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Thermal Stresses in a Circular Disk Due to an Instantaneous Point Heat Source*

By Yoitiro TAKEUTI**

In the previous paper, we obtained the thermal stress distributions in a circular disk due to an instantaneous line heat source. The present paper is concerned with corresponding results for the action of a point heat source which appears in the welding practice. Numerical work is carried out for the case of the thermal stresses in a circular disk due to an instantaneous hot spot on the middle point of radius. In the final part, we show that the results for the line heat source in the previous paper may be obtained by integrating the solution in this paper with respect to r or θ .

1. Nomenclature

χ : thermal stress function in two dimensions
 χ_0 : bi-harmonic function
 k : material constant, i.e. for plane stress problems, αE ; for plane strain problems, $\alpha E/(1-\sigma)$
 α : coefficient of linear thermal expansion
 E : Young's modulus
 σ : Poissons's ratio
 κ : thermal diffusivity
 t : time
 J_n : Bessel function of n -th order
 h : heat transfer coefficient
 a : radius of circular disk or cylinder
 Q : intensity of point heat source
 q : intensity of line heat source
 $f(r, \theta)$: temperature distribution function at initial state

p_s : introduced in Eq. (3)'

T : dimensionless time

ρ : dimensionless radius

μ_s : dimensionless characteristic quantity for p_s

r, θ : cylindrical coordinates

τ : temperature rise

$$\Delta = \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + (1/r^2)(\partial^2/\partial \theta^2)$$

2. Introduction

The problem of finding a complete solution of the thermoelastic equation for thermal stress in a disk is one of considerable analytical difficulty. Many of difficulties are removed when the problem becomes one of two dimensions which happens in a very long bar when the state of stress is known as plane strain, or in a very thin plate when the state of stress is known as plane stress. If we assume that the temperature does not vary over the thickness of the disk, we may assume that the stress and displacement due to heating also do not vary over the thickness. From this point of view, in the present paper, analysis is developed for

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** Professor, Division of Mechanical Engineering, Tokushima University, Tokushima.

the two dimensional thermal stress problems in a circular disk or infinitely long cylinder under a hot spot.

In the previous paper⁽¹⁾, a general procedure was developed for analyzing the unsteady thermal stress in a circular disk subjected to asymmetrical temperature distribution due to the action of an instantaneous heat source at initial state, employing the delta function. However, specific formulas and numerical results were obtained only for a line heat source on the concentric circular arc or on its radius.

It is the purpose of this paper to present the problem for a point heat source, which often appears in welding practice.

3. Analysis

Now the fundamental equation governing the two-dimensional thermal stress problems may be expressed in the known form,

$$\Delta\Delta\chi = -k\Delta\tau \dots\dots\dots (1)$$

It has been shown in the previous paper that the above thermal stress function for a thin circular disk or long cylinder due to the action of an asymmetrical temperature field at initial state may be given in cylindrical coordinate as

$$\chi = \chi_0 + k \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} p_s^{-2} e^{-\kappa p_s^2 t} J_n(p_s r) (A_{sn} \cos n\theta + B_{sn} \sin n\theta) \dots\dots\dots (2)$$

where

$$\left. \begin{aligned} A_{sn} &= \frac{\varepsilon_n p_s^2}{\pi a^2 (p_s^2 + h^2 - n^2 a^{-2}) \{J_n(p_s a)\}^2} \int_0^a \int_{-\pi}^{\pi} f(r', \theta') \cos n\theta' J_n(p_s r') r' dr' d\theta' \\ B_{sn} &= \frac{\varepsilon_n p_s^2}{\pi a^2 (p_s^2 + h^2 - n^2 a^{-2}) \{J_n(p_s a)\}^2} \int_0^a \int_{-\pi}^{\pi} f(r', \theta') \sin n\theta' J_n(p_s r') r' dr' d\theta' \\ \varepsilon_0 &= 1, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \dots\dots\dots = 2 \end{aligned} \right\} \dots\dots\dots (3)$$

