by the unbalanced whirl of journal is carried out by means of strain gauges.

As the result, it is made clear that the pressure of oil film is expressed as a sum of the pressure which is caused by the rotation of the journal at statical equilibrium position and one which is caused by the whirling motion of the journal. And it is also clarified that the varying pressure is always zero at the positions perpendicular to the velocity of the journal center if the journal is whirling in small circle.

In the theoretical investigation, based on these experimental results, we solve the Reynolds equation for an infinitely long bearing by making use

of the boundary conditions in which the pressure $p_o(\varphi)$ due to the rotation of the journal is zero at $\varphi=0$ and π , the pressure $p_r(\varphi)$ due to the radial velocity $\dot{k}$ of the journal center is zero at $\varphi=\pi/2$ and $3\pi/2$, and the pressure $p_t(\varphi)$ due to the tangential velocity $\kappa\dot{\theta}$ of the journal center is zero at $\varphi=0$ and π .

Analytical results are found in good agreement with the experimental ones.

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Optimum Control Based on an Evaluation Function Including the Initial Investment*

By Isao TODO**

The author proposes a synthetic criterion J of the form $J=J_m+kJ_e$ to evaluate a control system, where J_m is the evaluation function for the operation, J_e is that for the initial investment and k is a constant. It is shown that the maximum M of the manipulated variable can be included in J in the case of the optimization of a system under a constraining condition $|u|\leq M$, where u is the manipulated variable.

As the result, it is shown that the choice of M can have a considerable influence on the value of the evaluation function, and that a better solution will be obtained if M is considered as one of the design parameters, when the optimum control is intended. A control of a pure inertia model is considered as an example, for various types of J_m .

1. Introduction

The problem to find the optimum-control law so as to minimize or maximize the given functional, has been treated in general, by means of, for instance, the classical calculus of variations and the maximum principle. The author considers it may be necessary to examine whether the process as an integrated whole is optimum or not, in the realization of optimum controls.

The design of optimum controllers may become more complicated by the following reasons: cutting down the cost of construction of the plant, improper detection of signals, lack of information

available to control, and so on. Though the optimization problem under constraining conditions imposed on the manipulated variables has mainly been discussed up to date, the constraining conditions have usually been assumed to be fixed. But in engineering problems, these constraining conditions are to be chosen at the stage of the design and planning of a plant.

2. Nomenclature

- H : Hamiltonian function
- J, J^* : synthetic criterion proposed by the author
- J_m : evaluation function for operation
- J_e : evaluation function for initial investment
- k : constant
- M : maximum of u
- t^* : real time

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** Lecturer, Faculty of Engineering, Yokohama National University, Minami-ku, Yokohama.

t : normalized time
 T^* : real terminal time
 T : normalized terminal time
 u : manipulated variable
 w, w_1 : weighting factor
 x : state variable
 ϕ_i : i -th auxiliary vector function
 $\tau = T - t$: backward time
 Sub- and Superscripts
 B, C : condition at point B and point C, respectively
 ψ_i^0 : value of ψ_i at $\tau = 0$

3. A synthetic criterion function proposed by the author

Consider the system described by a set of first order differential equations:

$$dx_i/dt = f_i(x, u) \quad i=1, 2, \dots, n \dots \dots \dots (1)$$

where $x = (x_1, x_2, \dots, x_n)'$ is a state vector of the system, $u = (u_1, u_2, \dots, u_r)'$ is a control vector, and the prime denotes the transpose. The determination of an admissible control $u(t)$ which satisfies the constraining condition

$$u \in \Omega \dots \dots \dots (2)$$

and which, in addition, takes the system from a specified initial state at $t=0$ to a desired final state at $t=T$ while minimizing the functional

$$\int_0^T f_0(x, u) dt \dots \dots \dots (3)$$

is one of the typical optimum control problems⁽¹⁾⁽²⁾. In this paper, the criterion function given by Eq. (3), is defined as the evaluation function for the operation, denoted by J_m . A synthetic criterion function proposed by the author is expressed as

$$J = J_m + kJ_e \dots \dots \dots (4)$$

where J_e is the evaluation function for the initial investment, and k is a constant introduced to combine J_m with J_e . Namely, let J_m and J_e be the operating cost for one operation (or for a unit time) and the capital of investment, respectively. Then, k may be interpreted as the constant which converts J_e into the depreciation for one operation (or for a unit time).

