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A Study on the Elastodynamic Pump*

(1st Report, Normal Stress Effect)

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The elastodynamic extruder or pump was developed in 1959 by Maxwell and Scalora. This plastic extruder is based on the phenomenon called Weissenberg effect. In this paper pressure distributions for no discharge flow and discharge flow are investigated using the experimental apparatus similar to the above mentioned elastodynamic pump. Moreover pressure distributions are approximately analyzed by using the combination model of Rivlin's large deformation theory with the power law for the non-Newtonian viscosity, and the 3-constants Oldroyd model respectively. Those analytical results are compared with experimental data, and it is found that the former model may be more convenient than the latter model for practical purpose.

1. Introduction

Pumping action of an elastodynamic pump or extruder is based on the normal stress effect or Weissenberg effect. This pump was developed in 1959 by Maxwell and Scalora⁽¹⁾ as the plastic extruder. It is predictable that the elastodynamic pump may have the following applications as the plastic melt extruder. That is, elimination of bubbles, wire coating, drawing the pipe and blending of some polymers. On the other hand, this pump may be applied as the spinning pump for polymer solutions. According to Maxwell and Scalora, the elastodynamic extruder has some merits that the residence time in the extruder is short compared to the conventional extruders, heat transfer is facilitated, mixing and dispersion can be achieved in a short time, pressure pulsations at the die are avoided, and the construction is simple and has flexibility. On the other hand, as its shortcoming, they pointed out that the pressures developed in the extruder and the amount of discharge rate are considerably low. Maxwell and Scalora have carried out experimental investigations, but they have not tried theoretical treatments. The authors investigated the performance of the elastodynamic pump both experimentally and analytically. In this report, pressure distributions due to the normal stress effect are studied experimentally and analytically.

The pressure distributions are approximately obtained by using the 3-constants Oldroyd model, and the combination model of Rivlin's large deformation theory with the power law for the non-Newtonian viscosity respectively. Some experimental results are compared with the analytical results. Thus, it is found that in the working range of the pump, the latter model may be more convenient than the former for practical problems.

2. Experiment

2.1 Experimental apparatus and procedure

The general arrangement of experimental apparatus is shown in Fig. 1. The cover plate ② is adjusted by the screw to obtain the settled gap between that and the rotating disk ③. The experiments are carried out when the gap sizes are 1, 3 and 5 mm respectively. Pressure distributions are measured by the diaphragm type strain meter, in which the liquid material is introduced through tubes from pressure measuring holes ⑤, ⑥ etc. which are bored on the cover plate. Liquid material is fed into the apparatus through the hopper ⑦. Rotational speeds are varied in the range from 200 rpm to 800 rpm with 100 rpm steps. The diameters of rotating disks are 80, 109 and 144 mm respectively. Pressure distributions are measured for various rotational speeds, rotating disks and gap sizes.

For convenience, the case of no discharge flow through the nozzle located at the center of the cover plate is called "no flow." Such a case will correspond to the parallel plate type rheogoniometer at high range of shear-rate. Then the ex-

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trudate nozzle is mounted instead of the pressure measuring tap ⑤ located at the center of the cover plate ②, and then pressure distributions and flow rates are measured for several rotating disks, varying gap sizes and rotational speeds.

Used nozzles in this experiments are 3, 5 and 8 mm in diameter and 55 mm in length. For brevity, the case of discharge flow is called "flow". The temperature of fluid is measured by a thermocouple inserted in the cover plate, and it is maintained at $18^{\circ}\text{C} \pm 2^{\circ}\text{C}$ with cooling water through the jacket. Moreover, pressure distributions at low range of shear-rate are measured by using a Greensmith-Rivlin's parallel plate type rheogoniometer. Rotational speeds in this apparatus are 2~250 rpm and gap sizes 1, 3 and 5 mm. Manometer is used for the measurement of pressure distribution.

Three kinds solutions of polyisobutylene in decalin are used, and their physical properties are listed in Table 1. In order to obtain stabilized physical properties of the fluid, the solution is left at room temperature for about ten days, and it is circulated in the apparatus before the experiment. The values of rheological constants of the fluid are obtained by using a Couette's type viscometer for low rate of shear and a capillary viscometer for high rate of shear. The end correction of the capillary viscometer is made by Bagley's method⁽²⁾. Also, two constants λ_1 , λ_2 (λ_1 : relaxation time, λ_2 : retardation time) in the Oldroyd model are decided from the calculated

curve by Bird et al.⁽³⁾ Those values are shown in Table 1.

