An Analysis of a Multi-Degree-of-Freedom System
by a Dividing Method*

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values of \( X_i \) and \( F_{ij} \) respectively.

For instance, by substituting in some way the presumed connecting forces, \( F_{ij}^{(0)} \)'s, into the first equations of \( (3) \), the first solutions, \( X_i^{(1)} \)'s, of the partial systems are obtained. Then, by substituting these solutions into the second, \( F_{ij}^{(1)} \)'s are obtained. This process is repeated. Of course, we may start by assuming the solutions, \( X_i^{(1)} \)'s.

Though all that has been described above is applicable to general systems, in order to advance our theory we shall consider small vibrations of a linear system about its configuration of a stable equilibrium. Let us assume that all the coordinates, external forces and connecting forces are vibrating with a circular frequency, \( \omega \), and that, unless especially needed, these variables represent only their complex amplitudes.

In general, taking linear restoring forces and viscous resistances of the system into consideration, the equation of motion of the system \( i \) can be expressed in matrix notation as:

\[
M_i X_i^{(0)} = F_i + \sum_{j \neq i} F_{ij}^{(0)} \tag{4}
\]

where \( M_i \) is an \( n_i \times n_i \) matrix the elements of which are polynomials of the second degree in \( \omega \) (\( \omega = \sqrt{-1} \)), and where the connecting force, \( F_{ij} \), can be expressed as the sum of the two parts in the form:

\[
F_{ij} = L_{ij1} X_i^{(0)} + L_{ij2} X_j^{(0)} \tag{5}
\]

where \( L_{ij1} \) is an \( n_i \times n_j \) matrix and \( L_{ij2} \) is an \( n_i \times n_j \) matrix. The first term or the second term on the right-hand side is that part of the connecting force, \( F_{ij} \), which is uniquely determined by the motion of the system \( i \) or the system \( j \). From Eqs. (4) and (5), the following equations, corresponding to Eq. (3), for successive computation can be introduced:

\[
\begin{align*}
M_i X_i^{(0)} &= F_i + \sum_{j \neq i} F_{ij}^{(0)} \\
F_{ij}^{(0)} &= L_{ij1} X_i^{(0)} + L_{ij2} X_j^{(0)} \\
& \quad \text{(i=1, 2, ..., r,)} \\
F_{ij}^{(0)} &= L_{ij1} X_i^{(0)} + L_{ij2} X_j^{(0)} \tag{6}
\end{align*}
\]

The iteration process shown by Eq. (6) is based on the method, explained above, by which the solutions of the partial systems and the connecting forces are alternately computed. We can also contrive, in the following way, a more general dividing method which includes this as a special case.

First, with regard to the system \( i \), as the first approximation, we shall decide whether to ignore the influence of a connection with the system \( j \), that is, whether to regard \( F_{ij} \) as equal to zero, or to take the influence into consideration assuming that the system \( j \) is at a standstill. After deciding in this manner the first approximate situation of every system connected with the system \( i \), we shall call the system the first approximate system of the system \( i \). Let \( \sum_i \) and \( \sum_f \) denote summing-ups with the system \( j \) regarded as being at a standstill and with the system \( j \) ignored respectively. Then, \( \sum_i L_{ij1} X_i \), which is that part of the total connecting force, \( \sum_i F_{ij} \), acting on the system \( i \) which is determined by the motion of the system \( i \) itself, can be split into two parts:

\[
\sum_i L_{ij1} X_i = M_{ic} X_i + L_{ic} X_i, \quad M_{ic} = \sum_i L_{ij1}, \quad L_{ic} = \sum_i L_{ij2}
\]

Thus we obtain the following equation of motion of the first approximate system of the system \( i \):

\[
(M_i - M_{ic}) X_i = F_i \tag{7}
\]

For instance, in the case of Fig. 1, where the overall system is divided into three partial systems and where the first approximate system of each partial system is decided to be as shown in Fig. 1, with regard to System 1, Systems 2 and 3 are concerned with \( \sum_f \); with regard to System 2, System 1 is concerned with \( \sum_i \) and System 3, with \( \sum_f \); and with regard to System 3, Systems 1 and 2 are concerned with \( \sum_f \).

From the exact equation of motion of the system \( i \) of the form:

\[
(M_i - M_{ic}) X_i = F_i + L_{ic} X_i + \sum_{j \neq i} L_{ij2} X_j \tag{8}
\]

the following equations for successive computation can be introduced:

\[
\begin{align*}
(M_i - M_{ic}) X_i^{(0)} &= F_i + L_{ic} X_i^{(0)} + \sum_{j \neq i} L_{ij2} X_j^{(0)} \\
+ \sum_{j \neq i} L_{ij2} X_j^{(0)} (i=1, 2, ..., r) \tag{9}
\end{align*}
\]

Rewriting these equations in the form:

\[
\begin{align*}
(M_i - M_{ic}) X_i^{(0)} &= F_i + \sum_f F_{ij}^{(0)} \\
+ \sum_f L_{ij2} X_j^{(0)} (i=1, 2, ..., r) \tag{10}
\end{align*}
\]

we can say that the second terms on the right-hand side of Eq. (11) serve to correct the connecting forces, regarded as zeros in the first approximation, and that the third terms serve to correct the coordinates of the systems regarded as being at a standstill in the first approximation.

In this manner, successive computations follow.
a form according to which the solution of the first approximate system of each partial system is first computed, after which the boundary conditions of each partial system given in the form of connecting forces from other systems or in the form of coordinates of other systems are modified. If all partial systems are correlated with each other, it is evident, from the way of deciding the first approximate systems, that in all there exist \(2^{r-1}\) kinds of equations for successive computation. For instance, if the overall system is divided into two partial systems, we obtain four kinds of equations, and if it is divided into three partial systems, we obtain 64 kinds of equations.

3. Conditions necessary for the successive computation to converge

This chapter is devoted to a study of the conditions necessary for the successive computation to converge.

