On the Momentum-Integral Method of Solution for the Laminar Entrance-Flow Problems*

By Shingo ISHIWA**

For the laminar entrance-flow problem, the present author developed previously a new momentum-integral method of solution in which the independent parameter \( \Gamma = \frac{h^2}{U} \left( \frac{\partial^2 U}{\partial y^2} \right)_{y=h} \)
where \( h \) is the half gap width and \( U = [u]_{y=h} \), was introduced together with the usual Pohlhausen's parameter \( A = \frac{\partial^2}{\nu} \frac{dU}{dx} \). The method was applied to solve the problem of the radially outward entrance-flow between two parallel discs, and it was confirmed that the theory might describe satisfactorily the complicated characteristics of the fluid flow which involved separation and reattachment phenomena.

In the present paper, the author's method is further applied to analyze the most fundamental case, the two-dimensional flow between flat plates, and is compared with prior methods of solution. The result of the author's analysis agrees very well throughout the whole flow field with the exact solution given by a difference method: faults inherent in many other momentum-integral methods proposed to date are eliminated from the present method.

It can be emphasized from the present research that, in order to describe satisfactorily the laminar entrance-flow phenomenon in terms of a momentum-integral method, it is most indispensable to introduce \( \Gamma \) as an unknown variable to be solved.

1. Introduction

Because of its importance in many hydrodynamical applications, the entrance-flow problem has been very much investigated to date. Especially, the case of a steady and laminar flow of an incompressible fluid, i.e., the most fundamental case, has been calculated theoretically by many investigators with various methods of analysis. As the result, several methods of analysis have been found to give good results for flows in some simply shaped channels, such as a pipe and a straight channel formed by two parallel plates(1)(2)(5)(13). However, in spite of the prior efforts any satisfactory method of analysis seems to have not yet been proposed to date, the method which can describe accurately the complicated flow behaviour in a channel which changes its sectional area arbitrarily in the stream direction. From the standpoint of practical applications, it may be said that further investigations for the above-mention-

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* Received 12th October, 1965.
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potential flow has disappeared; and so forth.

These faults seen in the prior theories seem to result from the fact that, for the entrance-flow which belongs to the so-called internal flow phenomenon, those theories employed without any essential correction the techniques of the Pohlhausen's and subsequent methods which were originally constructed for the ordinary boundary-layer flow as an external flow phenomenon, and those theories did not take into consideration the important characteristics inherent only in the internal entrance-flow phenomenon. An important characteristic inherent only in the internal entrance-flow, in comparison with the external boundary-layer flow, will be that, in the downstream region where the potential flow has disappeared, viscosity affects the fluid motion at "every" portion, naturally including the outer edge of the boundary layer (note that this edge coincides with the center of the channel). In contrast with this, in the external boundary-layer flow the influence of viscosity clearly disappears at the edge and the outside of the boundary layer.

What is stated above can be rewritten as follows. Let \( x \) and \( y \) be the coordinates respectively along and normal to a wall surface, \( u \) and \( v \) the velocity components in the \( x \)- and \( y \)-directions respectively, \( \delta(x) \) the thickness of the boundary layer, \( h(x) \) the half channel-width and \( U(x) \) the mainstream velocity or the velocity at the center of the channel. Then, in the ordinary boundary-layer flow \( u=U(x) \) and \( \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \cdots = 0 \) at \( y=\delta(x) \), whereas in the entrance-flow in the region where the potential flow has disappeared \( u=U(x) \) and \( \frac{\partial u}{\partial y} = 0 \) (because of the symmetry of the flow pattern) at \( y=\delta(x) \equiv h(x) \) but \( \frac{\partial^2 u}{\partial y^2} \) at \( y=\delta(x) \equiv h(x) \) is generally not equal to zero since the influence of viscosity does not disappear there. Furthermore, \( \left( \frac{\partial^2 u}{\partial y^2} \right)_{y=h} \) should be considered to be an unknown function of \( x \) because the velocity profile of the fluid flow continues to change far downstream of the position \( x=x_f \), where the potential flow disappears. Namely, an important characteristic inherent in the entrance-flow phenomenon is that \( \left( \frac{\partial^2 u}{\partial y^2} \right)_{y=h} \) in the region where \( x>x_f \), is not known at the beginning, and is to be determined so as to satisfy the relations of the momentum-integral and the like.