2.2 Experimental results

Examples of experimental results are shown in Fig.2 in which (a) shows the pressure distributions in the case of "no flow", (b) that in "flow" and (c) the values of flow rate. From Fig. 2. (a), (b) it is found that the pressure in the case of "flow" may be slightly lower than "no-flow" for the same operating conditions. Also, the flow rate increases with a decreasing ratio of nozzle length to diameter. The investigations on flow rate will be discussed in the 2nd report.

3. Analytical treatment of "no-flow"

Now, cylindrical co-ordinates (r , θ , z) are used as shown in Fig. 1, and v_r , v_θ and v_z are velocity components in the co-ordinates directions respectively. The flow pattern in this case is axial symmetry, so that we put $\partial/\partial\theta=0$. Also,

Table 1 Physical constants of materials

Material remarks	No. 1	No. 2	No. 3
Molecular weight	10 430	100 000	60 000
Concentration in weight %	20	8	6
Specific weight (20°C)	0.90	0.905	1.04
n^*	1.35	1.83	2.70
μ_{psu}^* $\text{g}^n/\text{cm}^n \text{sec}^{2n-1}$	2.71×10^3	1.29×10^5	9.59×10^8
m^*	0.71	0.54	0
η_0 P	2.86×10^2	1.50×10^3	
λ_1 s	2.10×10^{-1}	2.45	
λ_2 s	1.39×10^{-1}	8.56×10^{-1}	

* Values in working range of the pump

Fig. 1 Experimental apparatus, coordinates and nozzle dimensions

the secondary flow which may occur due to the centrifugal force is neglected. Then, required conditions are written as follows:

$$v_r = v_z = 0, \quad v_\theta = f(r, z) \quad (1)$$

The stress tensor component τ_{ij} is expressed in the following form:

$$\tau_{ij} = -p\delta_{ij} + P_{ij} \quad (2)$$

where P_{ij} is the deviatoric stress tensor component, p the pressure and δ_{ij} the Kronecker delta. From these, the equations of motion become

$$-\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r P_{rr}) - \frac{1}{r} P_{\theta\theta} \quad (3)$$

$$0 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{\partial \tau_{\theta z}}{\partial z} \quad (4)$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial P_{zz}}{\partial z} \quad (5)$$

where ρ is the density of fluid. Now, the boundary conditions are

$$\left. \begin{aligned} v_\theta &= r\omega \text{ at } z=0 \\ v_\theta &= 0 \text{ at } z=H \end{aligned} \right\} \quad (6)$$

where ω is the angular velocity of rotating disk and H the gap size. Since $\partial p / \partial \theta = 0$, the flow pattern is considered as a drag flow. Hence the velocity distribution of v_θ in the z -direction is linear as follows:

$$v_\theta = \omega r (1 - z/H) \quad (7)$$

Here, the pressure distributions in the elastodynamic pump are analyzed by using (i) the viscoelastic fluid model combining the large deformation theory developed by Rivlin with the power law for non-Newtonian viscosity, and (ii) the 3-constants Oldroyd model respectively.

3.1 Combination model

According to the large deformation theory for simple shear in an incompressible elastic material

(a) Pressure distributions for the "no-flow"

(b) Pressure distributions for the "flow"

(c) Discharge flow rate

Fig. 2

developed by Rivlin, the relations between the components of the stress tensor and the strain ϵ are expressed as follows⁽⁴⁾:

$$\left. \begin{aligned} P_{\theta\theta} - P_{rr} &= c_1 \epsilon^2 \\ P_{zz} - P_{rr} &= -c_2 \epsilon^2 \\ \tau_{\theta z} &= (c_1 + c_2) \epsilon \end{aligned} \right\} \dots\dots\dots (8)$$

where denoting the strain energy per unit volume by W , and the two strain invariants by I_1 and I_2 , c_1 and c_2 are kinds of the modulus of elasticity expressed in the following formula:

$$c_1 = 2(\partial W / \partial I_1), \quad c_2 = 2(\partial W / \partial I_2) \dots\dots\dots (9)$$