Furthermore, in general, J_e can be described by the summation of three terms, that is

$$J_e = J_{e,f} + J_{e,d} + J_{e,p} \dots \dots \dots (5)$$

where

$J_{e,f}$: evaluation function for the controller cost

$J_{e,d}$: evaluation function for the primary elements

$J_{e,p}$: evaluation function for construction of the plant including the final control elements.

Hence, by substituting Eq. (5) into Eq. (4), we obtain

$$J = J_m + k(J_{e,f} + J_{e,d} + J_{e,p}) \dots \dots \dots (6)$$

There are several ways to determine the value

of $J_{e,f}$. In one case, $J_{e,f}$ can be determined by dividing the summation of the depreciation of the computer by the numbers of the controlled variables, when the computer is used in a time-sharing form. In another case, it can be done by summing the cost of resistances, condensers, relays, tubes and other components involved in the controller.

$J_{e,d}$ is the evaluation function for the primary elements to obtain the informations concerning the process. The more the number of detected state variables are, the larger the value of $J_{e,d}$ will be. However, in some cases, a controller can be relatively simple in its mechanism, when the information available for control increases.

The value of $J_{e,p}$ is the summation of the costs of construction, of final control elements and of other factors.

$J_m, J_{e,f}, J_{e,d}$ and $J_{e,p}$ are not independent, but are closely related to each other. If we design a control mechanism precisely to minimize J_m keeping $J_{e,d}$ and $J_{e,p}$ constant, $J_{e,f}$ will increase. However, as far as both $J_{e,d}$ and $J_{e,p}$ are kept constant, J_m can not be decreased below a certain minimum, even if the value of $J_{e,f}$ is increased infinitely. Fig. 1 shows a typical example of the relationship between $J_m, J_{e,f}$ and $J_{e,p}$. From Fig. 1, it is observed that the minimum of J_m , denoted by $(J_m)_{\min}$, decreases with the increase of $J_{e,f}$ up to a certain limit, and that J_m remains almost constant for a larger value of $J_{e,f}$. It is also seen that J_m and $(J_m)_{\min}$ tend to decrease with an increasing $J_{e,p}$.

4. An example of optimum control

As an example, we consider the control problem of the pure inertia model, as encountered in a single-axis attitude control systems. In the absence of disturbing torques, the motion of the vehicle is described by

$$\frac{d^2 X_1(t^*)}{dt^{*2}} = \frac{L(t^*)}{I} \dots \dots \dots (7)$$

Fig. 1 Typical relationship between $J_m, J_{e,f}$ and $J_{e,p}$ ($C_1 < C_2, J_{e,d}$ is kept constant)

where $X_1(t^*)$ is the attitude deviation from some fixed reference, $L(t^*)$ is the control torque, I is the moment of inertia of the vehicle and t^* is real time⁽³⁾. The optimum control problem for this case may be stated as the determination of an admissible control $L(t^*)$ so that the process is taken from an initial state:

$$X_1(0)=A, \quad X_2(0)=0 \dots\dots\dots(8)$$

to a final state:

$$X_1(T^*)=0, \quad X_2(T^*)=0 \dots\dots\dots(9)$$

minimizing J_m under the constraining condition given by

$$|L(t^*)| \leq L_0 \dots\dots\dots(10)$$

where

$$dX_1(t^*)/dt^* = X_2(t^*)$$

Putting

$$t = \sqrt{\frac{L_0}{IA}} t^*, \quad x_1(t) = \frac{X_1(t^*)}{A}, \quad u = \frac{L(t^*)}{L_0}$$

Eqs. (7) ~ (10) can be expressed in non-dimensional forms as follows:

the equation of motion;

$$\ddot{x}_1 = u \dots\dots\dots(7.a)$$

or

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u \dots\dots\dots(7.b)$$

the constraining condition;

$$|u| \leq 1 \dots\dots\dots(8.a)$$

the initial state;

$$x_1(0)=1, \quad x_2(0)=0 \dots\dots\dots(9.a)$$

the final state;

$$x_1(T)=0, \quad x_2(T)=0 \dots\dots\dots(10.a)$$

The reduction below is carried out based on Eqs. (7.a) ~ (10.a).