Taking account of what is mentioned above, the present author developed previously a new momentum-integral method of solution for the laminar entrance-flow problem, in which the non-dimensional parameter \( \Gamma = \frac{h^2}{U} \left( \frac{\partial^2 u}{\partial y^2} \right)_{y=h} \) together with the ordinary Pohlhausen's parameter \( \Lambda = \frac{\partial^2 U}{\partial x^2} \left( \frac{\partial^2 U}{\partial y^2} \right)_{y=h} \) was introduced. The new method was applied to calculate the characteristics of the radially outward flow between two parallel disks, from which it was confirmed that the new method was free from the faults seen in the prior methods and could describe satisfactorily the complicated flow behaviours including separation and reattachment phenomena.

The purpose of the present paper is to examine the applicability of the new method to different cases and to confirm the necessity and validity of introducing the new parameter \( \Gamma \) into the momentum-integral method for the entrance-flow problem, by applying the new method further to the simple and fundamental case, i.e., the two-dimensional entrance-flow in a straight channel formed by two parallel plates.

### 2. Results of analysis and discussions

In the previous paper the author developed, on the basis of what is stated in chapter 1, a new theory based on the momentum-integral method which covers generally the laminar meridional flow in an arbitrarily shaped narrow gap between two axi-symmetrically formed walls. The velocity profile was approximated in a polynomial in \( \frac{y}{\delta(x)} \) of the fourth-degree including the two shape factors \( \Lambda \) and \( \Gamma \), and the radius of the wall \( r \) and the gap-width \( 2h \) were both arbitrary functions of \( x \), where \( x \) is the meridional coordinate along a wall surface, the origin of \( x \) being positioned at the inlet to the gap.

Now, consider the case where \( r \) and \( h \) are both constants independent of \( x \). This corresponds to the axial flow between two concentric cylinders. Here, in the author's theory it is assumed that \( h \) is very small compared with the radius of the wall curvature, so that the axial flow between two concentric cylinders which is described in terms of the present theory is nothing but the two-dimensional flow between two parallel flat plates. From this it follows that the flow to be considered in this paper can be calculated immediately from the author's theory by putting

\[
r = r_0, \text{ const.}; \quad h = h_0, \text{ const.} \tag{1}
\]

Concerning further details of the theory and the method of numerical calculation, refer to the papers (16) and (17).
The results of the numerical calculations are shown in Table 1 and Figs. 1 and 2 for the case when the fluid flows into the gap with a uniform velocity $U_0$ at the inlet $x=0$. In these tables and figures the results of other methods of solution are shown together for the purpose of comparison.

Fig. 1 Velocity distribution $u/U_0$ in the inlet region

Fig. 2 Distribution of the additional pressure-loss $C(x)$
son. Here, comparison with some experimental results will also be advisable; however any experimental result for the flow which can be regarded as almost perfectly two-dimensional has not been obtained to date.

The analysis by Schlichting\(^{(4)}\) is such that for the flow near the entrance the boundary-layer equations are solved in a series form down to the third term and an approximate extrapolation to the fourth-order is added, and for the flow in the downstream region an eigen function in the first-order perturbation to the asymptotic Poiseuille flow at \(x = \infty\) is obtained to be joined smoothly to the solution near the entrance. Collins and Schowalter\(^{(5)}\) improved the accuracy of this Schlichting’s solution by means of extending for the upstream region the series solution to the sixth term and adding for the downstream region further two eigen functions. Bodoia and Osterle\(^{(1)}\) integrated the boundary-layer equations purely numerically by the use of a difference method.

The method of solution by Sparrow, Lin and Lundgren\(^{(13)}\) is an extension of the approximating technique of linearizing the nonlinear inertia terms in the boundary-layer equations, which was originated by Langhaar\(^{(11)}\). In this method, the boundary-layer equation is rewritten approximately as

\[
\varepsilon(x) U_b \frac{\partial \mu}{\partial x} = P(x) + \nu \frac{\partial^2 \mu}{\partial y^2} \] (\(U_b\): mean velocity)

and the pressure term \(P(x)\) and the weighting function \(\varepsilon(x)\) are determined from the relations of the momentum- and energy-integrals of the above equation. Schiller\(^{(6)}\) analyzed the entrance-flow in a pipe by a momentum-integral method employing the simple second-order polynomial approximation for the velocity profile, \(\frac{\mu}{U_b} = 2 \left( \frac{y}{h} \right) - \left( \frac{y}{h} \right)^2\).