When the strain ϵ is small, c_1 and c_2 are considered as constant, but they will, in general, depend on the strain. Now, the components of stresses in simple shear flow can be replaced by the simple shear deformation with introduction of recoverable shear strain after Weissenberg. That is

$$\left. \begin{aligned} P_{\theta\theta} - P_{rr} &= c_1 \lambda^2 \gamma^2 \\ P_{zz} - P_{rr} &= -c_2 \lambda^2 \gamma^2 \\ \tau_{\theta z} &= (c_1 + c_2) \lambda \gamma \end{aligned} \right\} \dots\dots\dots (10)$$

where γ is the rate of shear, and λ the apparent relaxation time. As mentioned above, since c_1 and c_2 may depend on the strain, they are assumed in the following forms over a wide range of the rate of shear in the viscoelastic fluid flow

$$c_1 = c_1^\circ \gamma^m, \quad c_2 = c_2^\circ \gamma^m \dots\dots\dots (11)$$

where c_1° , c_2° and m are rheological constants. At low rate of shear, $m=0$ hence c_1 and c_2 are constants. In general $m \geq 0$ from previous experimental results. Then, Eq. (10) becomes

$$\left. \begin{aligned} P_{\theta\theta} - P_{rr} &= c_1^\circ \lambda^2 \gamma^{m+2} \\ P_{zz} - P_{rr} &= -c_2^\circ \lambda^2 \gamma^{m+2} \\ \tau_{\theta z} &= (c_1^\circ + c_2^\circ) \lambda \gamma^{m+1} \end{aligned} \right\} \dots\dots\dots (12)$$

Now, according to the Weissenberg assumption⁽⁵⁾ and the experimental results made by Tamura, Kurata et al.⁽⁶⁾, the following expression is assumed.

$$P_{rr} = P_{zz} \dots\dots\dots (13)$$

Thus, Eq. (3) becomes

$$-\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} + \frac{\partial P_{zz}}{\partial r} - \frac{1}{r} (P_{\theta\theta} - P_{rr}) \dots\dots\dots (14)$$

And also, Eq. (12) becomes

$$P_{\theta\theta} - P_{rr} = G' \lambda^2 \gamma^{m+2}, \quad \tau_{\theta z} = G' \lambda \gamma^{m+1} \dots\dots\dots (15)$$

where $G' = c_1^\circ$

Using the power law for the non-Newtonian viscosity

$$\tau_{\theta z}^n = \mu_{psu} \gamma \dots\dots\dots (16)$$

where μ_{psu} and n are rheological constants and the dimension of μ_{psu} is given by $[g^n/cm^n \text{sec}^{2n-1}]$, the first formula of Eq. (15) becomes

$$P_{\theta\theta} - P_{rr} = \frac{\mu_{psu}^{2/n}}{G'} \gamma^{2/n-m} \dots\dots\dots (17)$$

The case of $m=0$ in Eq. (17) has been used by Metzner et al.⁽⁷⁾, Eq. (17) can be used for more kinds of viscoelastic fluids and a wider range of

rates of shear than the case of $m=0$. Since the left-hand member of Eq. (14) expresses the effect of centrifugal force, the following mean value of v_θ^2 is used instead of

$$\begin{aligned} \bar{v}_\theta^2 &= \frac{1}{H} \int_0^H v_\theta^2 dz = \frac{\omega^2 r^2}{H} \int_0^H (1-z/H)^2 dz \\ &= \frac{1}{3} \omega^2 r^2 \dots\dots\dots (18) \end{aligned}$$

The rate of shear is obtained from Eq. (7) as follows:

$$\gamma = \partial v_\theta / \partial z = -\omega r / H \dots\dots\dots (19)$$

Therefore, substituting Eqs. (17), (18), and (19) into Eq. (14), we have

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{\partial P_{zz}}{\partial r} - \frac{\mu_{psu}^{2/n} \omega^{2/n-m}}{G' H^{2/n-m}} r^{2/n-m-1} \\ &\quad + \frac{1}{3} \rho \omega^2 r \dots\dots\dots (20) \end{aligned}$$

In order to make the dimensionless form of Eq. (20), the following dimensionless quantities are defined

$$\left. \begin{aligned} r' &= \frac{r}{H}, \quad R_e^* = \frac{\rho \omega^{(2n-1)/n} H^2}{\mu_{psu}^{1/n}}, \\ N_e &= \frac{\mu_{psu}^{1/n} \omega^{1/n-m}}{G'}, \quad \tau_{zz}' = \frac{\tau_{zz}}{\mu_{psu}^{1/n} \omega^{1/n}} \end{aligned} \right\} \dots\dots\dots (21)$$

Substituting Eq. (21) into Eq. (20), the following expression is obtained.