5. The main problem considered in this paper

The maximum of normalized manipulated variable is equal to 1, as expressed in Eq. (8.a). However, we can shift the maximum of a constraining condition easily, by changing the structure of process, especially, the final control elements. Now consider the case when we have to increase $J_{e,p}$ by the amount of $\Delta J_{e,p}$ in order to shift the constraining condition imposed on u from 1 to M . In this case, a constraining condition is given by

$$|u| \leq M \dots\dots\dots(8.b)$$

instead of Eq. (8.a). On the other hand, $\Delta J_{e,p}$ can be regarded as a function of M , denoted by

$$\Delta J_{e,p} = G(M) \dots\dots\dots(11)$$

If the controller mechanism which minimizes J_m is installed in this system, we can consider $J_{e,d}$ and $J_{e,f}$ as independent of the increase of $J_{e,p}$ by $\Delta J_{e,p}$, since we need not alter the structures of a controller mechanism or primary elements. Therefore, by subtracting the terms independent

of M , from the right-hand side of Eq. (6), we obtain the criterion function J^* given by

$$J^* = (J_m)_{\min} + k \Delta J_{e,p} \dots\dots\dots(12)$$

In addition, in the case of the time optimum control of the systems characterized by Eqs. (7.a) ~ (10.a), $(J_m)_{\min}$ is given by

$$(J_m)_{\min} = 2/\sqrt{M} \dots\dots\dots(13)$$

[Eq. (13) is obtained by substituting $w=0$ into Eq. (30).] $(J_m)_{\min}$ given by Eq. (13) has a decreasing trend for the increase of M , as illustrated in Fig. 2.

Furthermore, if there exists a linear relationship between $\Delta J_{e,p}$ and M , as given by

$$\Delta J_{e,p} = cM \dots\dots\dots(11.a)$$

Eq. (12) can be rewritten as

$$J^* = 2/\sqrt{M} + kcM \dots\dots\dots(12.a)$$

and this J^* becomes minimum when

$$M = 1/(\sqrt{kc})^3 \dots\dots\dots(14)$$

We can find an example of constraining conditions imposed on manipulated variable in the case where the stroke of control valve, or the torque of a motor, is limited. We can increase M by adopting a piping system of larger diameters and by adjusting control valves.

In this paper, as an example, the control problem of a pure inertia model is considered, for various types of J_m .

6. The problems of unfixed terminal time

For the evaluation function J_m given by

$$J_m = \int_0^T f_0(x, u) dt \dots\dots\dots(3.a)$$

Fig. 2 J^* versus M

introducing a new coordinate x_0 characterized by
 $dx_0/dt = f_0(x, u)$ (15)
 and putting

$$X = (x_0, x_1, x_2)' \text{(16)}$$

$$F = (f_0, f_1, f_2)' \text{(17)}$$

we obtain a third order vector equation expressed by

$$dX/dt = F(x, u) \text{(18)}$$

Introducing an auxiliary vector $\phi = (\phi_0, \phi_1, \phi_2)'$ defined by the solution of the differential equation

$$\frac{d\phi_i}{dt} = -\sum_{j=0}^2 \left(\frac{\partial f_j}{\partial x_i} \right) \phi_j \quad (i=0, 1, 2) \text{(19)}$$

Eq. (19) can be reduced to

$$\frac{d\phi_i}{dt} = -\frac{\partial H}{\partial x_i} \quad (i=0, 1, 2) \text{(20)}$$

where H is the Hamiltonian function given by

$$H = (\phi, F) = \sum_{i=0}^2 \phi_i f_i \text{(21)}$$

Since ϕ_0 is constant as we see from Eq. (19), which is homogeneous with respect to ϕ_i ($i=0, 1, 2$) as well, we can assume $\phi_0 = -1$ without loss of generality. In the case of terminal time T unfixed, we can use the relationship of $\max H=0$. For the sake of simplicity, we use the following substitution. That is, $t = T - \tau$, where τ is the backward time. Hence, Eqs. (9.a) and (10.a) become, respectively,

$$(x_1)_{\tau=0} = 0, \quad (x_2)_{\tau=0} = 0 \text{(22)}$$

$$(x_1)_{\tau=T} = 1, \quad (x_2)_{\tau=T} = 0 \text{(23)}$$

$$\text{6.1 Case a : } J_m = \int_0^T (1 + w|u|) dt$$

Consider the evaluation function J_m defined by

$$J_m = \int_0^T (1 + w|u|) dt \text{(24)}$$

J_m given by Eq. (24) represents the evaluation function for the time optimum control, if we put $w=0$ in Eq. (24).

