argument about reducing Eq. (5) with assumption of $\lambda = \lambda_0$, associated with the formula of $h_i \propto v^2$.

(3) The general formula of the correction to the treatment according to item (2) is Eq. (16).

(4) Eq. (21) is the correction to Eq. (2) for our setup, provided the valve is operated at a constant speed.

(5) Fig. 6 is obtained by making corrections to Fig. 2, assuming the valve is shut off at a constant speed.

(6) The correction for the pipe line of hy-

draulic power plants is smaller than that for our model setup.

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Studies on the Convective Heat Transfer from Rotating Disk*

(1st Report, The Effect of Dissipative Energy on the Laminar Heat Transfer from a Disk Rotating in Uniform Forced Stream)

By IKUO MABUCHI**, TOSHIO TANAKA***, and YUKIHIRO SAKAKIBARA***

This paper deals with a theoretical investigation on the effect of the dissipative energy on the laminar heat transfer from a disk rotating in uniform forced stream. Under the conditions that the fluid is incompressible and its physical properties are constant, the energy equation is solved exactly for the cases where the variation of the surface temperature is expressed by the general quadratic distribution along the disk radius and the disk surface is thermally insulated.

1. Introduction

The laminar flow and heat transfer about a rotating disk were analysed at first† by von Kármán⁽¹⁾ and Millsaps and Pohlhausen⁽²⁾, respectively. Recently, the theoretical extensions to make clear the effects of compressibility⁽³⁾, Prandtl number⁽⁴⁾, variation of surface temperature^{(5) (6)}, etc. have been reported, but these studies are made for a disk rotating in a quiescent fluid.

The laminar flow field about a disk rotating in uniform stream was analysed at first†† by Han-nah⁽⁷⁾. Recently, Schlichting and Truckenbrodt⁽⁹⁾ obtained an approximate solution by applying the momentum-integral method. On the other hand, Yamaga⁽¹⁰⁾ extended the integral method to the heat transfer problem. Recently, the exact nu-

merical solution for heat transfer has been obtained by Tien and Tsuji⁽¹¹⁾. However, the effect of the dissipative energy is not considered in their analytical treatments for heat transfer.

The purpose of this report is to make clear the effect of the dissipative energy on laminar heat transfer from a disk rotating in uniform stream. In this analysis, usual boundary-layer approximation is applied for the thermal field near a disk and the variation of the surface temperature includes the following cases, i.e. (1) isothermal case (2) quadratic distribution (3) thermally insulated case.

2. Nomenclature

a, b : constant of velocity field defined by Eq. (5)

c : specific heat of fluid, and specific heat at constant pressure should be used for the gas.

E_{cr} : local Eckert number of rotating disk with axial forced flow defined by Eq. (37)

F, G, H : dimensionless function of velocity field

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† According to Dorfman⁽¹²⁾, a primary analysis for heat transfer was performed by Kibel (1947).

†† Recently, the exact solution was independently rediscovered by Tifford and Chu⁽⁸⁾.

defined by Eq. (10)

- k : thermal conductivity
 N_{ur} : local Nusselt number defined by Eq. (48)
 P_r : Prandtl number
 q_r : local surface heat flux
 r : radial co-ordinate (R : disk radius)
 R_{er} : local Reynolds number of rotating disk with axial forced flow defined by Eq. (50)
 t : fluid temperature
 t_∞ : temperature of uniform stream
 t_{aw} : surface temperature of disk thermally insulated
 t_w : disk surface temperature
 u, v, w : velocity components for r, ϕ, z
 z : normal co-ordinate
 Greek
 α_r : local heat transfer coefficient defined by Eq. (47)
 ζ : dimensionless normal distance defined by Eq. (9)
 λ : resultant angular velocity defined by Eq. (9)
 μ : viscosity of fluid
 ν : dynamic viscosity of fluid
 ϕ : azimuthal co-ordinate
 ω : angular velocity of rotating disk
 τ_0, τ_2 : dimensionless temperature defined by Eq. (10)
 τ_2^* : solution τ_2 for case of isothermal surface
 $\bar{\tau}_2^*$: particular solution τ_2 for case of quadratic distribution
 τ_2^{**} : solution τ_2 for case of thermally insulated disk

3. Transformation of basic equations

The physical model to be analysed is a steady, incompressible, laminar heat transfer from a disk, rotating at a constant angular velocity about its symmetrical axis and emerging in uniform stream toward a disk surface along the axis. The fluid is assumed to have constant physical properties. The co-ordinate system is shown in Fig. 1.