Also p_s is the positive root of

$$p_s J_n'(p_s a) + h J_n(p_s a) = 0 \dots\dots\dots (3)'$$

and radiation takes place at the surface $r=a$ into a medium at zero temperature. The additional terms of the right side of Eq. (2) correspond to a particular solution of the governing equation. If an instantaneous point heat source of strength Q acts at the point (r_0, θ_0) as shown in Fig. 1, we may express $f(r, \theta)$ by the form

$$f(r, \theta) = Q \delta(r - r_0) \delta(\theta - \theta_0) / r \dots\dots\dots (4)$$

Substituting Eq. (4) into (3), we obtain

$$A_{sn} \cos n\theta + B_{sn} \sin n\theta = E_{sn} Q J_n(p_s r_0) \cos n(\theta - \theta_0) \dots\dots\dots (5)$$

where

$$\left. \begin{aligned} E_{sn} &= \frac{\varepsilon_n p_s^2}{\pi a^2 (p_s^2 + h^2 - n^2 a^{-2}) \{J_n(p_s a)\}^2} \quad \text{if } 0 \leq h < \infty \\ &= \frac{\varepsilon_n}{\pi a^2 \{J_n'(p_s a)\}^2} \quad \text{if } h = \infty \end{aligned} \right\} \dots\dots\dots (5)'$$

Now, for this problem we adopt the following form of the bi-harmonic function so as to satisfy the condition that the solution must be finite at $r=0$.

$$\chi_0 = c_1 r^2 + \sum_{n=1}^{\infty} (c_{2n} r^n + c_{3n} r^{n+2}) \cos n(\theta - \theta_0) \dots\dots\dots (6)$$

Substituting Eqs. (5) and (6) into (2), the constants in Eq. (6) may be determined from the boundary conditions of zero tractions on the boundary $r=a$.

Fig. 1 Instantaneous point heat source

$$\left. \begin{aligned} c_1 &= -kQ \sum_{s=1}^{\infty} e^{-\kappa p_s^2 t} p_s^{-1} E_{s0} J_0'(p_s a) J_0(p_s r_0) / 2a \\ c_{2n} &= kQ \sum_{s=1}^{\infty} p_s^{-2} e^{-\kappa p_s^2 t} E_{sn} \frac{J_n(p_s r_0)}{2a^n} \\ &\quad \times \{na^{-2} J_n(p_s a) - p_s a^{-1} J_n'(p_s a)\} \\ c_{3n} &= kQ \sum_{s=1}^{\infty} p_s^{-2} e^{-\kappa p_s^2 t} E_{sn} \frac{J_n(p_s r_0)}{2a^n} \\ &\quad \times \{p_s a J_n'(p_s a) - (n+2) J_n(p_s a)\} \end{aligned} \right\} \dots\dots\dots (7)$$

Thus the thermal stress function for this case may be determined entirely.

$$\chi = kQ \sum_{s=1}^{\infty} e^{-\kappa p_s^2 t} p_s^{-2} \left[E_{s0} J_0(p_s r_0) \left\{ J_0(p_s r) - \frac{p_s}{2a} J_0'(p_s a) r^2 \right\} \right]$$

$$+ \sum_{n=1}^{\infty} E_{sn} J_n(p_s r_0) \cos n(\theta - \theta_0) \left[J_n(p_s r) + \frac{r^n a^{-n}}{2} \{ p_s a J_n'(p_s a) - (n+2) J_n(p_s a) \} \right. \\ \left. + r^2 a^{-2} \{ n J_n(p_s a) - p_s a J_n'(p_s a) \} \right] \dots \dots \dots (8)$$