The present two-dimensional flow between parallel plates can also be analyzed by the method quite similar to this Schiller’s one: the results are shown as the “Schiller-type”. According to this method the boundary-layer thickness \(\delta(x)\) is obtained in the closed form of

\[
\left( \frac{48 - \frac{7 \delta}{h}}{\delta} \right) \left( \frac{\delta}{h} \right) + 4 \log \left[ 1 - \frac{1}{3} \frac{\delta}{h} \right] = -10 \left( \frac{\nu x}{U_b h_b^2} \right) \]

Fig. 1 shows the velocity distributions \(\frac{\mu}{U_b}\) along the parallel lines at \(\frac{y}{h_b} = 0.1, 0.3, 0.5\) and 1.0. The non-dimensional quantity \(C(x)\) in Fig. 2 is defined by

\[
\frac{p_0 - p(x)}{\frac{1}{2} \rho U_b^2} = 6 \left( \frac{\nu x}{U_b h_b^2} \right) + C(x) \] .................(2)

where \(p(x)\) is the fluid pressure and \(p_0 = p(0)\). Here, as is well-known, the relation of Eq. (2) from which the term \(C(x)\) is dropped,

\[
\frac{1}{2} \rho U_b^2 = 6 \left( \frac{\nu x}{U_b h_b^2} \right),
\]

expresses the pressure distribution in the case when the flow assumes the Poiseuille parabolic velocity distribution from the start at the inlet section. \(C_m\) in Table 1 is the asymptotic value of \(C(x)\) at \(x = \infty\),

\[
C_m = \lim_{x \to \infty} C(x) \] ......................(3)

and this is called the additional pressure-loss coefficient.

Among the solutions of the theoretical analysis made to date, the most accurate solutions are the Bodoia and Osterle’s\(^{(1)}\) and the Collins and Schowalter’s\(^{(5)}\). In fact, the both results agree very well with each other as is shown in Table 1, though they are obtained by quite different methods of solution. The result of the present method best approximates these solutions not only in the values of \(C_m\) and \(C(x)\) but in the velocity distribution too, as can be seen from Table 1 and Figs. 1 and 2. The Schlichting’s analysis employs the same method of solution as the Collins and Schowalter’s, but his analysis results in noticeable errors especially for the values of \(C(x)\) and \(C_m\) because of insufficiency of the number of the terms calculated. In contrast with this, the present method gives much better results throughout the whole flow field than the Schlichting’s, though it is based on an approximated theory. The method, in which series and asymptotic solutions are joined to each other, is in principle precise, and may be considered to give good accuracy if a sufficient number of terms are calculated, as is shown in the Collins and Schowalter’s analysis.

<table>
<thead>
<tr>
<th>Method of solution</th>
<th>Additional pressure-loss coefficient</th>
<th>Velocities along the channel-center (U/U_b)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(C_m )</td>
<td>(\nu x/U_b h_b^2)</td>
</tr>
<tr>
<td>Present method</td>
<td>0.660</td>
<td>1.20</td>
</tr>
<tr>
<td>Bodoia-Osterle(^{(1)})</td>
<td>0.676</td>
<td>1.21</td>
</tr>
<tr>
<td>Collins-Schowalter(^{(5)})</td>
<td>0.678</td>
<td>1.20</td>
</tr>
<tr>
<td>Schlichting(^{(4)})</td>
<td>0.601</td>
<td>1.24</td>
</tr>
<tr>
<td>Sparrow-Lin(^{(13)})</td>
<td>0.650</td>
<td>1.23</td>
</tr>
<tr>
<td>Lundgren(^{(14)})</td>
<td>0.627</td>
<td>1.21</td>
</tr>
</tbody>
</table>
However, in this method of solution sometimes it is difficult or impossible to obtain the asymptotic solution for the downstream region especially when the section of the channel changes its area arbitrarily in the stream direction. From this, it can be said that this method has not generality in application. Next, the Sparrow, Lin and Lundgren's theory does not hold exactly in physical meanings; nevertheless their method gives better results than the Schlichting's. A merit of this method is that, at least in principle, it can be applied even to the case of a complicated form of the channel-section, not being restricted to simple channel-sections such as a circular pipe, a two-dimensional straight channel, etc. However, it must be noted that this method fails to describe accurately the flow behaviour in a channel of which sectional area varies arbitrarily in the stream direction and as the result where there appears the inertia effect strongly, because this theory does not treat exactly the inertia force of the fluid flow. In consequence this method is effective for the case where the form of the channel-section is complicated but is retained unchanged in the stream direction. The Schiller-type method, which is based on the very simple assumptions, gives results similar to the Bodola and Osterle's accurate solution for the region upstream of the position \( \frac{\nu x_f}{U_0 h^2} = 0.1038 \) where the potential flow disappears. However, this method results in a noticeable discontinuity at the position \( x=x_f \) and also gives a considerably shorter inlet length than in the actual phenomena, because in this method the velocity profile is assumed to be a similar parabolic distribution throughout the whole flow field. Further, it is clear that this method can not give an accurate description of the flow in a channel with a largely varying sectional area, because the flow noticeably changes its velocity profile there.