First, transforming Eq. (9), we have:
\[
X_i = (M_i - M_{el})^{-1} F_i + (M_i - M_{el})^{-1} (L_{ie} X_i + \sum_{j \neq i} L_{ij} X_j)
\]

(We except the frequencies, if any, at which \(|M_i - M_{el}| = 0\).)

By putting:
\[
X = \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix}, \quad A = \begin{pmatrix} (M_1 - M_{el})^{-1} F_1 \\ \vdots \\ (M_r - M_{el})^{-1} F_r \end{pmatrix}, \\
B = \begin{pmatrix} (M_1 - M_{el})^{-1} (L_{1i} L_{2i} L_{3i} \cdots L_{ri}) \\ \vdots \\ (M_r - M_{el})^{-1} (L_{1i} L_{2i} L_{3i} \cdots L_{ri}) \end{pmatrix}
\]

we have:
\[
X = A + BX
\]

(14)

Hence, denoting the exact solution of the overall system and the unit matrix by \(X^e\) and \(E\) respectively, we obtain:
\[
X^e = (E - B)^{-1} A
\]

(15)

(We except the frequencies, if any, at which \(|E - B| = 0\).)

The equations for successive computation (10) are identical with:
\[
X^{(b)} = A + BX^{(b-1)}
\]

(16)

which is introduced from Eq. (14).

If, and only if, there exists no viscous resistance in the first approximate system of the system \(i\), there exist frequencies at which \(|M_i - M_{el}| = 0\).

In such a case, when, and only when, the given forcing frequency is equal to one of the natural frequencies of the first approximate system of the system \(i\), \(|M_i - M_{el}|\) becomes zero and so this equation for successive computation can not be applied.

If there exist frequencies at which \(|E - B| = 0\), they are equal to the natural frequencies of the overall system in which there exists no viscous resistance. We now except these special frequencies at which there originally exists no stationary solution of the overall system.

The successive computation necessarily converges towards the exact solution if it ever converges. This follows from the fact that the solution of Eq. (14) is uniquely given by Eq. (15).

From Eq. (16), \(X^{(b)}\) can be expressed by \(X^e\) and \(X^{(1)}\) as follows:
\[
X^{(b)} = (E + B + B^2 + \cdots + B^{b-1}) A + B^{b-1} X^{(1)}
\]

\[
= X^e + B^{b-1} (X^{(1)} - X^e)
\]

(17)

Consequently, we obtain the following proposition by virtue of the theory of matrix:

"The necessary and sufficient condition for the successive computation to converge without distinction of the first given solution is that the absolute value of every eigenvalue of \(B\) be smaller than unity."

Even if some absolute values of eigenvalues of \(B\) are not smaller than unity, as long as the first solution is suitably given, the successive computation converges. Let \(\lambda_1, \ldots, \lambda_s\) denote the eigenvalues of \(B\) and \(\tau_1, \ldots, \tau_s\) the algebraic multiplicity of \(\lambda_i\). By putting Ker\((B - \lambda_i E)\tau_i \subset V_{i0}\), the \(n\)-dimensional vector space, \(V_i\), can be decomposed into a direct sum of the \(V_{i0} = \bigoplus_{\tau = 1}^{\infty} \bigoplus_{\lambda \tau} V_{i\tau}\). Now, putting Ker\((B - \lambda_i E) = V_{i0}\), \(V_{i0} = V_{i0} \oplus W_i\), where \(V_{i0}\) is the eigenspace corresponding to the eigenvalue, \(\lambda_i\), we can determine a subspace, \(W_i\), with which \(V_i = V_{i0} \oplus W_i\) and dim \(W_i = \tau_i\). Finally, we obtain a direct sum decomposition of \(V\) of the form \(V = V_{i0} \oplus \bigoplus_{i=1}^{s} W_{i0} \oplus \bigoplus_{i=1}^{s} W_{i}\). Then, corresponding to \(X^e, v_i \in V_{i0}\) and \(w_i \in W_i\) can be uniquely determined, so that we have the expression:
\[
X^e = \sum_{i=1}^{s} v_i + \sum_{i=1}^{s} w_i
\]

Suppose that \(|\lambda_i| \geq 1, \quad 1 \leq i \leq s\).

If we take \(X^{(1)}\) in the form:
\[
X^{(1)} = \sum_{i=1}^{s} v_i + \sum_{i=1}^{s} w_i \quad u_i \in V_{i0}
\]

\[
||B^k (X^{(1)} - X^e)|| = ||B^k \sum_{i=1}^{s} (u_i - v_i)||
\]

\[
= ||\sum_{i=1}^{s} \lambda_i^k (u_i - v_i)|| \leq \sum_{i=1}^{s} ||\lambda_i||^k ||u_i - v_i|| 
\]

(\(k \to \infty\))

Therefore, naming \(X^{(1)} - X^e\) the error vector, we obtain the following proposition:
"Even if there exist some eigenvalues of \( B \) the absolute values of which are not smaller than unity, if the first solution is given is such a way that the error vector is in the sum space of eigenspaces corresponding to eigenvalues the absolute values of which are smaller than unity, the successive computation converges."

We shall now consider cases where every geometric multiplicity of the eigenvalue, \( \lambda_i \), is equal to one. A case where \( n \) eigenvalues are different from each other is included in these cases. We may say that under this restriction the approach loses little generality because of the fact that, in actual vibrating systems, it is quite unusual for there to exist an eigenvalue of \( B \) with an algebraic multiplicity larger than one.

By putting:

\[
V = \sum_{i=1}^{n} \lambda_i \nu_i, \quad X = \sum_{i=1}^{n} \nu_i, \quad X^{(1)} = \sum_{i=1}^{n} u_i;
\]

we have:

\[
B^{(X)} = \sum_{i=1}^{n} \lambda_i \nu_i
\]

Now that \( \lambda_i \nu_i \) and \( V \) are a direct factor of \( V \), \( B^{(X)} = 0 \) is equivalent to \( \lambda_i \nu_i = 0 \) with respect to every \( i \). We thus obtain the following proposition:

"If every geometric multiplicity of the eigenvalue of \( B \) is equal to one, the necessary and sufficient condition for the successive computation to converge is that the first solution be given in such a way that the error vector is in the sum space of eigenspaces corresponding to eigenvalues the absolute values of which are smaller than unity."