Then, the values of the thermal stress components are given by

$$\widehat{rr} = (1/r^2)(\partial^2 \chi / \partial \theta^2) + (1/r)(\partial \chi / \partial r) \\ = kQ \sum_{s=1}^{\infty} e^{-\kappa p_s^2 t} p_s^{-1} E_{s0} J_0(p_s r_0) \{ r^{-1} J_0'(p_s r) - a^{-1} J_0'(p_s a) \} \\ + kQ \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} e^{-\kappa p_s^2 t} p_s^{-2} E_{sn} J_n(p_s r_0) \cos n(\theta - \theta_0) [\{ p_s r^{-1} J_n'(p_s r) \\ + r^{n-2} a^{-n} n(1-n) \{ p_s a J_n'(p_s a) - (n+2) J_n(p_s a) \} / 2 \\ + r^n (2-n)(n+1) a^{-n+2} \{ n J_n(p_s a) - p_s a J_n'(p_s a) \} / 2 \}] \dots \dots \dots (9)$$

$$\widehat{\theta\theta} = \partial^2 \chi / \partial r^2 = kQ \sum_{s=1}^{\infty} e^{-\kappa p_s^2 t} p_s^{-1} E_{s0} J_0(p_s r_0) \{ p_s J_0''(p_s r) \\ - a^{-1} J_0''(p_s a) \} + kQ \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} e^{-\kappa p_s^2 t} p_s^{-2} E_{sn} J_n(p_s r_0) \cos n(\theta - \theta_0) [p_s^2 J_n''(p_s r) \\ + r^{n-2} a^{-n+2} (n+1)(n+2) \{ n J_n(p_s a) - p_s a J_n'(p_s a) \} / 2 + r^{n-2} a^{-n} n(n-1) \\ \times \{ p_s a J_n'(p_s a) - (n+2) J_n(p_s a) \} / 2] \dots \dots \dots (10)$$

$$\widehat{r\theta} = -(\partial / \partial r) \{ (1/r)(\partial \chi / \partial \theta) \} = kQ \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} e^{-\kappa p_s^2 t} p_s^{-2} E_{sn} J_n(p_s r_0) n \sin n(\theta - \theta_0) \\ \times [r^{n-2} (n-1) a^{-n} \{ p_s a J_n'(p_s a) - (2+n) J_n(p_s a) \} / 2 + r^{n-2} a^{-n+2} (n+1) \{ n J_n(p_s a) \\ - p_s a J_n'(p_s a) \} / 2 + \{ r p_s J_n'(p_s r) - J_n(p_s r) \} / r^2] \dots \dots \dots (11)$$

4. Numerical example

As a numerical example, we consider the state of stress in the case of an infinitely circular cylinder or thin disk with the following data,

$$r_0 = a/2, \quad \theta_0 = 0$$

and the surface $r=a$ is kept at zero temperature.

In this case, since p_s is the positive root of

$$J_n(p_s a) = 0 \dots \dots \dots (12)$$

From the consideration in the previous paper the temperature distribution in this problem is given by

$$\tau = Q \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} e^{-\kappa p_s^2 t} \frac{\varepsilon_n}{\pi \{ J_n'(p_s a) \}^2} J_n(p_s a/2) J_n(p_s r) \cos n\theta \dots \dots \dots (13)$$

Introducing into Eqs. (9), (10) and (11) dimensionless variables ρ , T and μ_s such that

$$r/a = \rho, \quad \kappa t/a^2 = T, \quad p_s a = \mu_s \dots \dots \dots (14)$$

we reduce the above equations as to the stress components to a non-dimensional form as follows:

$$\widehat{rr}/L = \sum_{s=1}^{\infty} e^{-\mu_s^2 T} \frac{J_0(\mu_s/2)}{\mu_s \{ J_0'(\mu_s) \}^2} \left\{ \frac{J_0'(\mu_s \rho)}{\rho} - J_0'(\mu_s) \right\} + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} e^{-\mu_s^2 T} J_n(\mu_s/2) \\ \times \frac{\cos n\theta}{\mu_s^2 \{ J_n'(\mu_s) \}^2} [2 \{ \mu_s \rho^{-1} J_n'(\mu_s \rho) - \rho^{-2} n^2 J_n(\mu_s \rho) \} + \mu_s J_n'(\mu_s) \rho^n \\ \times \{ \rho^{-2} n(1-n) - (2-n)(1+n) \}] \dots \dots \dots (15)$$