Now, according to the present method, the position where the potential flow disappears is \( \frac{\nu x_f}{U_0 h^2} = 0.0586 \). As can be seen from Figs. 1 and 2, both the pressure and the velocity of the fluid flow change very smoothly through this position. This will be an advancement from the prior momentum-integral methods. Next, the value of \( C \) at \( x=x_f \) is 0.546 while the asymptotic value of \( C \) at \( x \to \infty \), \( C_\infty = \lim_{x \to \infty} C(x) \), is 0.660: the additional pressure-loss increases more than 20% after the potential flow has disappeared. This means that the flow characteristics change largely even after the potential flow has disappeared. Thus the present method can describe satisfactorily the asymptotic behaviour of the fluid flow downstream of the position \( x=x_f \). It is to be noted that almost all of the prior momentum-integral methods fail to describe this asymptotic flow behaviour. The above-mentioned facts, that the present method results in a very smooth change of fluid flow through the position \( x=x_f \), and also that the method can describe satisfactorily the asymptotic flow behaviour in the downstream region, are obviously due to the introduction of \( \Gamma = \frac{k^2}{U} \left( \frac{\partial^2 U}{\partial x^2} \right)_{x=h} \) into the theory as an independent parameter to be solved.

3. Conclusions

For the laminar entrance-flow problem, the present author developed previously a momentum-integral method of solution in which the new independent parameter \( \Gamma = \frac{k^2}{U} \left( \frac{\partial^2 U}{\partial x^2} \right)_{y=h} \), together with the usual Pohlhausen's parameter \( A = \frac{\nu}{\nu} \frac{dU}{dx} \) (or \( \kappa = \frac{\nu}{\nu} \frac{dU}{dx} \)), was introduced. The method was applied to solve the problem of the radially outward entrance-flow between two parallel disks, and it was confirmed that the theory may describe satisfactorily the complicated characteristics of the fluid flow which involves separation and reattachment phenomena.

In the present paper, the author's method has been applied further to analyze the most fundamental case, the two-dimensional flow between parallel flat plates, and has been compared with prior methods of solution. The result of the author's analysis agrees well throughout the whole flow field with the exact solution given by a difference method, and it can be confirmed that the faults inherent in many other momentum-integral methods proposed to date are eliminated from the present method.

It can be emphasized from the present research that, in order to describe satisfactorily the laminar entrance-flow phenomenon in terms of a momentum-integral method, it is most indispensable to introduce \( \Gamma \) into the theory as an unknown variable to be solved.

Originally, the author's new theory was developed in the paper (16) for the general case where fluid flows meridionally in an arbitrarily shaped narrow gap between two axi-symmetrically formed walls, in which general case the above-mentioned radial flow between parallel discs and two-dimensional flow between parallel plates are included. The same theory can of course be applied also to other cases than those mentioned.
above, for example, to the flows in a pipe and in a circular duct with arbitrarily varying sectional area, and the like. In this connection, for the case of a pipe Hishida\(^{10}\) analyzed the flow in the region where potential flow exists assuming a fourth-order polynomial velocity profile: the present method can be joined to this Hishida's calculation to obtain the solution for the farther downstream region.