The basic equations for such a system are given as continuity:

Fig. 1 Physical model and co-ordinate

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad \dots\dots\dots (1)$$

momentum:

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = \left(u \frac{\partial u}{\partial r} \right)_{z \rightarrow \infty} + \nu \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{u}{r} \right) + \frac{\partial^2 u}{\partial z^2} \right\} \quad \dots\dots\dots (2)$$

$$u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \nu \left\{ \frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{v}{r} \right) + \frac{\partial^2 v}{\partial z^2} \right\} \quad \dots\dots\dots (3)$$

energy†:

$$u \frac{\partial t}{\partial r} + w \frac{\partial t}{\partial z} = \frac{\nu}{P_r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial t}{\partial r} \right) + \frac{\partial^2 t}{\partial z^2} \right\} + \frac{\mu}{\rho c} \left\{ 2 \left(\frac{\partial u}{\partial r} \right)^2 + 2 \left(\frac{u}{r} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial r} - \frac{v^2}{r} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right)^2 \right\} \quad \dots\dots\dots (4)$$

The boundary conditions for the velocity field are given as

$$\left. \begin{aligned} u(r, 0) &= 0, \quad v(r, 0) = r\omega, \quad w(r, 0) = 0 \\ u(r, \infty) &= ar, \quad v(r, \infty) = 0, \\ w(r, \infty) &= -2az - b \end{aligned} \right\} \quad \dots\dots\dots (5)$$

where a is a constant [$a = (2/\pi)(W_\infty/R)$] obtained from the potential flow theory for axisymmetrical stagnation flow and has the same dimension as the angular velocity. The boundary conditions for thermal field are given as

$$t(r, 0) = t_w, \quad t(r, \infty) = t_\infty \quad \dots\dots\dots (6)$$

The variation of surface-surrounding fluid temperature difference ($t_w - t_\infty$) with distance from the disk axis is generally specified by

$$T_w(0, r) = t_w - t_\infty = \sum_{i=0}^n a_{2i} r^{2i} \quad \dots\dots\dots (7)$$

The thermal field which satisfies the above conditions has a solution in form:

$$T(z, r) = t - t_\infty = \sum_{i=0}^n A_{2i}(z) r^{2i} \quad \dots\dots\dots (8)$$

where $A_{2i}(z)$ is a function of z only. It should be also noted that temperature level is so chosen that $A_{2i}(\infty) = 0$.

Now, the partial differential equation system may be reduced to the ordinary one, if the following similarity transformation††, which was introduced by Hannah, is used.

$$\zeta = (\lambda/\nu)^{1/2} z, \quad \lambda = (a^2 + \omega^2)^{1/2} \quad \dots\dots\dots (9)$$

† Under the assumption of incompressible fluid, the work of compression is negligible and an analysis for compressible fluid will be discussed in further report, where the effect of heating by compressive work on the present results for gas will be clarified.

†† Tifford and Chu, who analysed independently, adopted $\zeta = \{(a + \omega)/\nu\}^{1/2} z$ as an independent variable.

$$\left. \begin{aligned} u &= \lambda r F(\zeta), \quad v = \omega r G(\zeta), \quad w = (\lambda \nu)^{1/2} H(\zeta) \\ \tau_0(\zeta) &= (k/\mu \nu \lambda) A_0(\zeta) \\ \tau_{2i}(\zeta) &= (k \nu^{i-1}/\mu \lambda^{i+1}) A_{2i}(\zeta) \quad (i=1, 2, \dots) \end{aligned} \right\} \dots\dots\dots(10)$$

The basic equations (1) through (4) after the above transformation become as follows:

$$H' + 2F = 0 \dots\dots\dots(11)$$

$$F'' - HF' = F^2 - (\omega/\lambda)^2 G^2 - (a/\lambda)^2 \dots\dots\dots(12)$$

$$G'' - HG' - 2FG = 0 \dots\dots\dots(13)$$

$$\tau_0'' - P_r H \tau_0' = -4\tau_2 - 4F^2 - 2H'^2 \dots\dots\dots(14)$$

$$\tau_2'' - P_r (H \tau_2' + 2F \tau_2) = -16\tau_4 - (\omega/\lambda)^2 G'^2 - F'^2 \dots\dots\dots(15)$$

$$\left. \begin{aligned} \tau_{2i-2}'' - P_r \{ H \tau_{2i-2}' + (2i-2) F \tau_{2i-2} \} \\ = -(2i)^2 \tau_{2i} \end{aligned} \right\} \dots\dots\dots(16)$$

$$\tau_{2n}'' - P_r (H \tau_{2n}' + 2n \tau_{2n}) = 0$$

The primes in Eqs. (11) through (16) denote differentiation with respect to ζ .