In actual problems, as the first solution we usually take the solutions of the first approximate

\[
B_1 = \begin{pmatrix}
M_1 - M_{1e} & 0 \\
M_2 - M_{2e} & M_2 - M_{2e} \\
0 & M_1 - M_{1e}
\end{pmatrix}^{-1},
\]

we have \( B = B_1 B_2 \). Therefore, using the relation (18), we can examine the absolute values of the eigenvalues of \( B \).

\[
B_1 B_1^* = \begin{pmatrix}
(M_1 - M_{1e})^*(M_1 - M_{1e}) & 0 \\
(M_2 - M_{2e})^*(M_2 - M_{2e}) & (M_1 - M_{1e})^*(M_2 - M_{2e})
\end{pmatrix}^{-1}
\]

Since \( (M_1 - M_{1e})^*(M_1 - M_{1e}) \) and \( (M_2 - M_{2e})^*(M_2 - M_{2e}) \) are positive definite Hermitian matrices, when we denote the smallest eigenvalue of the former by \( \alpha_{11} \) and the largest one of the latter by \( \beta_{11} \), \( \beta_{11} \) is equal to the largest among the \( \beta_{11} \) values; in other words, it is equal to the inverse number of the smallest among the \( \alpha_{11} \) values.

\[
\text{Max}_{\alpha_{kk}} |\alpha_{kk}| \leq |A_0| \leq m \text{Max}_{\alpha_{kk}} |\alpha_{kk}|, \quad |A_0|^* = \text{The largest eigenvalue of } A_0 A_0^*. \tag{21}
\]
From these two equations, it follows that:

The largest eigenvalue of $A_{k_{i}}A_{k_{j}}^{*} \leq (m_{\text{Max}}|a_{k_{i}k_{j}}|)^{2}$.................................(22)

Therefore, denoting the $(h, k)$ elements of $(M_{i} - M_{ke})^{-1}$ and $B_{k}$ by $(M_{i} - M_{ke})^{-1}_{h_{k}}$, and $(B_{k})_{h_{k}}$ respectively, we obtain:

$\hat{\beta}_{h_{k}} \leq (m_{\text{Max}}|a_{k_{i}k_{j}}|)^{2}$.................................(23)

After examining the magnitudes of $\hat{\beta}_{h_{k}}$ in this manner, if we find $\hat{\beta}_{1} \hat{\beta}_{2} < 1$, we can conclude that the successive computation converges.

Now, let us introduce normal coordinates of each first approximate system. By denoting the natural frequencies, the normal masses, the normal spring constants and the quantities representing viscous resistances of the first approximate system of the system $i$ by $\omega_{i_{i_{i}}}$, $m_{i_{i_{i}}}$, $k_{i_{i}}$ ($l=1, 2, \ldots, n_{i}$) and $c_{i_{i_{i}}}$, ($l_{i_{i}}, k_{i}=1, 2, \ldots, n_{i}$) respectively, $M_{i} - M_{ke}$ can be expressed in the following form:

$M_{i} - M_{ke} = \begin{pmatrix}
-m_{i_{i}}\omega_{i_{i}}^{2} + \text{jo} c_{i_{i}} & m_{i_{i}}\omega_{i_{i}}^{2} - \text{jo} c_{i_{i}} \\
\text{jo} c_{i_{i}} & \text{jo} c_{i_{i}}
\end{pmatrix}$.................................(24)

First, let us consider cases where no viscous resistance exists within the first approximate systems. Let $\lambda_{i_{l}}$ ($l=1, 2, \ldots, n_{i}$) denote the eigenvalues of $\lambda_{M_{i} - M_{ke}}^{-1}((M_{i} - M_{ke})^{-1})$; then we obtain:

$\lambda_{i_{l}} = \frac{1}{m_{i_{l}}\omega_{i_{l}}^{2} - \text{jo} c_{i_{l}}} \quad (l=1, 2, \ldots, n_{i})$.................................(25)

Since the elements of $B_{k}$ are polynomials of the first degree in $\text{jo}$, in general, from Eq. (23), we can find such positive numbers as $a$ and $b$, with which:

$\hat{\beta}_{h_{k}} \leq a^{2} + b^{2}$.................................(26)

If $a$ and $b$ increase, respectively, the coefficients of viscous resistances and the spring constants of the parts with which the partial systems are connected, which we shall briefly call the connecting parts, increase. From this, we obtain the following sufficient condition for the successive computation to converge:

$\frac{2a^{2}c_{i_{l}}^{2}}{(1 - \text{jo} c_{i_{l}})^{2}} + \frac{b^{2}}{k_{i_{l}}^{2}} < 1 \quad (l=1, 2, \ldots, n_{i}; i=1, 2, \ldots, r)$.................................(27)

where:

$a_{i_{l}} = \omega_{i_{l}}/\omega_{i_{l}}$, $c_{i_{l}} = 2\sqrt{m_{i_{l}}k_{i_{l}}}$

(1=1, 2, ..., n_{i}).................................(28)

We put $a=0$ when no viscous resistance exists in the connecting parts.

If extremely small viscous resistances exist within the first approximate system, approximately:

$\lambda_{i_{l}} = \frac{1}{k_{i_{l}}[(1 - \text{jo} c_{i_{l}})^{2} + (2c_{i_{l}}\text{jo} c_{i_{l}})^{2}]}$.................................(29)

where:

$c_{i_{l}} = c_{i_{l}i_{l}}/c_{i_{l}} \quad (l=1, 2, \ldots, n_{i})$.................................(30)

Therefore, we obtain the following approximate sufficient condition for the successive computation to converge:

$\frac{2a^{2}c_{i_{l}}^{2}}{c_{i_{l}}^{2}} + \frac{b^{2}}{k_{i_{l}}^{2}} < 1$

(1=1, 2, ..., n_{i}; i=1, 2, ..., r).................................(31)

where we put $a=0$ if no viscous resistance exists in the connecting parts.