The Hamiltonian function H is given by

$$H = -(1 + w|u|) + \phi_1 x_2 + \phi_2 u$$

Fig. 3 Time behavior of the optimum control for case a

Integration of equations $\dot{\phi}_1=0$, $\dot{\phi}_2=-\phi_1$, which are obtained from Eq. (20), yields

$$\phi_1 = \phi_1^0, \quad \phi_2 = \phi_2^0 - \phi_1^0 \tau \text{(25)}$$

H is maximum if

$$u = \begin{cases} M & (\phi_2^0 - \phi_1^0 \tau)/w \geq 1 \\ 0 & |(\phi_2^0 - \phi_1^0 \tau)/w| < 1 \\ -M & (\phi_2^0 - \phi_1^0 \tau)/w \leq -1 \end{cases} \text{(26)}$$

as illustrated in Fig. 3, where

$$\tau_B = \frac{\phi_2^0 + w}{\phi_1^0}, \quad \tau_C = \frac{\phi_2^0 - w}{\phi_1^0} \text{(27)}$$

From the conditions at $\tau=0$, that is, $H=0$, $x_1=0$, $x_2=0$, we obtain

$$\phi_2^0 = \frac{wM+1}{M} \text{(28)}$$

Application of Eq. (22), (23) to this case yields

$$\tau_B + \tau_C = T, \quad \tau_B \tau_C = \frac{1}{M} \text{(29)}$$

Calculation of Eq. (24), from the above relationships, yields

$$(J_m)_{\min} = 2\sqrt{\frac{1}{M} + 2w} \text{(30)}$$

Fig. 4 shows the plots of $(J_m)_{\min}/(J_m)_{\min(M=1)}$ versus M , where $(J_m)_{\min(M=1)}$ is the value of $(J_m)_{\min}$ for $M=1$. In general, the decreasing trend of $(J_m)_{\min}$ with the increase of M is more remarkable for a smaller w , except for the case of $w=\infty$.

$$\text{6.2 Case b : } J_m = \int_0^T (w_1 + |x_1| + w|u|) dt$$

When J_m is given by

$$J_m = \int_0^T (w_1 + |x_1| + w|u|) dt \text{(31)}$$

the time behavior of the optimum control is of bang-bang type accompanied with $u=0$, as illustrated in Fig. 5, where

$$\left. \begin{aligned} \tau_B &= \phi_1^0 + \sqrt{(\phi_1^0)^2 + 2(\phi_2^0 + w)} \\ \tau_C &= \phi_1^0 + \sqrt{(\phi_1^0)^2 + 2(\phi_2^0 - w)} \end{aligned} \right\} \text{(32)}$$

Calculation of Eq. (31), from the above relation-

Fig. 4 $(J_m)_{\min}/(J_m)_{\min(M=1)}$ versus M for case a

ships, yields

$$(J_m)_{\min} = \sqrt{(2w_1+1)\left(\frac{2w_1+1}{M}\right) + 4w} \dots\dots\dots(33)$$

Fig. 6 and Fig. 7 show the plots of $(J_m)_{\min}/(J_m)_{\min(M=1)}$ versus M , for various values of w_1 in the case of $w=1.0$, and for various values of w for the case of $w_1=1.0$, respectively. In general, the decreasing trend of $(J_m)_{\min}$ with the increase of M is more remarkable for a larger w_1 and for a smaller w .

6.3 Case c: $J_m = \int_0^T (1+wu^{2n})dt$

When J_m is given by

$$J_m = \int_0^T (1+wu^{2n})dt \quad (n=1, 2, \dots) \dots\dots\dots(34)$$

H is maximum if

$$u = \begin{cases} M & \left(\frac{\psi_2^0 - \psi_1^0 \tau}{2nw} \right)^{1/(2n-1)} \geq M \\ \left(\frac{\psi_2^0 - \psi_1^0 \tau}{2nw} \right)^{1/(2n-1)} & \left| \left(\frac{\psi_2^0 - \psi_1^0 \tau}{2nw} \right)^{1/(2n-1)} \right| \leq M \\ -M & \left(\frac{\psi_2^0 - \psi_1^0 \tau}{2nw} \right)^{1/(2n-1)} \leq -M \end{cases} \dots\dots\dots(35)$$