Similarly, the boundary conditions, Eq. (5) are transformed as follows.

$$\left. \begin{aligned} F(0) &= 0, \quad G(0) = 1, \quad H(0) = 0 \\ F(\infty) &= (a/\lambda), \quad G(\infty) = 0 \\ H(\infty) &= -2(a/\lambda)\zeta - C \end{aligned} \right\} \dots\dots\dots(17)$$

$$\left. \begin{aligned} \tau_0(0) &= \tau_{00}, \quad \tau_0(\infty) = 0 \\ \tau_2(0) &= \tau_{20}, \quad \tau_2(\infty) = 0 \end{aligned} \right\} \dots\dots\dots(18)$$

where τ_{00} , τ_{20} are $\tau_{00} = (k/\mu \nu \lambda) a_0$, $\tau_{20} = (k/\mu \lambda^2) a_2$, ... and C is $C = b(\lambda \nu)^{-1/2}$.

The first three equations, (11), (12) and (13) with the boundary condition, Eq. (17) were numerically integrated by Hannah for the case with $a/\omega = 0.5, 1$ and 2 . The case with $a/\omega = \infty$ represents the axisymmetrical stagnation flow, which has been studied by Homann⁽¹⁴⁾. The other limiting case with $a/\omega = 0$ represents the induced flow due to a rotating disk in quiescent fluid, which has been studied by Cochran⁽¹³⁾.

The object of this paper is to solve the energy equations (14) through (16) taking the dissipative energy into account. Now, in the actual conditions in which turbine disks operate, the variation of radial surface temperature is nearly quadratic. From this standpoint, we shall discuss only the case with a general quadratic temperature distribution in the form

$$T_w(0, r) = a_0 + a_2 r^2 \dots\dots\dots(19)$$

In order to solve by means of numerical techniques the energy equations, we shall need the values of the velocity functions H and F as the input data, but the numerical solutions reported by Hannah are three significant figures. Therefore, to raise the accuracy of numerical computation, we recomputed by using HIPAC-103 electronic

computer and obtained numerical solutions having five significant figures and in addition, computed the case with $a/\omega = 0.1, 0.25$ to make clear in detail the influence of axial flow.

4. Exact solution of closed form ($P_r=1$)

When the angular velocity of a disk is large, or small but the velocity of forced flow is large, the rotating Reynolds number defined by Eq. (50) becomes large. Therefore, the assumption of boundary-layer approximation for thermal field is valid with sufficient accuracy. Then, the energy equations become as follows

$$\tau_0'' - P_r H \tau_0' = 0 \dots\dots\dots(14)'$$

$$\tau_2'' - P_r (H \tau_2' + 2F \tau_2) = -F'^2 - (\omega/\lambda)^2 G'^2 \dots\dots\dots(15)'$$

Eq. (14)' corresponds to the isothermal disk without dissipative effect. On the other hand, Eq. (15)' with $a/\omega = 0$ was studied by Kibel or Millsaps and Pohlhausen, but no exact solution is available for the case $a/\omega \neq 0$. In this section, an exact solution in closed form is sought for $P_r=1$ and its results are found useful to check the accuracy of numerical solution in the following section.

The fluids with $P_r=1$ correspond to gases. Thus, as a rotating Eckert number defined by Eq. (37) is brought close to unity, the effect of compressibility will appear. Consequently, in the energy equation, the heating term $u(\partial p/\partial r)$ by the compression work cannot be neglected. Then, there is in such case a limitation to the application of the result for gas, but numerical results for $P_r > 2$, corresponding to liquids that are incompressible, may be expected to be sufficiently correct.

Now, $\tau_2(\zeta)$ satisfies $\tau_2 \rightarrow 0$ ($\zeta \rightarrow \infty$) according to boundary condition, Eq. (18). Taking into account this fact, we have next transformation

$$F(\zeta) = (a/\lambda) + F^*(\zeta) \dots\dots\dots(20)$$

After this transformation, Eqs. (12), (13) and (15)' are respectively changed as

$$2(a/\lambda)F^* + F^{*2} + HF^{*'} - (\omega/\lambda)^2 G^2 = F^{*''} \dots\dots\dots(21)$$

$$2(a/\lambda)G + 2F^*G + G'H = G'' \dots\dots\dots(22)$$

$$\tau_2' H + 2(a/\lambda)\tau_2 + 2F^*\tau_2 = \tau_2'' + F^{*2} + (\omega/\lambda)^2 G'^2 \dots\dots\dots(23)$$