From what we have presented so far, we may summarize the qualitative inclinations of the convergence characteristic as follows:

1. The successive computation usually converges, except in the frequency ranges of the neighborhoods of the natural frequencies of the first approximate systems; if the forcing frequency is larger enough than the largest natural frequencies of the first approximate systems, it converges without fail.

2. In the neighborhoods of the natural frequencies of the first approximate systems, the convergence is doubtful.

3. The smaller the coefficients of viscous resistances or the spring constants of the connecting parts, as compared with those within the first approximate systems, the better the convergence characteristic.

4. The convergence speed becomes larger as the forcing frequency is located farther from the natural frequencies of the first approximate systems.

The natural frequencies, $\omega_{i_{l}} \sim \omega_{e_{r}}$, of the overall system are given by the roots of the following algebraic equation in $\omega^{2}$:

$|E - B_{k}| = 0$.................................(32)

where $B_{k}$ is a matrix that is introduced from Eq. (13) by omitting the terms of the polynomials of the first degree in $\text{jo}$ from each element of $M_{i}$, $M_{ke}$, $L_{i_{l}}$ and $L_{i_{l}e}$. Though in extremely special cases some of the natural frequencies of the overall sys-
system may be equal to those of the first approximate systems, we shall except these cases, because we have already discussed the natural frequencies of the first approximate systems. In other words, the natural frequencies of the overall system are those at which unity is equal to an eigenvalue of $B$. It goes without saying that $\omega_1 \sim \omega_n$ do not depend on the way of dividing the overall system into the partial systems or the way of deciding the first approximate systems. Therefore, if there exists no viscous resistance, or if there exist extremely small ones in the system, regardless of the way of introducing the equations for successive computation, there exists an eigenvalue of $B$ close to unity in the neighborhoods of the natural frequencies of the overall system, so that the successive computation does not converge or the convergence speed is small if it does converge. By taking into account the paragraphs (1) $\sim$ (4) described above, we may sum up as follows:

(5) In order to make the frequency range in which the successive computation converges wider and in order to make the convergence speed larger, it is desirable to decide the way of dividing the overall system into partial systems and the way of deciding the first approximate systems so that the natural frequencies of the first approximate systems are located as close to the corresponding natural frequencies of the overall system as possible.

We shall now consider a case where the overall system is composed of two partial systems. In this case there exist four kinds of equations for successive computation. If the coefficients of viscous resistances and the spring constants of the connecting parts are small enough as compared with those within the first approximate systems, as has been explained above, every iteration process has a good characteristic of convergence. Also, if one of the two partial systems has a large inertia and the other has a small one, if with regard to the former the latter is ignored and with regard to the latter the former is regarded as being at a standstill, in the first approximation, a good iteration process can be introduced.

As a simple example, we shall take a spring-mass system with two degrees of freedom, as is shown in Fig. 2.

The equations of motion can be written in the form:

$$\begin{align*}
(1-\sigma^2+Rv^2)s_1 - Rv^2s_2 &= x_{11}, \\
-s_1 + (1-\sigma^2/R^2)x_2 &= x_{12},
\end{align*}$$

where:

$m_1$, $m_2$: mass, $k_1$, $k_2$: spring constant,

Table 1: Natural frequencies of the overall system and the first approximate systems

<table>
<thead>
<tr>
<th>Natural frequencies of the overall system</th>
<th>First order</th>
<th>Second order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{n1}$</td>
<td>0.95</td>
<td>10.49</td>
</tr>
<tr>
<td>System of $m_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Natural frequencies of the first approximate system</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_{n1}$</td>
<td>(a) 1.00</td>
<td>0</td>
</tr>
<tr>
<td>(b) 3.32</td>
<td>10.00</td>
<td></td>
</tr>
<tr>
<td>(c) 1.00</td>
<td>10.00</td>
<td></td>
</tr>
<tr>
<td>(d) 3.32</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Eigenvalues of $B$ and successive computations for $\sigma=15, \ x_{11}=0$, $x_{22}=1$, $X'=3.88 \times 10^{-2}$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>(a) 0.459</th>
<th>(b) 0.194</th>
<th>(c) -0.188</th>
<th>(d) -0.042</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^{(1)}$</td>
<td>$x_1 \times 10^{-4}$</td>
<td>$x_2 \times 10^{-3}$</td>
<td>$x_1 \times 10^{-4}$</td>
<td>$x_2 \times 10^{-3}$</td>
</tr>
<tr>
<td>$X^{(2)}$</td>
<td>2.95</td>
<td>-4.41</td>
<td>2.74</td>
<td>-8.00</td>
</tr>
<tr>
<td>$X^{(3)}$</td>
<td>3.42</td>
<td>-7.85</td>
<td>3.88</td>
<td>-8.29</td>
</tr>
<tr>
<td>$X^{(4)}$</td>
<td>3.82</td>
<td>-8.25</td>
<td>3.82</td>
<td>-8.25</td>
</tr>
</tbody>
</table>

($X^{(1)}$ are taken as solutions of the first approximate systems)
$z_1$, $z_2$: coordinate, $F_1$, $F_2$: external force, $R=m_1/m_1$, $\omega_1^2=K_1/m_1$, $\omega_2^2=K_2/m_2$, $\nu=\omega_2/\omega_1$, $x_{10}=F_1/K_1$, $x_{20}=F_2/K_2$, $\sigma=\omega_2/\omega_1$

We shall perform numerical computations, assuming that $R=0.1$ and $\nu=10$. By computing the natural frequencies, we see first that the natural frequencies of the first approximate systems are nearly equal to those of the overall system in the case of (c).

For $\sigma=15$, in every case the iteration converges. Judging from the magnitudes of $|\lambda|$, the cases (b) and (c) seem to have good characteristics of convergence; this is confirmed by an actual computation by putting $x_{10}=0$ and $x_{20}=1$.