It is possible, if necessary, to improve the accuracy of the present method furthermore. For example, if we employ the energy-integral equation in place of the compatibility condition at the wall, 

\[
\frac{1}{\rho} \frac{dp}{dx} - \rho \left( \frac{\partial u}{\partial y} \right)_{y=0}^2, \quad \text{as in the Tani's method}\(^{15}\) which succeeded in giving good results for the ordinary boundary-layer problem, more precise solutions will be able to be obtained for the flow including separation and reattachment phenomena.

Acknowledgements

The author wishes to acknowledge the kind instructions by Professor S. Uchida and Professor Y. Furuya of Nagoya University, and by Dr. E. Hori of Mechanical Engineering Research Laboratory, Hitachi Ltd., and also to thank Miss T. Mizunuma for her numerical works for this study.

References

(6) L. Schiller: Z. AMM, Bd. 2 (1922), S. 96.
(20) K. Pohlhausen: Z. AMM, Bd. 1 (1921), S. 252, etc.

Discussion

Y. Tanida: (1) It seems to the present questioner that the momentum-integral method does not always give accurate result for the position where potential flow disappears (in this paper \(\frac{\nu_x}{U_0 h_0^2} = 0.0586\)). An explanation will be appreciated for this point.

(2) It is interesting that in the author's previous papers\(^{16}(17)\) the occurrence of a separation near the entrance is indicated theoretically for the radial flow between parallel discs. However, the following question arises here: Is it permissible to assume that \(\frac{\partial y}{\partial y} \sim 0\) for the separation region?

(3) Also, when separation occurs near the entrance, the flow at the inlet section will be distorted to some extent, by the separation. If so, then it follows that the flow distortion at the inlet will again change the condition of the flow downstream of the inlet. Is it not necessary to correct the assumed condition \(u = U_0, \text{const.}\) at the inlet \(x = 0\)?

E. Sugino: (4) The present theory indicates that the position \(\frac{\nu_x}{U_0 h_0^2} = 0.136\) chosen in Table 1?

(5) An illustration of the solution for a flow including separation and reattachment phenomena will be appreciated.

(6) The author's analysis is valid only when \(h < r\) so that this method can not calculate the case of an axial flow between concentric cylinders where it is not permissible to put \(h < r\). It will be advisable to develop an analysis for such a case too.

M. Hishida: (7) In applying the present method to the entrance-flow in a pipe, we encounter some difficulties. For example, in the relation between the pressure and the velocity on the axis, which is obtained from the boundary-layer equa-
tions as
\[ \frac{dU}{dx} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \left( \frac{\partial^2 u}{\partial y^2} \right)_{y=h} - \nu \left( \frac{1}{h-y} \frac{\partial u}{\partial y} \right)_{y=h} \]
the third term in the right hand side becomes 0/0. The writer's question is whether the present method has insurmountable difficulties in calculating the inlet flow in a circular pipe.

**Author's closure**

(1) Strictly speaking, there is no definite boundary which distinguishes boundary layer from mainstream, so it is impossible to determine physically definitely the position where the potential flow disappears. The "position where the potential flow disappears" can only be defined in the approximate momentum-integral method. Accordingly, if we wish to examine the accuracy of a momentum-integral method by comparing it with a more exact method of solution, comparison should be made between the values of the fluid pressure, velocity and the like, and not of the position where the potential flow disappears. The present method in which the new parameter \( I = \frac{h^2}{U} \left( \frac{\partial u}{\partial y} \right)_{y=h} \) is introduced gives good results for those quantities, as has already been confirmed.

(2) For the ordinary boundary-layer flow, the nature of the separation phenomenon has been examined and analyzed mathematically by many investigators. In general, the separation point of the ordinary boundary-layer flow is a mathematical singularity of the boundary-layer equations. On the other hand, the separation of the entrance-flow is different in nature from this ordinary boundary-layer separation. This problem has been investigated in detail in the author's papers (16) and (17) [especially in (17)], and in conclusion it has been demonstrated that in the entrance-flow the separation and the reattachment phenomena have not any singular nature, and, since breakaway of flow from the wall does not occur\(^{(11)}\), the boundary-layer approximation (to assume that \( u > v \) or \( \frac{\partial p}{\partial y} = 0 \)) holds true even for a region where a reversed flow exists. For further details refer to the author's papers (17) and (16).