Let us multiply Eq. (21) by F^* and Eq. (22) by $(\omega/\lambda)\{(\omega/\lambda)G + \tilde{k}\}$. Furthermore, if we add two equations obtained to Eq. (23), we get

$$\begin{aligned} H[\tau_2 + (1/2)\{(\omega/\lambda)^2 G^2 + F^{*2}\} \\ + \tilde{k}(\omega/\lambda)G + (a/\lambda)\{(\omega/\lambda)G + F^*\}]' \\ + 2F^*[\tau_2 + (1/2)\{(\omega/\lambda)^2 G^2 + F^{*2}\} + \tilde{k}(\omega/\lambda)G \\ + (a/\lambda)\{(\omega/\lambda)G + F^*\}]' + 2(a/\lambda)[\tau_2 \\ + (1/2)\{(\omega/\lambda)^2 G^2 + F^{*2}\} + \tilde{k}(\omega/\lambda)G \\ + (a/\lambda)\{(\omega/\lambda)G + F^*\}] \\ = [\tau_2 + (1/2)\{(\omega/\lambda)^2 G^2 + F^{*2}\} + \tilde{k}(\omega/\lambda)G \\ + (a/\lambda)\{(\omega/\lambda)G + F^*\}]'' \dots\dots\dots(24) \end{aligned}$$

Thus, the solution of this problem is

$$\tau_2 = -(1/2)\{(\omega/\lambda)^2 G^2 + F^{*2}\} - \tilde{k}(\omega/\lambda)G - (a/\lambda)\{(\omega/\lambda)G + F^*\} \dots (25)$$

where $\tilde{k}$ is an arbitrary constant and determined from the boundary conditions.

4.1 Isothermal surface

The boundary condition is

$$\zeta=0, \tau_2=0 \dots (26)$$

It then follows from Eq. (25) that

$$\tilde{k}(\omega/\lambda) = (a/\lambda)\{(1/2)(a/\lambda) - (\omega/\lambda)\} - (1/2)(\omega/\lambda)^2 \dots (27)$$

Let us indicate the solution $\tau_2(\zeta)$, which expresses the dissipative effect for an isothermal disk, by $\tau_2^*(\zeta)$. Now, substituting Eq. (27) into Eq. (25) and using the transformation $F^* \rightarrow F$, we obtain

$$\tau_2^*(\zeta) = (1/2)\{(\omega/\lambda)^2(G-G^2) + (a/\lambda)^2(1-G) - F^2\} \dots (28)$$

On the other hand, the formal solution for $\tau_0(\zeta)$ is given by

$$\tau_0(\zeta) = \tau_{00} \left[1 - \int_0^\zeta \exp\left(\int_0^\zeta H d\zeta\right) d\zeta / \int_0^\infty \exp\left(\int_0^\zeta H d\zeta\right) d\zeta \right] \dots (29)$$

4.2 Pure quadratic surface temperature

The boundary condition is

$$\zeta=0, \tau_2(0) = \tau_{20}$$

It then follows from Eq. (25) that

$$\tau_{20} = (1/2)\{(a/\lambda)^2 - (\omega/\lambda)^2 - 2(a/\lambda)(\omega/\lambda)\} - \tilde{k}(\omega/\lambda) \dots (30)$$

Eq. (25) may be also rewritten by using τ^* defined in the preceding section as

$$\tau_2(\zeta) = \tau_2^*(\zeta) + (1/2)\{(a/\lambda)^2 - (\omega/\lambda)^2 - 2(a/\lambda)(\omega/\lambda)\}G - \tilde{k}(\omega/\lambda)G \dots (31)$$

Eliminating $\tilde{k}$ from both Eqs. (30) and (31), we get

$$\tau_2(\zeta) = \tau_2^*(\zeta) + \tau_{20}G(\zeta) \dots (32)$$

4.3 Thermally insulated surface

The boundary condition is

$$\zeta=0, d\tau_2/d\zeta=0 \dots (33)$$

It then follows from Eq. (25) that

$$\tilde{k} = -\{(\omega/\lambda) + (a/\lambda)\} \dots (34)$$

Let us describe the solution τ_2 , which gives the temperature profiles near a disk surface insulated thermally, by $\tau_2^{**}(\zeta)$. Substituting Eq. (34) into Eq. (25), we have

$$\tau_2^{**}(\zeta) = (\omega/\lambda)^2 G - (1/2)\{(\omega/\lambda)^2 G^2 + F^2 - (a/\lambda)^2\} \dots (35)$$

It is clear from this equation that in the case with $P_r=1$, the insulated surface temperature has a constant value, i.e. $\tau_2^{**}(0)=0.5$, regardless of the value a/ω .

