') \right] \]
\[ \frac{2\pi^2}{g_3^2 + g_9^2} \left[ \cos(\mu y - \nu) \cos(\mu' y - \nu) \right] Q \sqrt{g_3^2 + g_9^2} \]

\[ \text{(A)} \]
Thus, after $T$ and $V$ are obtained, it is possible to get the frequencies of any mode by Ritz's method. When the locations of holes are symmetrical about mid-line of plate, $S$, $C$, $S'$, $C'$, $G$, and $K$ are same in the absolute values and their signs are same or opposite.

**Numerical calculations and the comparison with the experimental results**

For the numerical example, the simplest case, i.e., the plate, with one hole only in its centre, of which all edges are clamped, is studied for the fundamental mode, assuming $\nu=0.3$. In this case, $I=m=1/2$, and

$$S(nl, pl) = -C(nl, pl) = 1/\sin \frac{g_n}{2} \sin \frac{g_p}{2},$$

$$-S'(nl, pl) = C'(nl, pl) = 1/\sinh \frac{g_n}{2} \sinh \frac{g_p}{2},$$

for the case that both $n$ and $p$ are odd,

$$S(nl, pl) = C(nl, pl) = 1/\cos \frac{g_n}{2} \cos \frac{g_p}{2},$$

$$S'(nl, pl) = C'(nl, pl) = 1/\cosh \frac{g_n}{2} \cosh \frac{g_p}{2},$$

for the case that both $n$ and $p$ are even,

$$G(nl, pl) = 1/\cos \frac{g_n}{2} \cos \frac{g_p}{2},$$

for the case both are even

$$G(nl, pl) = 0,$$

for the case that both or either of $n$ or $p$ is odd, and

$$K(nl, pl) = -\frac{1}{\sin \frac{g_n}{2} \sinh \frac{g_p}{2}},$$

for the case both are odd,

$$K(nl, pl) = 0$$

for the case that both or either of $n$ or $p$ is even.

To obtain the first approximation, we put $n=p=s$, $t=1$, and the numerical values used in calculation are as follows,

$$S = -C = 2.035 \times 10^5,$$

$$-S' = C' = 0.03593599,$$

$$K = -0.2704873, \quad G = 0, \quad g_1 = 4.730041,$$

$$\lambda _1 = -1.017809$$

Fig. 1 shows the relations between the non-dimensional frequency $p^2 \pi^2 a^4 g D$ and the non-dimensional radius $r$. The magnitude of errors is the same as that of the holeless plate and about 1%. For the square plate, the variations of frequency are largest and for $r=0.2$, the frequency increases 62%. As $g (=a/b)$ approaches zero, so do the variations. Fig. 2 shows the experimental results with the calculated ones. The dimensions of plate are also shown in Fig. 2, and the hole has been bored successively larger and the frequencies are measured with the electric resistance strain gauge and electro-magnetic oscillograph.

To the radius of which the dimension is as large as one-sixth of the shorter side, both results approximately agree.

The author wishes to express his gratitude to Mr. T. Ogasawara, Mr. F. Nakamura, and Mr. S. Tsuchida for their cooperative efforts in carrying out the experiments.
Appendix

(1) To obtain the equation (A):
The modification of Parseval integral due to Bessel is (3),
\[ \pi J_0(\sqrt{x^2+y^2}) = \int_0^{\pi} e^{iy\cos \theta} \cos(\theta \sin \theta) \, d\theta \]
In the case that \( y \) and \( z \) are real, the integration of the imaginary part of the right hand in the above equation is zero, so
\[ = \int_0^{\pi} \cos(\theta \cos \theta) \cos(\theta \sin \theta) \, d\theta \]
After differentiating with \( z \), we obtain
\[ = \int_0^{\pi} \cos(\theta \cos \theta) \cos(\theta \sin \theta) \sin \theta \, d\theta = \frac{\pi z}{\sqrt{x^2+y^2}} J_1(\sqrt{x^2+y^2}) = \pi z Q(\sqrt{x^2+y^2}) \]
Using the above expression and considering \( x = r \cos \theta + l \), we obtain the equation (A).