$$\frac{\partial \tau_{zz}'}{\partial r'} - N_e r'^{2/n-m-1} + \frac{1}{3} R_e^* r' = 0 \dots\dots\dots (22)$$

Now, denoting the height of fluid column in the manometer by h the following relation is obtained.

$$\rho g h = -\tau_{zz} = p - P_{zz} \dots\dots\dots (23)$$

Here putting

$$\frac{\rho g h}{\mu_{psu}^{1/n} \omega^{1/n}} = -\frac{\tau_{zz}}{\mu_{psu}^{1/n} \omega^{1/n}} = -\tau_{zz}' \equiv Z \dots\dots\dots (24)$$

Eq. (22) becomes

$$\frac{\partial Z}{\partial r'} = -N_e r'^{2/n-m-1} + \frac{1}{3} R_e^* r' \dots\dots\dots (25)$$

Integrating Eq. (25) with respect to r' and then putting $Z=Z_0$ at $r'=0$ we have

$$Z_0 - Z = \frac{n}{2-nm} N_e r'^{2/n-m} - \frac{R_e^*}{6} r'^2 \dots\dots\dots (26)$$

For many viscoelastic fluids

$$\frac{\partial \ln(Z_0 - Z)}{\partial r'} > 0$$

Therefore, from Eq. (11) and Eq. (26), the relation between m and n is found as follows:

$$\frac{2}{n} > m \geq 0 \dots\dots\dots (27)$$

Thus, the pressure distribution for the "no-flow" developed by the normal stress effect in the elastodynamic pump is given by Eq. (26), and the second term of the right-hand member of Eq. (26) is the term expressing the centrifugal force effect. Moreover, the relation between the rheological constant n depending on the non-Newtonian viscosity and

m depending on the elastic behavior is found by Eq. (27).

3.2 3-constants Oldroyd model

The components of stress tensor in the simple shearing flow of the 3-constants Oldroyd fluid are related to the rate of shear by⁽⁸⁾

$$\left. \begin{aligned} P_{rr} &= -\left[\frac{2}{3}\lambda_1 F(r) - \frac{2}{3}\lambda_2 \eta_0\right] r^2 \\ P_{zz} &= -\left[\frac{2}{3}\lambda_1 F(r) - \frac{2}{3}\lambda_2 \eta_0\right] r^2 \\ P_{\theta\theta} &= \left[\frac{4}{3}\lambda_1 F(r) - \frac{4}{3}\lambda_2 \eta_0\right] r^2 \\ \tau_{\theta z} &= r F(r) \\ F(r) &= \eta_0 \left(1 + \frac{2}{3}\lambda_1 \lambda_2 r^2\right) / \left(1 + \frac{2}{3}\lambda_1^2 r^2\right) \end{aligned} \right\} \dots\dots(28)$$

where η_0 is the lower limiting viscosity, λ_1 the relaxation time and λ_2 the retardation time. We put

$$\left. \begin{aligned} r' &= \frac{r}{H}, \quad R_e = \frac{\rho \omega H^2}{\eta_0}, \quad \lambda^* = \frac{\lambda_2}{\lambda_1} \\ N_e &= \lambda_1 \omega, \quad \tau_{zz}' = \frac{\tau_{zz}}{\eta_0 \omega} \equiv -Z \end{aligned} \right\} \dots\dots\dots(29)$$

Then, the pressure distribution for the 3-constants Oldroyd fluid is obtained by the same analytical procedure as section 3.1. That is

$$\begin{aligned} Z_0 - Z &= \frac{3}{2} \frac{(1 - \lambda^*)}{N_e} \ln \left(1 + \frac{2}{3} N_e^2 r'^2\right) \\ &\quad - \frac{1}{6} R_e r'^2 \dots\dots\dots(30) \end{aligned}$$

4. Comparison with the experimental results and the analytical results

The pressure distributions due to the normal stress effect for the two models in the case of no-flow are approximately given by Eqs. (26) and (33) respectively. Now, the correlations between these analytical results and experimental data are examined.