5. Numerical exact solution ($P_r \neq 1$)

5.1 Isothermal surface

From Eqs. (7) and (8), the dimensionless temperature profile may be expressed as

$$(t-t_\infty)/(t_w-t_\infty) = A_0(z)/a_0 + r^2 A_2(z)/a_0$$

Rearranging this equation by Eq. (10), we get

$$(t-t_\infty)/(t_w-t_\infty) = \tau_0(\zeta)/\tau_{00} + (\mu\lambda^2 r^2 / ka_0) \tau_2^*(\zeta) = \tau_0(\zeta)/\tau_{00} + P_r E_{cr} \tau_2^*(\zeta) \dots (36)$$

where E_{cr} is a local Eckert number of a rotating disk with forced flow and defined by

$$E_{cr} = r^2(a^2 + \omega^2)/gc(t_w - t_\infty) \dots (37)$$

Now, the formal solution for $\tau_0(\zeta)$ is given by

$$\tau_0(\zeta) = \tau_{00} \left[1 - \int_0^\zeta \exp\left(\int_0^\zeta HP_r d\zeta\right) d\zeta / \int_0^\infty \exp\left(\int_0^\zeta HP_r d\zeta\right) d\zeta \right] \dots (38)$$

On the other hand, the solution $\tau_2^*(\zeta)$ must be numerically integrated under the following boundary condition

$$\tau_2^*(0)=0, \tau_2^*(\infty)=0 \dots (39)$$

The numerical calculation, applying Runge-Kutta's integral technique, was performed by means of HIPAC 103 electronic computer.

Now, according to Eq. (36), the temperature profile can be described by the dissipative term plus the usual heat transfer term not related with disk radius. The variations of $\tau_2^*(\zeta)$ by the Prandtl number and velocity ratio (a/ω) are shown in Fig. 2. It must be noted that at $P_r \leq 1$, the maximum value of τ_2^* in the case with $a/\omega=0$ is larger than the one in the case with $a/\omega=\infty$, but at $P_r > 1$, this relation is reversed.

Representative examples indicating the effect of axial flow on the temperature profile are shown in

Fig. 2 Variation of solution τ_2^* that indicates the effect of dissipative energy by Prandtl number

Figs. 3 (a) and (b), in which $E_{cr}=0.4$ for $P_r=1, 5$ and $E_{cr}=0.2$ for $P_r=10, 100$ are selected as the parameter. In addition, the case with $E_{cr}=0$ is plotted for convenience of comparison and expressed by the broken curves. As shown by Eq. (36), the dissipative effect depends on the same dimensionless product ($P_r E_{cr}$) as the Couette flow. In Couette flow, τ_2^* is independent of the Prandtl number, while, in rotating disk, τ_2^* depends on the Prandtl number. Owing to this conclusive fact, the dimensionless temperature profiles in the cases with $P_r=5, E_{cr}=0.4$ and with $P_r=10, E_{cr}$

Fig. 3 Influence of axial flow velocity on temperature profile (isothermal surface)

$=0.2$ are mutually dissimilar, even if the value ($P_r E_{cr}$) is equally $P_r E_{cr}=2$ in each case.

A typical example clarifying the effect of the rotating Eckert number on the temperature profiles is shown in Fig. 4 for $P_r=10, a/\omega=1$. It is noted in Fig. 4 that at $P_r E_{cr}=6$, the heat flows from the fluid side to a disk surface, since the temperature level in the boundary-layer due to the dissipative energy is higher than the one at a disk surface.

5.2 Pure quadratic surface temperature

As the differential equation Eq. (15)' is linear, the solution is given by the solution $\tau_2^*(\zeta)$, which was obtained in the preceding section, plus a new solution $\bar{\tau}_2^*(\zeta)$, i.e.

$$\tau_2(\zeta) = \tau_2^*(\zeta) + \bar{\tau}_2^*(\zeta) \quad (40)$$

Substituting Eq. (40) into Eq. (15)', we obtain a new differential equation

$$\bar{\tau}_2^{*''} - P_r(H\bar{\tau}_2^{*'} + 2F\bar{\tau}_2^*) = 0 \quad (41)$$

This equation satisfies the following boundary condition

$$\bar{\tau}_2^*(0) = \tau_{20}, \quad \bar{\tau}_2^*(\infty) = 0 \quad (42)$$

The solution $\bar{\tau}_2^*(\zeta)$ is similar in character to the solution $\tau_0(\zeta)$ in the preceding section, and expresses a usual heat transfer term without the dissipative effect, when the surface temperature is a pure quadratic distribution, i.e. $T_w(0, r) = a_2 r^2$.

Putting $P_r=1$ in Eq. (41) and comparing with Eq. (13), we have a relation $\bar{\tau}_2^*/\tau_{20} = G$, which equals to Eq. (32). This result means that Reynolds analogy between shearing stress and heat transfer is valid, regardless of the value a/ω for the case of a rotating disk having a quadratic surface temperature.