For $\sigma=8$, the iteration converges in the cases of (b) and (c), but does not converge in the cases of (a) and (d). The case (c), where $|\lambda|$ is small, has a good characteristic of convergence.

For $\sigma=4$, the iteration converges only in the case of (c).

We may say that the inclinations of the above computations are in good agreement with the general explanation of the convergence characteristic.

5. The dynamic influence coefficient

The connecting force vector, $F_{ij}$, the components of which are generalized forces, can be expressed by the vector, $P_{ij}$, the components of which are components in the directions of the rectangular coordinates of the actual forces and couples acting mutually between the system $i$ and the system $j$, in the form:

$$ F_{ij}=N_{ij}P_{ij} \tag{34} $$

where $N_{ij}$ is an $n_s$, $s_j$ matrix provided that $P_{ij}$ is an $s_j$-dimensional column vector. $P_{ij}$ can also be expressed by an $s_i$, $s_i$, matrix, $T_{ij}$, and an $s_i$, $s_j$, matrix, $T_{ji}$, in the form:

$$ P_{ij}=T_{ij}X_i+T_{ij}X_j \tag{35} $$

where:

$$ N_{ij}T_{ij}=L_{ij}, \quad N_{ij}T_{ji}=L_{ij} \tag{36} $$

With regard to the system $j$ ignored in the first approximation, we solve the equation:

$$(M_i-M_{ij})X_i=N_{ij}P_{ij}; \quad P_{ij}=egin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{37}$$

in which only one component of $P_{ij}$ is taken to be equal to unity and in which all the other components are zeros. With regard to the system $j$ regarded as being at a standstill in the first approximation, we solve the equation:

$$(M_i-M_{ij})X_i=L_{ij}X_j; \quad X_j=egin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{38}$$

in which only one component of $X_j$ is taken to be equal to unity and in which all the other components are zeros. We shall define these solutions as the dynamic influence coefficients; in other words, the dynamic influence coefficients are the elements of $(M_i-M_{ij})^{-1}N_{ij}$ or $(M_i-M_{ij})^{-1}L_{ij}$.

We can transform the equations for successive computation (11) into the following form:

$$ X_i^{(n)}=(M_i-M_{ij})^{-1}F_i+\sum_{j=1}^{r}(M_i-M_{ij})^{-1}T_{ij}X_j^{(n-1)}$$

$$+\sum_{j=1}^{r}(M_i-M_{ij})^{-1}L_{ij}X_j^{(n-1)} \quad (i=1,2,\ldots,r),$$

$$ P_{ij}^{(n)}=T_{ij}X_i^{(n)}+T_{ij}X_j^{(n)} \quad (j|i; \quad i=1,2,\ldots,r) \tag{39} $$

From these expressions, it can be said that $(M_i-M_{ij})^{-1}N_{ij}P_{ij}^{(n-1)}$ and $(M_i-M_{ij})^{-1}L_{ij}X_j^{(n-1)}$ on the right-hand side of the first equations are linear combinations of vectors the components of which are the dynamic influence coefficients, where the components of $P_{ij}^{(n-1)}$ and $X_j^{(n-1)}$ are the respective coefficients in these linear combinations.

According to this dividing method, if some physical constant of the system is changed, a large

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6917</td>
<td>$\pm 1.543j$</td>
<td>0.233 $\pm 0.826j$</td>
<td>$0.6917$</td>
<td>$7.845$</td>
</tr>
</tbody>
</table>

Table 4 Eigenvalues of $B$ for $\sigma=4$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.724j</td>
<td>$\pm 0.869j$</td>
<td>$0.690 \pm 0.869j$</td>
<td>$0.690 \pm 0.869j$</td>
<td>$1.82$</td>
</tr>
</tbody>
</table>

Table 3 Eigenvalues of $B$ and successive computations for $\sigma=8$, $x_{10}=0$, $x_{20}=1$, $X^{(1)}=-3.44 \times 10^{-1}$

( $X^{(1)}$ are taken equal to the solutions of the first approximate systems )

<table>
<thead>
<tr>
<th>$X^{(1)}$</th>
<th>$X^{(1)}$</th>
<th>$X^{(2)}$</th>
<th>$X^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>$0.10^{-1}$</td>
<td>$2.78$</td>
<td>$0.10^{-1}$</td>
<td>$2.78$</td>
</tr>
<tr>
<td>$-3.43$</td>
<td>$1.84$</td>
<td>$-3.43$</td>
<td>$1.83$</td>
</tr>
<tr>
<td>$-3.45$</td>
<td>$1.83$</td>
<td>$-3.45$</td>
<td>$1.83$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$X^{(1)}$</th>
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<th>$X^{(2)}$</th>
<th>$X^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>$0.10^{-1}$</td>
<td>$-1.56$</td>
<td>$0 \times 10^{-1}$</td>
<td>$-1.56$</td>
</tr>
<tr>
<td>$2.48$</td>
<td>$-4.01$</td>
<td>$2.95$</td>
<td>$-4.01$</td>
</tr>
<tr>
<td>$6.75$</td>
<td>$-8.21$</td>
<td>$7.66$</td>
<td>$-8.29$</td>
</tr>
</tbody>
</table>
portion of the equations for successive computation remains unchanged, while only a small portion of them, that is concerned with that constant directly, is changed. Therefore, this method can be effectively utilized for investigating the influence of the constant on the solutions.