$$\widehat{\theta\theta}/L = \sum_{s=1}^{\infty} e^{-\mu_s^2 T} \frac{J_0(\mu_s/2)}{\mu_s \{ J_0'(\mu_s) \}^2} \{ \mu_s J_0''(\mu_s \rho) - J_0'(\mu_s) \} + \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} e^{-\mu_s^2 T} \\ \times J_n(\mu_s/2) \frac{\cos n\theta}{\mu_s^2 \{ J_n'(\mu_s) \}^2} [2 \mu_s^2 J_n''(\mu_s \rho) + \mu_s J_n'(\mu_s) \rho^n \{ \rho^{-2} n(n-1) - (n+1)(n+2) \}] \dots \dots \dots (16)$$

$$\widehat{r\theta}/L = \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} e^{-\mu_s^2 T} \frac{J_n(\mu_s/2) n \sin n\theta}{\mu_s^2 \{ J_n'(\mu_s) \}^2} [2 \rho^{-2} \{ \rho \mu_s J_n'(\mu_s \rho) \\ - J_n(\mu_s \rho) + \mu_s J_n'(\mu_s) \rho^n \{ \rho^{-2} (n-1) - (n+1) \} \}] \dots \dots \dots (17)$$

where

$$L = kQ/\pi a^2$$

The dimensionless curves in Figs. 2~11 show the distributions of temperature and radial, circumferential, and shearing stress in the circular disk.

5. Remarks

In the rest of the paper we mention that the solutions in the previous paper are readily obtained

Fig. 2 Variation of temperature on the diameter for $\theta=0$ Fig. 4 Stress distribution of $\hat{r}r$ on the $\theta=0$ and π Fig. 3 Temperature distribution on the diameter for $\theta=0$ Fig. 5 Stress distribution of $\hat{\theta}\theta$ on $\theta=0$ and π

Fig. 6 Stress distribution of $\hat{r}r$ on $\theta = \pi/4$ and $5\pi/4$ Fig. 9 Stress distribution of $\hat{r}r$ on $\theta = \pi/2$ and $3\pi/2$ Fig. 7 Stress distribution of $\hat{\theta}\theta$ on $\theta = \pi/4$ and $5\pi/4$ Fig. 10 Stress distribution of $\hat{\theta}\theta$ on $\pi/2$ and $3\pi/2$ Fig. 8 Stress distribution of $\hat{r}\theta$ on $\theta = \pi/4$ and $5\pi/4$ Fig. 11 Stress distribution of $\hat{r}\theta$ on $\pi/2$ and $3\pi/2$

from the result in this paper. For example, it will be observed that when the line heat source of intensity q acts at the concentric circular arc ($\theta_1 \leq \theta \leq \theta_2$) with radius r_0 on the circular disk we may easily find the thermal stress function replacing Q in Eq. (8) by $q/r_0(\theta_2 - \theta_1)$ and integrating $\chi_p r_0 d\theta_0$ for the range (θ_1, θ_2) with respect to θ_0 .

$$\begin{aligned} \chi_l = & kq \sum_{s=1}^{\infty} e^{-\kappa p_s^2 t} E_{s0} J_0(p_s r_0) p_s^{-2} \left\{ J_0(p_s r) - \frac{r^2 p_s}{2a} J_0'(p_s a) \right\} \\ & + \frac{kq}{2(\theta_2 - \theta_1)} \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} e^{-\kappa p_s^2 t} E_{sn} J_n(p_s r_0) p_s^{-2} n^{-1} [2J_n(p_s r) \\ & + r^n a^{-n} \{ p_s a J_n'(p_s a) - (n+2) J_n(p_s a) \} + r^{n+2} a^{-(n+2)} \{ n J_n(p_s a) \\ & - p_s a J_n'(p_s a) \}] \{ \sin n(\theta - \theta_1) - \sin n(\theta - \theta_2) \} \dots \dots \dots (18) \end{aligned}$$