Finally, it is easy to see that the dimensionless temperature profile can be computed by

$$\begin{aligned} (t-t_\infty)/(t_w-t_\infty) &= A_2(z)/a_2 \\ &= (\mu\lambda^2/ka_2)\tau_2^*(\zeta) + \bar{\tau}_2^*(\zeta)/\tau_{20} \\ &= P_r E_{cr} \tau_2^*(\zeta) + \bar{\tau}_2^*(\zeta)/\tau_{20} \quad (43) \end{aligned}$$

Fig. 4 Influence of Eckert number on temperature profile (isothermal surface)

5.3 General quadratic surface temperature

The solution for a general quadratic surface temperature, which includes an isothermal or pure quadratic surface as a special case, can be obtained by combining the solutions for the special cases

$$t - t_\infty = A_0(z) + r^2 A_2(z) \\ = (\mu\nu\lambda/k)\tau_0(\zeta) + (\mu\lambda^2 r^2/k)\tau_2(\zeta)$$

Substituting Eq. (40) for $\tau_2(\zeta)$ into the above equation and rearranging by the relations, $a_0 = (\mu\nu\lambda/k)\tau_{00}$, $a_2 = (\mu\lambda^2/k)\tau_{20}$, we have

$$t - t_\infty = a_0(\tau_0/\tau_{00}) + a_2 r^2 (\bar{\tau}_2^*/\tau_{20}) \\ + P_r E_{cr} (t_w - t_\infty) \tau_2^*$$

Thus, it is easy to indicate that the dimensionless temperature profile may be calculated by

$$\frac{t - t_\infty}{t_w - t_\infty} = \frac{a_0}{t_w - t_\infty} \left\{ \frac{\tau_0(\zeta)}{\tau_{00}} - \frac{\bar{\tau}_2^*(\zeta)}{\tau_{20}} \right\} \\ + \frac{\bar{\tau}_2^*(\zeta)}{\tau_{20}} + P_r E_{cr} \tau_2^*(\zeta) \dots \dots \dots (44)$$

Particularly, by putting $a_0/(t_w - t_\infty) = 1$, Eq. (44)

(a) Eckert number $E_{cr} = 0$

(b) Eckert number $E_{cr} = 0.1$

Fig. 5 Influence of parameter in surface temperature, i.e. $a_0/(t_w - t_\infty)$ on temperature profile

reduces to Eq. (16) in the isothermal case and similarly, by putting $a_0/(t_w - t_\infty) = 0$, the one reduces to Eq. (43) in the pure quadratic case. Therefore, the value $a_0/(t_w - t_\infty)$ changes generally in the interval $0 \leq a_0/(t_w - t_\infty) \leq 1$. But, if we consider the case where the disk temperature decreases with the disk radius, the value $a_0/(t_w - t_\infty)$ changes in the interval $1 \leq a_0/(t_w - t_\infty) < \infty$. The temperature profiles calculated for the various values of $a_0/(t_w - t_\infty)$ are shown in Figs. 5 (a) and (b). In these figures, the solid line corresponds to the case with $P_r = 20$ and the broken line to $P_r = 50$. There is

Fig. 6 Variation of solution $\tau_2^{**}(\zeta)$ near a thermally insulated disk by both Prandtl number and axial flow velocity

Fig. 7 Influence of Prandtl number on temperature profile near a thermally insulated disk

the heat flow toward a disk from a fluid side, when the parameter $a_0/(t_w - t_\infty)$ is larger than unity.

5.4 Thermally insulated surface

The solution for $P_r \neq 1$ can be obtained by numerically integrating Eq. (15)' with the boundary condition Eq. (33). The distributions of $\tau_2^{**}(\zeta)$ for the cases with $a/\omega = 0, \infty$ are shown in Fig. 6 for the representative Prandtl numbers, i.e. $P_r = 1, 10, 100$. The value $\tau_2^{**}(0)$ for $P_r > 1$ increases with an increase of the value a/ω , but at $P_r = 1$, $\tau_2^{**}(0)$ is constant regardless of the value a/ω .

Denoting the surface temperature at an ad-

iabatic disk to be attained due to the frictional heating by τ_{aw} , we get from Eqs. (8) and (10)

$$\frac{t_{aw} - t_\infty}{(1/2)r^2(a^2 + \omega^2)/gc} = 2P_r \tau_2^{**}(0) \dots \dots \dots (45)$$

Normalizing Eq. (8) by the above equation, we get

$$(t - t_\infty)/(t_{aw} - t_\infty) = \tau_2^{**}(\zeta)/\tau_2^{**}(0) \dots \dots \dots (46)$$

This equation gives the temperature profile near an adiabatic disk. The numerical results for $\tau_2^{**}(0)$ are summarized in Table 1. Fig. 7 shows the temperature profiles for a representative velocity ratio $a/\omega = 0.5$.