6. A dividing method for a case where connections of the partial systems are attended with the equations of constraint containing coordinates

In the cases where connections of the partial systems are attended with the constraints containing coordinates, such as when a spring constant with which two partial systems are connected is infinity or when two rigid bodies are connected with a spherical joint, as forces and couples acting among the partial systems, we must take into account the constraining forces (forces and couples) in addition to those determined by the motions of the partial systems. Let \( n_{ij} \) and \( F_{0ij} \) denote, respectively, the number of constraints attendant upon the connection of the system \( i \) with the system \( j \) and a constraining force vector the components of which are generalized forces acting on the system \( i \) from the system \( j \). Suppose that the equations of motion of the system \( i \) are expressed in the following form:

\[
\begin{align*}
& \mathbf{f}_i(X_i, f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n, f_0, \ldots), \\
& \mathbf{f}_{0i}(X_i, f_i) = 0 \quad (i = 1, 2, \ldots, n), \\
& \mathbf{f}_{ij}(X_i, X_j) = 0 \quad (h = 1, 2, \ldots, n_{ij} ; j \neq i), \\
& \mathbf{f}_i = \mathbf{\psi}_i(X_i, X_j) \quad (j \neq i)
\end{align*}
\]

where the second equations stand for the constraining conditions.

The first approximate system of each partial system can be decided in the same manner as in Chapter 2. In the first equations of (40), by putting \( f_i = 0 \) and \( f_{0j} = 0 \) with respect to the system \( j \) ignored in the first approximation, and in the second and third, by putting \( X_j = 0 \) with respect to the system \( j \) regarded as being at a standstill in the first approximation, the equations of the first approximate system of the system \( i \) are obtained.

In general, from these equations, the solutions of the first approximate systems can be obtained provided that the number of the constraining forces (the number of independent scalars) acting between the system \( i \) and the system \( j \) is equal to \( n_{ij} \).

The equations for the successive computation of the solution can be introduced from Eq. (40), if the way of dividing the overall system into the partial systems and the way of deciding the first approximate system of each partial system satisfy some further conditions for suitability. For instance, (1) with regard to the system \( i \), if more than one system exists the connections with which are attended with the constraints containing coordinates, the number of the system regarded as being at a standstill in the first approximation must be one or zero (if one, we denote such a system \( j \) as \( j_0 \) concerning the systems of that kind, (2) if the constraints containing coordinates are attendant upon the connection of the system \( i \) with the system \( j \), when with regard to the former, the latter is regarded as being at a standstill in the first approximation, with regard to the latter, the former is ignored in the first approximation. If these conditions are satisfied, the following equations for successive computation may generally be introduced:

\[
\begin{align*}
& \mathbf{f}_i(X_i^{(0)}, f_1, f_2, \ldots, f_{i-1}^{(a-1)}, f_{i+1}^{(a-1)}, \ldots), \\
& \mathbf{f}_{0i}(f_1, f_{0i}^{(a-1)}, \ldots), \\
& \mathbf{f}_{ij}(X_i^{(a)}, X_j^{(a-1)}) = 0 \quad (h = 1, 2, \ldots, n_{ij} ; i = 1, 2, \ldots, r), \\
& \mathbf{f}_i^{(a)} = \mathbf{\psi}_i(X_i^{(a)}, X_j^{(a)}) \quad (j \neq i ; i = 1, 2, \ldots, r)
\end{align*}
\]

where \( \mathbf{f}_{0ij}^{(a)}(j \neq i) \) in the first equations is replaced by \( \mathbf{f}_{0ij}^{(a-1)} \) as occasion demands.

We shall now consider the case shown in Fig. 3, where two rigid bodies are connected at one common point, \( P_0 \). We assume that both of them can rotate freely about \( P_0 \) and that the constraining force is the one acting at \( P_0 \) with each other. Then, the equations of motion can be written in the following form:

\[
\begin{align*}
&M_i X_i = F_i + L_i W, \\
&M_i X_i = F_i + L_i W, \\
&T_i X_i = T_i X_i
\end{align*}
\]

where \( W \) is a three-dimensional column vector that expresses the constraining force acting on

![Overall system](image.png)

Fig. 3 System composed of two rigid bodies.
the rigid body 2 from the rigid body 1 at $p_0$, and where $L_1$ and $L_2$ are matrices, with six rows and three columns, by which the constraining force is transformed to generalized forces. The third equation stands for the constraining condition where $T_1$ and $T_2$ are matrices with three rows and six columns. The following equations for successive computation are introduced by deciding the first approximate systems of the partial systems to be as shown in Fig. 4:

\[
\begin{align*}
M_1X_1^{(0)} &= F_1 + L_1W^{(0)} \\
M_2X_2^{(0)} &= F_2 + L_2W^{(0)} \\
T_1X_1^{(0)} &= T_2X_2^{(0)}
\end{align*}
\]  

(43)

When we begin our computation with the system of the rigid body 1, we compute $X_1^{(1)}$ from the first equation by putting $W^{(1)} = 0$; then, by substituting this into the second and third equations, we compute $W^{(2)}$ and $X_2^{(2)}$. This process is then repeated. When we start with the system of the rigid body 2, we compute $X_2^{(2)}$ and $W^{(2)}$ from the second and third equations by putting $X_1^{(0)} = 0$; then, by substituting this $W^{(2)}$ into the first equation, we compute $X_1^{(2)}$. This process is then repeated.

The problem of the convergence of this successive computation results in that of the eigenvalues of a matrix, as in Chapter 3. In Eq. (43), by solving the first for $X_1^{(1)}$ the second for $X_2^{(2)}$, and by substituting these solutions into the third, we obtain:

\[
W^{(4)} = A_{00} + B_{00}W^{(3)}
\]  

(44)

where:

\[
A_{00} = (T_2M_2^{-1}L_2)^{-1} \times (T_2M_2^{-1}F_2 - T_1M_1^{-1}F_1), \\
B_{00} = (T_2M_2^{-1}L_2)^{-1}(T_1M_1^{-1}L_1)
\]  

(45)

We except the frequencies, if any, at which $|M_1| = 0$, $|M_2| = 0$, or $|T_1M_1^{-1}L_1| = 0$.