For

Fig. 5 Time behavior of the optimum control for case b

Fig. 6 $(J_m)_{\min}/(J_m)_{\min(M=1)}$ versus M for case b with $w=1.0$

$$M \leq \left\{ \frac{1}{(2n-1)w} \right\}^{1/(2n)} \dots\dots\dots(36)$$

the time behavior of the optimum control has the saturated region as illustrated in Fig. 8, which shows the case of $n=1$, where

$$\left. \begin{aligned} \tau_B &= \frac{-2nwM^{2n-1} - \psi_2^0}{\psi_1^0} \\ \tau_C &= \frac{2nwM^{2n-1} - \psi_2^0}{\psi_1^0} \end{aligned} \right\} \dots\dots\dots(37)$$

Thus, $(J_m)_{\min}$ is obtained as

$$(J_m)_{\min} = 2\sqrt{\frac{(4n-1)(1+wM^{2n})^2 - 4n^2w^2M^{4n}}{(4n-1)M}} \dots\dots\dots(38)$$

For

$$M \geq \left\{ \frac{1}{(2n-1)w} \right\}^{1/(2n)} \dots\dots\dots(39)$$

the optimum control is given by

$$u = \left(\frac{\psi_2^0 + \psi_1^0 \tau}{2w} \right)^{1/(2n-1)} \dots\dots\dots(40)$$

where

$$\left. \begin{aligned} \psi_1^0 &= -4 \frac{w^{1/(2n)}}{n^{1/(2(2n-1))}} \sqrt{\frac{(2n-1)}{2(4n-1)}} \\ \psi_2^0 &= 2w^{1/(2n)} \end{aligned} \right\} \dots\dots\dots(41)$$

Thus, we obtain

$$(J_m)_{\min} = 4n \{ (2n-1)w \}^{1/(4n)} \sqrt{\frac{2}{(2n-1)(4n-1)}} \dots\dots\dots(42)$$

which is independent of M .

Fig. 7 $(J_m)_{\min}/(J_m)_{\min(M=1)}$ versus M for case b with $w_1=1.0$

Fig. 8 Time behavior of the optimum control for case c with $n=1$

Consider the case $n=1$. For $M \leq 1/\sqrt{w}$, $(J_m)_{\min}$ decreases with the increase of M and, in addition, this trend is more pronounced for a smaller w , while for $M \geq 1/\sqrt{w}$, $(J_m)_{\min}$ is constant with respect to M , as illustrated in Fig. 9 which is obtained from Eqs. (38) and (42). Fig. 10 shows the optimum trajectories for this case with $w=0.2$. From Fig. 10, it is observed that the optimum trajectory is the same for all $M \geq \sqrt{5}$.

It may be stated that the region, in which $(J_m)_{\min}$ is smaller than $(J_m)_{\min(M=1)}$, is smaller for a larger n .

7. The problems of fixed terminal time

7.1 Case d: $J_m = \int_0^T |u| dt$

The optimum control for the evaluation function J_m given by

$$J_m = \int_0^T |u| dt \quad (43)$$

is of the bang-bang type. Namely, for $MT^2 \geq 4$,

Fig. 9 $(J_m)_{\min}/(J_m)_{\min(M=1)}$ versus M for case c with $n=1$

Fig. 10 Optimum trajectories for case c with $n=1$ and $w=0.2$

the optimum control is given by

$$u = \begin{cases} M & 0 \leq \tau \leq \tau_c \\ 0 & \tau_c < \tau < \tau_B \\ -M & \tau_B \leq \tau \leq T \end{cases} \quad (44)$$

where

$$\begin{aligned} \tau_B &= \frac{T}{2} \left(1 + \sqrt{1 - \frac{4}{MT^2}} \right) \\ \tau_c &= \frac{T}{2} \left(1 - \sqrt{1 - \frac{4}{MT^2}} \right) \end{aligned} \quad (45)$$

Thus, $(J_m)_{\min}$ is obtained by

$$(J_m)_{\min} = MT \left(1 - \sqrt{1 - \frac{4}{MT^2}} \right) \quad (46)$$

For $MT^2 < 4$, this system is uncontrollable, and therefore the optimum control does not exist. Fig. 11 shows the plots of $(J_m)_{\min}/(J_m)_{\min(M=1)}$ versus M .