In above treatise the subscripts l and p denote the stress function for the line and point heat source. With this stress function we find

$$\begin{aligned} \widehat{rr} = & \frac{kq}{\pi a^2} \sum_{s=1}^{\infty} \frac{e^{-\kappa p_s^2 t} p_s J_0(p_s r_0)}{(p_s^2 + h^2) J_0^2(p_s a)} \left\{ \frac{J_0'(p_s r)}{r} - \frac{J_0'(p_s a)}{a} \right\} + \frac{4kq}{\pi a^2 (\theta_2 - \theta_1)} \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \\ & \times e^{-\kappa p_s^2 t} \frac{J_n(p_s r_0)}{n(p_s^2 + h^2 - n^2 a^{-2}) J_n^2(p_s a)} \left[\frac{r^n}{2a^{n+2}} (1+n)(2-n) \{ n J_n(p_s a) - p_s a J_n'(p_s a) \} \right. \\ & + \frac{r^{n-2}}{2a^n} n(1-n) \{ p_s a J_n'(p_s a) - (2+n) J_n(p_s a) \} \\ & \left. + r^{-2} \{ p_s r J_n'(p_s r) - n^2 J_n(p_s r) \} \right] \cos \frac{n(2\theta - \theta_1 - \theta_2)}{2} \sin \frac{n(\theta_2 - \theta_1)}{2} \dots \dots \dots (19) \end{aligned}$$

$$\begin{aligned} \widehat{\theta\theta} = & \frac{kq}{\pi a^2} \sum_{s=1}^{\infty} \frac{e^{-\kappa p_s^2 t} p_s J_0(p_s r_0)}{(p_s^2 + h^2) J_0^2(p_s a)} \{ p_s J_0''(p_s r) - a^{-1} J_0'(p_s a) \} \\ & + \frac{4kq}{\pi a^2 (\theta_2 - \theta_1)} \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-\kappa p_s^2 t} J_n(p_s r_0)}{n(p_s^2 + h^2 - n^2 a^{-2}) J_n^2(p_s a)} \\ & \times \left[p_s^2 J_n''(p_s r) + \frac{r^{n-2}}{2a^n} n(n-1) \{ p_s a J_n'(p_s a) - (2+n) J_n(p_s a) \} \right. \\ & \left. + \frac{r^n}{2a^{n+2}} (n+1)(n+2) \{ n J_n(p_s a) - p_s a J_n'(p_s a) \} \right] \cos \frac{n(2\theta - \theta_1 - \theta_2)}{2} \sin \frac{n(\theta_2 - \theta_1)}{2} \dots \dots \dots (20) \end{aligned}$$

$$\begin{aligned} \widehat{r\theta} = & \frac{4kq}{\pi a^2 (\theta_2 - \theta_1)} \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-\kappa p_s^2 t} J_n(p_s r_0)}{(p_s^2 + h^2 - n^2 a^{-2}) J_n^2(p_s a)} \left[\frac{r^n}{2a^{n+2}} (n+1) \{ n J_n(p_s a) \right. \\ & - p_s a J_n'(p_s a) \} + \frac{r^{n-2}}{2a^n} (n-1) \{ p_s a J_n'(p_s a) - (2+n) J_n(p_s a) \} \\ & \left. + r^{-2} \{ p_s r J_n'(p_s r) - J_n(p_s r) \} \right] \sin \frac{n(2\theta - \theta_1 - \theta_2)}{2} \sin \frac{n(\theta_2 - \theta_1)}{2} \dots \dots \dots (21) \end{aligned}$$

By a similar integration procedure, the case of the line heat source on the radius may be found to be $\chi_l = \int_0^a \chi_p dr_0$.