The cases where $|M_1| = 0$, $|M_2| = 0$, or $|T_2M_2^{-1}L_2| = 0$ are, respectively, equivalent to those where there exists no viscous resistance in the system of the rigid body 1 expressed by the equation $M_1X_1 = 0$ (the first approximate system of the rigid body 1); in the system of the rigid body 2 expressed by the equation $M_2X_2 = 0$, and in the system of the rigid body 2 expressed by the equations $M_2X_2 = L_2W$ and $T_2X_2 = 0$ (the first approximate system of the rigid body 2). Moreover, the frequencies at which $|M_1| = 0$, $|M_2| = 0$, or $|T_1M_1^{-1}L_1| = 0$ are equal to the respective natural frequencies of these systems in such situations. At these special frequencies this successive computation cannot be applied.

From Eq. (42), the exact solution of $W$ can be written as $(E - B_{00})^{-1}A_{00}$, denoting the unit matrix by $E$. Therefore, it is clear that the successive computation converges towards the exact solution if it converges at all. The necessary and sufficient condition for the successive computation to converge without distinction of the first given $W^{(0)}$ is that the absolute value of every eigenvalue of the matrix, $B_{00}$, shown above be smaller than unity. $B_{00}$ is not changed by the linear transformation of coordinates.

By putting:

\[
B_{00} = T_1M_1^{-1}L_1, \\
B_{00} = T_2M_2^{-1}L_2
\]  

(46)

we can examine the absolute value of the eigenvalue, $\lambda$, of $B_{00}$, as in Chapter 4.

We obtain the following equation, denoting the $(h, k)$ elements of $B_{00}$ and $B_{00}^{-1}$ by $(B_{00})_{hk}$ and $(B_{00}^{-1})_{hk}$ respectively:

\[
|\lambda| < \text{Max} \left| (B_{00})_{hk} \text{Max} \left| (B_{00}^{-1})_{hk} \right| \right|
\]  

(47)

If no viscous resistance exists in the system, or if there exist only extremely small ones, the convergence of the successive computation is doubtful in the neighborhoods of the natural frequencies of the first approximate systems and of the overall system.

All these things can be discussed in similar fashion by replacing the positions of the rigid bodies 1 and 2 with each other.

We shall now deal with the case taken up in Chapter 4, assuming that the spring constant with which two masses are connected together is infinity. The following equations for successive computation may be introduced by deciding the first approximate systems to be as shown in Fig. 5:
\[ (1 - \sigma^2)\dot{x}_1^{(a)} = x_1^{(a)} - w^{(a)}, \]
\[ \ddot{x}_1^{(a)} = x_1^{(a)} / R + w^{(a)} / R, \]
\[ (1 - \sigma^2)\dot{x}_2^{(a)} = x_2^{(a)} / R + w^{(a)} / R, \]
\[ \ddot{x}_2^{(a)} = x_2^{(a)} / R + w^{(a)} / R, \]
\[ \dot{w}^{(a)} = -\frac{R \dot{x}_1^{(a)} - 1}{1 - \sigma^2} x_1^{(a)} + \frac{R \dot{x}_2^{(a)} - 1}{1 - \sigma^2} x_2^{(a)} + w^{(a)} - \frac{R \dot{w}^{(a)}}{1 - \sigma^2}, \]
\[ \text{where: } x_1^{(a)} = \frac{F}{k} \text{ and } w = \frac{W}{h}. \]

The relation between \( w^{(a)} \) and \( w^{(a-1)} \) then becomes:
\[ w^{(a)} = -\frac{R \dot{x}_1^{(a)} - 1}{1 - \sigma^2} x_1^{(a)} + \frac{R \dot{x}_2^{(a)} - 1}{1 - \sigma^2} x_2^{(a)} + w^{(a)}, \]
\[ \frac{R \dot{w}^{(a)}}{1 - \sigma^2}. \]

The necessary and sufficient condition for the successive computation to converge without distinction of \( w^{(a)} \) is:
\[ \left| \frac{R \dot{x}_1^{(a)} - 1}{1 - \sigma^2} \right| < 1 \]
This indicates that, for instance, if the two masses are equal in magnitude, the frequency range in which the successive computation converges is very narrow, as is shown by \( \sigma^2 < 1/2 \), but if \( R > 1/2 \), in which case the contrary, quite narrow, is shown by \( 10/11 < \sigma^2 < 10/9 \). If the successive computation is carried out according to the form of displacement for the mass 1 by replacing the positions of the masses 1 and 2 with each other, the frequency ranges in which it converges and does not converge are reversed except the point of \( |R \sigma^2/(1 - \sigma^2)| = 1 \).

7. A dividing method with regard to natural frequencies

In order to investigate qualitatively and quantitatively how a system constant influence the natural frequencies of the system, it is very often unsatisfactory to carry out merely a large number of numerical computations. In such a case, the natural frequencies can sometimes be obtained by dividing the overall system into two partial systems, one of which includes the constant, or in such a way that the constant is exactly included in the connecting parts; computing the solutions of the forced vibrations of the two partial systems separately with the help of one physical quantity, and then comparing these two solutions.

We shall write the equations of motion of the two partial systems in the following form:
\[ M_{11} \dot{X}_1 + M_{12} \dot{X}_2 = 0, \]
\[ M_{21} \dot{X}_1 + M_{22} \dot{X}_2 = 0, \]
\[ M_{11} \Delta_{11} = M_{21} \]
(Transposed matrix of \( M_{21} \))

The frequency equation then becomes:
\[ [M_{11} M_{21}] = 0 \]
\[ [M_{21} M_{21}]. \]

Now, let us consider the conditions under which Eq. (49) can be transformed into the following form, using the physical quantity, \( q \), with which two partial systems are connected:
\[ (M_{11} + K_{12}) X_1 = q P_1, \]
\[ (M_{22} + K_{22}) X_2 = q P_2. \]

where, denoting the degrees of freedom of the partial systems 1 and 2 by \( n_1 \) and \( n_2 \) respectively, \( P_1 \) and \( P_2 \) are, respectively, \( n_1 \)-dimensional and \( n_2 \)-dimensional column vectors, and \( Q_1 \) and \( Q_2 \) are, respectively, \( n_1 \)-dimensional and \( n_2 \)-dimensional row vectors.