7.2 Case e: $J_m = \int_0^T (|x_1| + w|u|) dt$

The optimum control for the case given by

$$J_m = \int_0^T (|x_1| + w|u|) dt \quad (47)$$

is similar to the case of 7.1 and is given by Eq. (44), for $MT^2 \geq 4$. In this case, we obtain

$$(J_m)_{\min} = T \left(\frac{1}{2} + wM - wM \sqrt{1 - \frac{4}{MT^2}} \right) \quad (48)$$

Eq. (48) is reduced to

$$(J_m)_{\min} = \frac{T}{2} + w \left(\int_0^T |u| dt \right)_{\min} \quad (48.a)$$

where $\left(\int_0^T |u| dt \right)_{\min}$ is the minimum value of $\int_0^T |u| dt$, and is identical to Eq. (46), of which plots are shown in Fig. 11. Therefore, in general, the decreasing trend of $(J_m)_{\min}$ with the increase of M is more remarkable for a larger w and for a smaller T .

7.3 Case f: $J_m = \int_0^T u^{2n} dt$

Consider the case given by

Fig. 11 $(J_m)_{\min}/(J_m)_{\min(M=1)}$ versus M for case d

$$J_m = \int_0^T u^{2n} dt \quad (n=1, 2, \dots) \quad (49)$$

For $MT^2 < 4$, this system is uncontrollable.

For

$$2(4n-1)/(2n-1) \geq MT^2 \geq 4 \quad (50)$$

the time behavior of the optimum control is given by

$$u = \begin{cases} -M & 0 \leq \tau \leq \tau_c \\ \left(\frac{\phi_2^0 + \phi_1^0 \tau}{2n} \right)^{1/(2n-1)} & \tau_c \leq \tau \leq \tau_B \\ M & \tau_B \leq \tau \leq T \end{cases} \quad (51)$$

where

$$\left. \begin{aligned} \tau_B &= \frac{2nM^{2n-1} - \phi_2^0}{\phi_1^0} \\ \tau_c &= \frac{-2nM^{2n-1} - \phi_2^0}{\phi_1^0} \end{aligned} \right\} \quad (52)$$

$$\left. \begin{aligned} \phi_1^0 &= -\frac{4nM^{2n-1}}{T} \frac{1}{\sqrt{(4n-1)\left(1 - \frac{4}{MT^2}\right)}} \\ \phi_2^0 &= 2nM^{2n-1} \frac{1}{\sqrt{(4n-1)\left(1 - \frac{4}{MT^2}\right)}} \end{aligned} \right\} \quad (53)$$

Thus, we obtain

$$(J_m)_{\min} = M^{2n} T \left(1 - 2n \sqrt{\frac{1 - \frac{4}{MT^2}}{(4n-1)}} \right) \quad (54)$$

For

$$MT^2 \geq 2(4n-1)/(2n-1) \quad (55)$$

the optimum control is given by

$$u = \left(\frac{\phi_2^0 + \phi_1^0 \tau}{2n} \right)^{1/(2n-1)} \quad (56)$$

where

$$\left. \begin{aligned} \phi_1^0 &= -\frac{2^{2n+1}n}{T^{4n-1}} \left(\frac{4n-1}{2n-1} \right)^{2n-1} \\ \phi_2^0 &= \frac{2^{2n}n}{T^{4n-2}} \left(\frac{4n-1}{2n-1} \right)^{2n-1} \end{aligned} \right\} \quad (57)$$

The optimum control for the case given by Eq. (55) has not saturated regions and is independent of M . Thus, we obtain

Fig. 12 $(J_m)_{\min}/(J_m)_{\min}(M=1)$ versus M for case f with $n=1$

$$(J_m)_{\min} = \frac{2^{2n}(4n-1)^{2n-1}}{(2n-1)^{2n} T^{4n-1}} \quad (58)$$

Fig. 12 shows the plots of $(J_m)_{\min}/(J_m)_{\min}(M=1)$ versus M , for the case $n=1$. For $6/T^2 \geq M \geq 4/T^2$, the decreasing trend of $(J_m)_{\min}$ with the increase of M is remarkable for a smaller T , but, for $M \geq 6/T^2$, $(J_m)_{\min}$ remains constant with respect to M . In general, the decreasing trend of $(J_m)_{\min}$ is smaller for a larger n .