(2) In the calculations of \( T_2 \) and \( V_2 \) using the equation (A), in the case that the integrands contain hyperbolic sine and cosine, \( \mu \) and \( \nu \) are imaginary, and so the values inside the brackets of \( Q \) are complex and it is inconvenient to calculate numerically. But, as they are always the pair of \( Q(\sqrt{p+iq}) \) and \( Q(\sqrt{p-ij}) \), they are reduced to \( T_2 \) and \( V_2 \) as mentioned above, by reason of the following,
\[ \frac{n_2(x)}{x} = \frac{1}{2} [J_{s+1}(x) + J_{1-s}(x)] \]
After multiplying both sides by \( x^{s+1}, \)
\[ \sum_{n=-\infty}^{\infty} \frac{n_2(x)}{x} x^{s+1} = \frac{1}{2} \sum_{n=-\infty}^{\infty} [J_{s+1}(x) t^{s+1} + J_{1-s}(x) t^{s+1}] = \frac{1}{2} \left[ e^{\frac{x}{2} \left( t - \frac{1}{t} \right)} + e^{-\frac{x}{2} \left( t - \frac{1}{t} \right)} \right] \]
putting \( x = u + iv \)
\[ = \frac{1}{2} (1 + t^2) e^{\frac{u}{2} \left( t - \frac{1}{t} \right)} \]
\[ = \frac{1}{2} (1 + t^2) \sum_{m=-\infty}^{\infty} J_m(u) t^m \sum_{l=-\infty}^{\infty} J_l(v) t^l \]
Comparing the coefficients of \( t^{s+1} \) in both sides,
\[ \frac{n_2(x)}{x} = \sum_{p=-\infty}^{\infty} J_{p-q}(u) \frac{P_{2s+2}(v)}{v} \]
Moreover, the right hand of this equation is convergent.

In the addition theorem of Gegenbauer (4), putting \( n = 1 \), and the argument is \( \pi/2, \)
\[ Q(\sqrt{Z^2+z^2}) = J_1(\sqrt{Z^2+z^2}) \sqrt{Z^2+z^2} \]
\[ = \sum_{m=0}^{\infty} 2(-1)^m J_{2m+1}(Z) J_{2m+1}(Z) x \]
After putting \( Z = g + i h, x = u + iv \), and using the above equation,
\[ Q(\sqrt{(g+ik)^2+(u+iv)^2}) \]
\[ = \sum_{m=0}^{\infty} (-1)^m \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (2m+1-q) (2m+1-p) \frac{J_{2m+1-q}(g)}{g} \frac{J_{2m+1-p}(u)}{u} \times J_{2m+1}(h) J_{2m+1}(v) \]
The parts in which \( p+q \) is even are the real part of \( Q \), and the parts in which \( p+q \) is odd are the imaginary part. Considering
\[ I_0(x) = (-1)^n I_n(-x), \]
\[ R(Q(\sqrt{Z^2+z^2})) = R(Q(\sqrt{Z^2+z^2})) \]
\[ \partial Q(\sqrt{Z^2+z^2}) = -\partial Q(\sqrt{Z^2+z^2}) \]
where \( \partial \) is the conjugate complex.
Similarly, in the case that \( Z \) is real and \( x \) is complex
\[ R(Q(\sqrt{Z^2+z^2})) = R(Q(\sqrt{Z^2+z^2})) \]
\[ \partial Q(\sqrt{Z^2+z^2}) = -\partial Q(\sqrt{Z^2+z^2}) \]
That is, the real parts have the same sign to each other and the imaginary parts have the opposite signs, and they are equal in the absolute values. Thus, in $T_2$ and $V_2$, $Z^2 + z^2$ are used in place of $Z^2 + z^2$ or $Z^2 + z^2$.

3) $S$, $C$, $S'$, $C'$, $K$, and $G$ mentioned above can be expressed with $\phi$, $\psi$, $\Phi$, and $\Psi$ in the author's paper (2) as follows,

$$
\begin{align*}
S(nl, pl) &= \phi_0 \phi_p + \psi_0 \psi_p \\
C(nl, pl) &= -\phi_0 \phi_p + \psi_0 \psi_p \\
S'(nl, pl) &= -\Phi_0 \Phi_p + \Psi_0 \Psi_p \\
C'(nl, pl) &= \Phi_0 \Phi_p + \Psi_0 \Psi_p \\
K(nl, pl) &= \phi_0 \phi_p \\
G(nl, pl) &= \phi_0 \Phi_p 
\end{align*}
$$

where

$$
\begin{align*}
\phi_0(l) &= \sin h g_d l + \lambda \cos g_d l, \\
\psi_0(l) &= \cos h g_d l - \lambda \sin g_d l \\
\Phi_0(l) &= \sinh h g_d l + \lambda \cosh g_d l, \\
\Psi_0(l) &= \cosh h g_d l + \lambda \sinh g_d l
\end{align*}
$$

References


4) loc. cit., p. 362.