(3) It has been confirmed in the paper (17) that in the entrance-flow, a breakaway of flow from the wall can not take place even if separation occurs. And, in the actual radial flow between parallel discs, the gap-width is in general very small compared with the length from the inlet to the separation point. From these two points it may be said that the potential flow at the inlet section is practically unaffected by the separation.

In practical applications, the influence of the boundary layer existing at the inlet will be more important than the problem under consideration. For this, refer to the numerical examples in the author's paper (16) from which it can be seen that the boundary layer existing at the inlet influences largely the flow in the downstream region.

(4) In the cases of a pipe, a two-dimensional straight channel, etc., the inlet region may be considered to extend to the position at which the flow "nearly" perfectly assumes a Poiseuille parabolic velocity distribution. Strictly speaking, however, it is impossible to determine that position physically definitely because theoretically the approach to the Poiseuille flow is only attained asymptotically. (It should be noted here that the prior momentum-integral methods, especially the Schiller-type methods, are unsatisfactory in this respect because they fail to describe the above-mentioned asymptotic flow behaviour in the downstream region.)

Practically, it is usual to define the inlet region as extending to the position at which the velocity at the channel-center reaches some 98% or 99% of that of the fully developed Poiseuille flow. The position of \( \frac{v}{U_h} = 0.136 \) in Table I, is not of the author's choice, but of one by Collins and Schowalter: in their analysis, at this position the velocity at the channel-center reaches just 98% of that of the Poiseuille flow.

(5) The author would like the questioner to refer to the author's paper (16), where some numerical examples for the radial flow between parallel discs which includes the separation and reattachment phenomena are shown.

(6) The author agrees in opinion with the questioner. The case in question is a problem to be solved hereafter.

(7) The case in question can be solved in the following way.

The Taylor expansion of \( \frac{\partial u}{\partial y} \) with respect to \( y \) about \( y = h \) is, noting that \( \frac{\partial u}{\partial y} \bigg|_{y=h} = 0 \),
\[
\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \bigg|_{y=h} (h-y) + \frac{1}{2!} \frac{\partial^3 u}{\partial y^3} \bigg|_{y=h} (h-y)^2 \quad \cdots
\]
Hence, the third term of the right hand side of the relation in question becomes
A Simplified Analysis of Cascade Flow in Mixed-Flow Pumps and Turbines*

By Hisaaki Daiguji**

The direct problem on the cascade flow in mixed- and radial-flow pumps and turbines has been solved by various methods. Among these the method of using the finite difference calculus and the relaxation method is sufficiently applicable to cascades with arbitrary blade shapes and with varying passage-heights and passage-angles. In this paper, such a numerical method is improved in the way of division of the stream function and on three relaxation procedures to save the labour for calculation and to increase the accuracy of the solution. Then three numerical examples are presented to examine the accuracy, to compare with the experimental data and to show the applicability of the method.

1. Introduction

The methods of analysis on the flow through the cascade in mixed- and radial-flow turbo-machines with arbitrary blade shapes and with varying passage-heights have been presented by A. J. Acosta(1) and H. Murai(2) who obtained the potential function by a method applying conformal transformation and a method of singularities, respectively, and by J. D. Stanitz(3) and C. H. Wu(4) who solved numerically a boundary value problem for the stream function by using the finite difference calculus(5).

The method developed by Wu for the compressible flow is simplified for the incompressible flow in pumps and turbines taking into account the linearity of the differential equation. Namely, the numerical calculation in the presented method employing the Southwell’s relaxation method(6) is modified in (i) the relaxation in the upstream and downstream regions, (ii) the treatments of the profile contours, and (iii) the calculation for the flow round a corner, to save the labour for calculation and to increase the accuracy of the solution.

The method is applicable to the cascade in mixed-flow (radial-flow) turbomachines with arbitrary blade shapes and varying passage-heights and passage-angles. However, it is necessary that the geometry of cascade be sufficiently smooth and that the passage-height be reasonably small. When the procedures of calculation are suitably planned, it is possible to determine more easily the approximate flows and to examine in detail the local flows.

Nomenclature

The following nomenclature is used in the paper:

- $B$: passage-height at radius $R$ 
- $b$: passage-height 
- $c_p$: static pressure coefficient 
- $g$: gravitational acceleration 
- $H$: theoretical head 
- $h$: net spacing, Fig. 3