8. Conclusion

In this paper, a synthetic criterion J including the initial investment is presented, and it is shown that the design of M is determined by making use of J in the case of the optimization of a system under a constraining condition $|u| \leq M$ imposed on the manipulated variable u .

Concerning the relationship between $(J_m)_{\min}$ and M , the following conclusions are obtained:

(1) In the case where the optimum control is of the bang-bang type, or of the bang-bang type accompanied with $u=0$, $(J_m)_{\min}$ decreases with the increase of M . On the other hand, in the case where the optimum control includes the continuous actions, $(J_m)_{\min}$ decreases with the increase of M up to a certain value of M , but remains constant for the larger value of M .

(2) In the case when the terminal time T is not fixed, the decreasing trend of $(J_m)_{\min}$ with the increase of M is more remarkable for the heavier weighting imposed on the time optimum control. In the case when T is fixed, the decreasing trend of $(J_m)_{\min}$ with the increase of M is more pronounced for a smaller T .

The choice of M is mainly shown in this paper. However, the approach proposed may be extended to the problems of how the plants should be designed to achieve the integrated optimization including the operation.

As the result, it has been shown that it is necessary to consider M as one of the design parameters and to examine whether the synthetic criterion given by Eq. (4) is satisfied or not, in the realization of optimum controls.

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Discussion

T. MITSUMAKI: (1) Optimum design of plants at the stage of plant design is being attempted recently. However, the criteria for optimum in this case are mainly based on past experiences. This paper is of much interest, for treating the optimization of the total system from the plant design to the operation.

In the example of this paper, J_m and J_e are treated separately. Is it possible to solve this kind of problem, by formulating a single equation in general? It seems that the development of the general approach is necessary to solve these problems described above. I would like to have your opinion about this point.

T. TERANO: (2) I appreciate this paper which pointed out the influences of the constraining conditions on the optimal value of the criterion function, in a concrete form.

It is an interesting idea to design a control systems by taking the initial investment into account. However, I am afraid it might be difficult to know the relationship between M and J_e . In the optimum design, there can be more important factors other than J_m and J_e , for instance, the expenditure, the maintainability, the reliability, the time required for construction, and so on.

R. MORI: (3) For the optimization approach proposed in this paper, the designation of the initial state and the final state is necessary. Therefore, I am afraid that this approach might be difficult to apply to the design problems of the constant value control systems encountered frequently in process control. In computer control systems, the optimum condition is maintained by adjusting the system parameters, and the system parameters should be treated as the manipulated variable u . The author's method seems to be insufficient for dealing with such a kind of problem. In addition, the formulation of the criterion function for such cases seems to become complicated, since J_m is described as the summation of the integrals over each period, and J_e is, by definition, obtained by integrating from zero to infinity in time.

(4) In the example shown in this paper, the constraining condition is given as the saturation of force. In process control, it is necessary to estimate the suitable values of saturation, for the position, the velocity and the acceleration, in the choice of components. Is there any approach to determine which of these variables is dominant

as a constraint by saturation?

(5) In process control, the variation of gain of each element under constant overall loop-gain will also influence the cost of the system. How can this fact be introduced into the optimization approach you proposed?

Author's closure

(1) J_m and J_e are treated separately, since one of the aims of this paper is to show the influence of increasing M on J_m . However, the development of the general approach will be required, especially, for the problems of determining many other design parameters at the same time. But it may be a difficult problem, which is related closely to the sensitivity of the system.

In some cases, J_m should be modified so as to include the evaluation for the static optimization in addition to that for the dynamic optimization.

(2) The general trend concerning the relationship between J_e and M in the case of a motor or a control valve, can be known from its specification or its catalogue. If we have to choose one type out of some products given in advance, we had better select the one which gives the minimum value of J , comparing all J 's for given products.

(3) In this paper, the case when both the initial and final states are defined has been treated. However, the approach proposed may be extended to the problems of the constant value control systems or the computer control systems, by introducing the evaluation function J_m in a stochastic form such as the integral of square error, or by introducing the evaluation function J_m representing the evaluation for the static optimization in addition to that for the dynamic optimization.

In many computer control systems, J_e can be well determined in advance from the data concerning the cost of computer and the period of the depreciation. The factors which are not determined in advance, can be included in J_m .

(4) The author would like to study this point in the near future.

(5) The approach proposed will be extended to these problems, by modifying J_m as the author mentioned in his reply (3), if the informations concerning the disturbances to the process are known beforehand.