Table 1 $\tau_2^{**}(0)$

a/ω	P_r							
	0.7	1.0	2.0	5.0	10	20	50	100
0	0.633	0.500	0.322	0.183	0.119	0.0779	0.0439	0.0284
0.1	0.619	0.500	0.330	0.190	0.125	0.0818	0.0462	0.0298
0.25	0.609	0.500	0.339	0.203	0.133	0.0877	0.0499	0.0323
0.5	0.602	0.500	0.346	0.209	0.141	0.0939	0.0540	0.0352
1.0	0.599	0.500	0.350	0.214	0.146	0.0980	0.0569	0.0373
2.0	0.597	0.500	0.351	0.216	0.148	0.0995	0.0581	0.0382
∞	0.597	0.500	0.352	0.217	0.148	0.1001	0.0585	0.0385

Table 2

(a) $-\tau_0'(0)/\tau_{00}$

a/ω	P_r							
	0.7	1.0	2.0	5.0	10	20	50	100
0	0.323	0.396	0.565	0.853	1.134	1.485	2.091	2.687
0.1	0.356	0.424	0.586	0.869	1.149	1.501	2.108	2.707
0.25	0.408	0.476	0.638	0.924	1.209	1.567	2.189	2.803
0.5	0.483	0.558	0.733	1.038	1.341	1.724	2.387	3.043
1.0	0.573	0.658	0.856	1.193	1.535	1.961	2.679	3.424
2.0	0.633	0.726	0.941	1.311	1.675	2.134	2.928	3.711
∞	0.665	0.762	0.986	1.372	1.752	2.230	3.055	3.870

(b) $\tau_2^{*'}(0)$

a/ω	P_r							
	0.7	1.0	2.0	5.0	10	20	50	100
0	0.326	0.308	0.272	0.226	0.194	0.163	0.129	0.106
0.1	0.338	0.321	0.286	0.239	0.205	0.173	0.136	0.113
0.25	0.367	0.350	0.314	0.264	0.228	0.193	0.152	0.126
0.5	0.419	0.400	0.360	0.306	0.265	0.226	0.179	0.148
1.0	0.489	0.467	0.421	0.358	0.311	0.266	0.212	0.176
2.0	0.537	0.513	0.463	0.394	0.343	0.294	0.234	0.195
∞	0.563	0.537	0.485	0.414	0.360	0.308	0.246	0.205

(c) $-\tau_2^{*'}(0)/\tau_{20}$

a/ω	P_r							
	0.7	1.0	2.0	5.0	10	20	50	100
0	0.515	0.616	0.847	1.238	1.621	2.100	2.927	3.742
0.1	0.545	0.641	0.866	1.255	1.637	2.118	2.949	3.768
0.25	0.603	0.699	0.925	1.319	1.709	2.201	3.052	3.893
0.5	0.697	0.800	1.042	1.462	1.878	2.403	3.311	4.209
1.0	0.817	0.933	1.204	1.672	2.134	2.717	3.725	4.720
2.0	0.899	1.025	1.318	1.824	2.322	2.949	4.035	5.107
∞	0.943	1.075	1.380	1.907	2.426	3.079	4.208	5.323

6. Heat transfer coefficient

The local heat transfer coefficient is defined by

$$q_r = \alpha_r (t_w - t_\infty) \quad (47)$$

where q_r is the local surface heat flux. Also, the local Nusselt number is defined by

$$N_{ur} = \frac{q_r}{t_w - t_\infty} \frac{r}{k} = - \frac{r}{t_w - t_\infty} \left(\frac{\partial t}{\partial z} \right)_{z=0} \quad (48)$$

Substituting Eq. (44) into Eq. (48), we get

$$\frac{N_{ur}}{\sqrt{R_{er}}} = \frac{a_0}{t_w - t_\infty} \left\{ \frac{\bar{\tau}_2^{*'}(0)}{\tau_{20}} - \frac{\tau_0^{*'}(0)}{\tau_{00}} \right\} - \frac{\bar{\tau}_2^{*'}(0)}{\tau_{20}} - P_r E_{cr} \tau_2^{*'}(0) \quad (49)$$

where R_{er} is a local Reynolds number for a rotating disk with the forced flow and defined by

$$R_{er} = r^2 (a^2 + \omega^2)^{1/2} / \nu \quad (50)$$

The values of $\tau_0^{*'}(0)$, $\tau_2^{*'}(0)$ and $\bar{\tau}_2^{*'}(0)$ for $P_r=1$ are respectively sought from Eqs. (28), (29) and (32), that is

$$\left. \begin{aligned} \tau_0^{*'}(0)/\tau_{00} &= - \left[\int_0^\infty \exp \left(\int_0^\zeta H d\zeta \right) d\zeta \right]^{-1} \\ \tau_2^{*'}(0) &= -G'(0)/2 \\ \bar{\tau}_2^{*'}(0)/\tau_{20} &= G'(0) \end{aligned} \right\} \quad (51)$$