From Eqs. (49) and (51), we obtain the equations:
\[ P_1 Q_1 = -M_{11}, \quad K_{12} = M_{12}, \quad K_{21} = M_{21}, \quad P_2 Q_2 = -M_{22} \]

We see that all the row vectors of \( M_{11} \) must be equal in the ratios of their components in order for such \( P_1 \) and \( Q_1 \) that satisfy the equation \( P_1 Q_1 = -M_{11} \) to exist. If this condition is fulfilled and if \( M_{11} \) has the form of:
\[ M_{11} = \begin{pmatrix} p_1 & (q_1, q_2, \ldots, q_{n_1}) \\ \vdots & \vdots \\ p_{n_1} & (q_1, q_2, \ldots, q_{n_1}) \end{pmatrix} \]
we may, for instance, take:
\[ P_1 = \begin{pmatrix} p_1 \\ \vdots \\ p_{n_1} \end{pmatrix}, \quad Q_1 = (q_1, q_2, \ldots, q_{n_1}). \]

The equation \( P_2 Q_2 = -M_{22} \) is satisfied by putting:
\[ P_2 = \pm Q'_2, \quad Q_2 = \pm P'_2 \]

\( K_{12} \) and \( K_{21} \) may then be determined by the equations:
\[ K_{12} = \pm \begin{pmatrix} p_1 & p_2 & \cdots & p_1 & p_{n_1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_{n_1} & p_2 & \cdots & p_1 & p_{n_1} \end{pmatrix}, \quad q_1 & q_2 & \cdots & q_{n_1} & q_{n_1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ q_{n_1} & q_2 & \cdots & q_1 & q_{n_1} \end{pmatrix} \]

(In these equations, one of the alternative signs applies.)

Dividing the overall system into two partial systems, if the correlating terms are of the form of Eq. (53), by introducing the physical quantity, \( q \), we can reduce the problem of obtaining the natural frequencies to that of solving the forced vibrations of the partial systems. In Eq. (51), by giving \( q \) an appropriate value, \( q_0 \), we solve the first equation for \( X_1 \) and the second for \( X_2 \), and then compute \( q \) from the third into which these solutions are substituted. The solutions of the frequency equation (50) are obtained by searching for the frequencies, for instance, on the \( q - \sigma^2 \) plane graphically or by using computers, at which the computed \( q \) value is exactly equal to \( q_0 \).

As an example, let us consider again the spring-
mass system with two degrees of freedom, considered in Chapter 4. The equations of motion of free vibrations are:
\[
\begin{align*}
(-m_1\omega^2 + k_1)x_1 - k_2x_2 &= 0, \\
(-m_2\omega^2 + k_2)x_2 - k_3x_3 &= 0
\end{align*}
\]
By dividing the overall system into the system of \(m_1\) and that of \(m_2\), and by putting:
\[
M_{12} = -k_2 = (-\sqrt{k_2})/\sqrt{k_2}, \quad P_1 = -\sqrt{k_2}, \quad q_1 = \sqrt{k_2}
\]
we have: \(P_1 = \sqrt{k_2}, \quad Q_1 = \sqrt{k_2}\)
Then, taking \(P_2 = -Q_2\), \(Q_2 = -P_2\), we have:
\[
P_2 = -\sqrt{k_2}, \quad Q_2 = -\sqrt{k_2}, \quad K_{12} = -k_2, \quad K_{23} = -k_2
\]
Consequently, the equations which correspond to Eq. (51) become:
\[
\begin{align*}
(-m_1\omega^2 + k_1)x_1 = q\sqrt{k_2}, \\
(-m_2\omega^2)x_2 = q(-\sqrt{k_2}),
\end{align*}
\]
We can say that these equations express a normal vibration, in which the spring, \(k_2\), exerts a force on the masses 1 and 2 the magnitude of which is equal to \(q\sqrt{k_2}\). Since \(m_1\) and \(k_1\) are included only in the first equation, and \(m_2\), only in the second, this method can effectively be utilized in order to investigate the influence of these system constants on the natural frequencies. In studying the influence of \(k_2\), it is useful to put \(q = \text{const.}/\sqrt{k_2}\); \(k_2\) thereby vanishes from the first and the second equations.

Here, we shall give an example of numerical computations done by keeping \(m_1\), \(k_1\) and \(k_2\) constant and by changing only \(m_2\). By taking \(q = \sqrt{k_2}\), from Eq. (56) we have:
\[
\begin{align*}
x_1 &= \frac{k_2}{-m_1\omega^2 + k_1} - \frac{R\omega^2}{-\omega^2 + 1}, \\
x_2 &= \frac{k_2\omega^2}{-m_2\omega^2 + k_2} \frac{\nu^2}{\omega^2} (1 + x_1 = x_2)
\end{align*}
\]
In Fig. 6, the points \(I_1\) and \(I_1'\) give the natural frequencies for \(R = 0.1\) and \(\nu = 10\). The natural frequencies given by the points \(I_2\) and \(I'_2\) correspond to the cases where only \(m_2\) is multiplied by 4 or 9 respectively.

8. Conclusion
Let us now itemize the characteristic features of the dividing method concerning forced vibrations.

(1) The analysis starts with partial systems with small degrees of freedom; the dimensions of the system to be treated can be made larger as occasion demands by connecting the partial systems successively.

(2) The treatment of each partial system does not differ from the ordinary form of the simultaneous differential equation, and the individual analysis of each partial system has a fully systematic meaning.

(3) We can decide the equations for successive computation so that the coordinates and the connecting forces in question appear explicitly in the iteration process.

4. We can obtain qualitative and quantitative bases for reducing, from the practical point of view, the given problem to that of a system with smaller degrees of freedom.

The dividing methods explained in this paper can be effectively applied in the study of actual vibration problems in various fields.

In the near future, we shall report on the extension of these methods to cases where the overall system contains systems with an infinite number of degrees of freedom and on an analysis of the vibrating systems of an automobile.

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References