$$\begin{aligned} \chi_l = & \frac{2kq}{a} \sum_{s=1}^{\infty} e^{-\kappa p_s^2 t} p_s^{-3} E_{s0} M_0(p_s a/2) \left\{ J_0(p_s r) - \frac{p_s r^2}{2a} J_0'(p_s a) \right\} \\ & + \frac{kq}{a} \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} e^{-\kappa p_s^2 t} p_s^{-3} E_{sn} M_n(p_s a/2) \left[2J_n(p_s r) + \frac{r^n}{a^n} \{ J_n'(p_s a) p_s a \right. \\ & \left. - (n+2) J_n(p_s a) \} + \frac{r^{n+2}}{a^{n+2}} \{ n J_n(p_s a) - p_s a J_n'(p_s a) \} \right] \cos n\theta \dots \dots \dots (22) \end{aligned}$$

$$\text{where} \quad 2M_n(p_s a/2) = p_s \int_0^a J_n(p_s r_0) dr_0 \dots \dots \dots (22)'$$

With this stress function we obtain the final values of the thermal stress components for this case. These results agree with those of previous paper.

Finally, considering the boundary conditions on the surface strictly, it is evident that the above analysis should have to apply for the case of an infinitely long bar only. For the case of a thin circular disk all formulae developed in this paper must be considered as mere approximation, because the solution can be obtained by making the assumption that the heat transfers on both sides of the upper and lower surface are small.

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Reference

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Compressive Strength of Mild Steel at High Strain Rate and at High Temperature*

By Kichinosuke TANAKA** and Motohiro KINOSHITA***

This paper deals with the dynamic behaviour of 0.03 %C steel at high temperature (up to 700 °C) within the range of strain rates of $10 \sim 10^3$ /sec. The effects of temperature and strain rate on the compressive strength were clarified experimentally and the particular behaviour at the temperature near blue brittleness was explained using the Cottrell's theory. The dynamic stress to cause a certain amount of strain rises with the increase of strain rate at room temperature, but the stress is scarcely affected by the strain rate at 200~600 °C, and then the stress increases again with the strain rate at 700 °C. The blue brittleness occurs at higher temperature with the increase of strain rate. For the static test and the dynamic tests of 10^2 /sec, 3×10^2 /sec and 10^3 /sec of strain rate, the blue brittleness temperature is about 200, 550, 600 and 700 °C respectively.

1. Introduction

It is a wellknown fact that the strength of materials is affected by the rate of deformation. But, the effects of temperature and strain rate on the strength of metals are not yet clarified sufficiently. This paper describes the relation among stress, strain and strain rate on annealed 0.03 % carbon steel. The experiments are performed under the dynamic compressive loads at room temperature, 200, 300, 400, 500, 600 and 700 °C.

M. J. Manjoine has studied systematically on the dynamic tensile strength of mild steel at high temperatures⁽¹⁾. For the purpose of hot workings, J. Alder and others⁽²⁾ obtained the dynamic strength at rather high temperatures. In Japan, S. Hashizume⁽³⁾, I. Gokyu and others⁽⁴⁾ got the dynamic strength over a fairly wide range of temperatures. The former is for 0.15 %C steel and the latter is for 0.02~0.09 %C steel. The authors

have reported on the dynamic strength of 0.03 % C steel at room temperature previously⁽⁵⁾. And to treat the behaviours near blue brittleness temperature about which this paper is concerned, not only the results obtained at high temperature, but also the results at low temperature⁽⁶⁾ are full of suggestions.

It seems very important to compare the results on steel described in this paper with the results on aluminium⁽⁷⁾, but the comparison is not included in this paper.

2. Experimental method

2.1 Experimental apparatus

The apparatus used for the experiment is as follows. A projectile apparatus, a striking bar, an input bar, an output bar and a throw-off bar are installed in alignment. The specimen is mounted between the input bar and the output bar in contact with both and is surrounded by an electric furnace. The striking bar is projected from the projectile apparatus by the force of two coil springs and hits the input bar. This longitudinal impact produces compressive stress, the

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** Professor, Faculty of Engineering, Kyoto University, Sakyo-ku, Kyoto.

*** Assistant, Faculty of Engineering, Kyoto University.