On the other hand, these values for $P_r \neq 1$ can be sought from the numerical solutions and summarized at Table 2 (a), (b) and (c).

Eq. (49) is a general expression for the heat transfer coefficient on the disk, rotating at a constant angular velocity and emerging in uniform stream of incompressible fluid. Especially, for a disk rotating in a quiescent fluid with $P_r=1$, we obtain

$$G(0) = -0.616, \quad \left[\int_0^\infty \exp \left(\int_0^\zeta H d\zeta \right) d\zeta \right]^{-1} = -0.396$$

from Table 2. Then, Eq. (49) reduces to Dorfman's expression⁽¹²⁾ as follows:

Fig. 8 Relation between E_{cr} and a/w when surface heat transfer is zero

$$\frac{N_{ur}}{\sqrt{R_{er}}} \Big|_{a/w=0} = 0.616 - 0.220 \frac{a_0}{t_w - t_\infty} - 0.308 E_{cr}$$

This result for gas with $P_r=1$ holds exactly, since the heating effect by compressive work ($u \partial p / \partial r$) is negligible in the case $a/w=0$. It is clear from Eq. (49) that in the non-dissipative case, the surface heat transfer becomes zero at $a_0/(t_w - t_\infty) = 2.8$ and the heat flow toward a disk may be realized at $a_0/(t_w - t_\infty) > 2.8$ [see Fig. 5 (a), (b)]. On the other hand, the parameter $a_0/(t_w - t_\infty)$ varies between 0 and 1 in the usual case where the surface temperature increases with the disk radius, but also the heat flow toward a disk may be realized at large values of the Eckert number (see Fig. 4). Consequently, it is interesting to make clear the limiting condition, at which the surface heat transfer becomes zero. The Eckert number at such a condition is named the critical Eckert number and denoted by E_{cr}^* .

The relation between E_{cr}^* and a/w computed from Eq. (49) is shown in Fig. 8, in which the Prandtl number ($P_r > 2$) is used as the parameter.

Fig. 9 Influence of axial flow velocity on local Nusselt number

Since the influence of compressive work ($u\partial p/\partial r$) appears for gases in the presence of the axial flow when E_{cr} becomes large, the plot for gas will be restricted in small interval of a/ω . It is apparent from Fig. 8 that the critical Eckert number for the case of isothermal surface is larger than the one for the case of pure quadratic surface. In other words, the relative effect of dissipative term on temperature profiles is more remarkable in the case of isothermal surface than in the case of non-isothermal surface.

Next, let us consider the influence of axial forced flow on the Nusselt number. The representative example is shown in Fig. 9, in which $N_{ur}|_{a=0}$ denotes the local Nusselt number for a disk rotating in the fluid at rest. It is interesting that the local Nusselt number ratio, $N_{ur}/N_{ur}|_{a=0}$ becomes smaller than unity in some interval of a/ω . This fact can be explained by the fact that the rate of decrease in the surface temperature gradient due to the dissipative effect depends on both the velocity ratio a/ω and the Prandtl number. For instance, let us consider the case with $E_{cr}P_r=5$ that may be realized in two cases, i.e. $P_r=100$, $E_{cr}=0.05$ and $P_r=20$, $E_{cr}=0.25$. In the former, $N_{ur}/N_{ur}|_{a=0}<1$ can be attained in the interval $0<a/\omega\leq 0.2$, while in the latter in the interval $0<a/\omega\leq 2$.

It is also apparent from Fig. 9 that at $E_{cr}=0$, the Nusselt number ratio hardly increases at very small value a/ω , but the increment rate of the Nusselt number ratio becomes remarkable with an increase of the value a/ω , but tends to a saturated state. Furthermore, the influence of the surface temperature distribution on the Nusselt number

ratio is not always large in the case $E_{cr}=0$, but such an influence becomes remarkable, especially at high Prandtl number in the presence of the dissipative effect. Looking at the curves for $P_r=100$, $E_{cr}=0.1$ and $P_r=20$, $E_{cr}=0.25$ in Fig. 9, we can understand the above fact.

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