4.1 For the combination model

According to Eq. (26), the plots of $Z_0 - Z + \frac{1}{6} R_e r'$ vs. r' are shown in Figs. 3~5. The gradient of the straight line gives the value of $\frac{2}{n} - m$, and the interception of the ordinate at $r'=1$ gives the value of $\frac{n}{2-nm} N_e$. The used fluid in Figs. 3 (a) and Fig. 3 (b) is material No. 1. Fig. 3 (a) shows the pressure distributions in a low range of rates of shear by the rheogoniometer. As shown in Table 1, the value of n for the material No. 1 is 1.35. And the gradient $\frac{2}{n} - m$ of the straight line shown in Fig. 3 (a) has a value of 1.47 for $N_e < 4.2 \times 10^{-1}$ ($r' < 70$), and hence $m=0$. This fact

means that c_1 in Eq. (10) is a constant in a low range of the rates of shear. Fig. 3 (b) shows the pressure distributions in a high range of the rates of shear by the elastodynamic pump. The gradient $\frac{2}{n} - m$ has a value of 0.76, therefore $m=0.71$. It means that c_1 depends upon the rate of shear in its high range. And also, above values satisfy Eq. (27). Both Fig. 4 (a) and Fig. 4 (b) show the pressure distributions when the material No. 2 is used.

Fig. 4 (a) shows the results in a low range of the rate of shear and also $m=0$. Fig. 4 (b) shows them in a high range of the rate of shear and $m=0.54$. The case of material No. 3 is shown in Fig 5. In this fluid, m equals to zero over all

Fig. 3 Dimensionless pressure distributions for "no-flow" (material No. 1)

the experimental range of the rate of shear. It means c_1 is independent of the rate of shear. These results are satisfied by Eq. (27) with all materials used. Though this model has some dubious points, for example a directly measuring method of c_1 or

Fig. 4 Dimensionless pressure distributions for "no-flow" (material No. 2)

Fig. 5 Dimensionless pressure distributions for "no-flow" (material No. 3)

(a)

(b)

(c)

Fig. 6 Comparison of the analytical predictions and the experimental data (material No. 2)

G' is unknown at the present time, it is very effective for the engineering problems necessitating a high range of the rates of shear, because this model is very simple and has good correlation with experimental results for a wide range of the rates of shear.

4.2 For the 3-constants Oldroyd model

According to Eq. (30), examples of the plots of $Z_0 - Z + \frac{R_e}{6}$ vs. r' are shown in Fig. 6 and Fig. 7. As found in Fig. 6 (a) and Fig. 6 (b) for an extremely low rate of shear, the discrepancy between Eq. (30) and the experimental results is not so large, but as shown in Fig. 6 (c) and Fig. 7, for a high rate of shear it is very large. Therefore, it may be considered that the Oldroyd model is inadequate to quantitative explanations of the normal stress effect for a high range of the rates of shear. On the other hand, a steady pipe flow was analyzed using the Oldroyd model by Bird et al.⁽³⁾, and it was found that the Oldroyd model does not agree with experimental results in a high range of shear rates.

5. Analytical treatment of "flow"

From the above mentioned facts, the combination model is utilized as the constitutive equation in order to analyze the flow pattern in the state of so-called "flow". In this case, the nozzle diameter is less than 1/10 times the rotating disk diameter, and the gap H is very small, and then except for a small region near the nozzle, the flow

Fig. 7 Comparison of the analytical predictions and the experimental data (material No. 2)

may be considered to be 2-dimensional. Cylindrical co-ordinates (r, θ, z) and velocity components (v_r, v_θ, v_z) are the same as those in the preceding paragraph. Then, the general form of the equations of motion and the equation of continuity are as follows:

$$\rho \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r P_{rr}) + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{P_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} \right\} \dots (31)$$

$$\rho \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r} \frac{\partial P_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} \right\} \dots (32)$$

$$\rho \left(v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial P_{zz}}{\partial z} \right\} \dots (33)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (\rho r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0 \dots (34)$$

Now, except for a small region near the delivery nozzle, it may be assumed that

$$v_\theta \gg v_r, \quad v_z \approx 0 \dots (35)$$

because the volumetric flow rate Q is small. As the flow state is considered to be axially symmetric, $\partial/\partial\theta=0$.

Hence Eqs. (31) and (34) become

$$-\frac{\rho v_\theta^2}{r} = -\frac{\partial p}{\partial r} + \frac{\partial P_{rr}}{\partial r} + \frac{P_{rr} - P_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} \dots (36)$$

$$\rho \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{\partial \tau_{\theta z}}{\partial z} \dots (37)$$

$$0 = \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{\partial P_{zz}}{\partial z} \dots (38)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r v_r) = 0 \dots (39)$$

To obtain the exact solutions of Eqs. (36) and (39) for the present problem is very difficult. Thus the problem is approximately analyzed only by Eqs. (36) and (39) for the purpose of obtaining the pressure distributions of r -direction due to the normal stress effect. The flow pattern in this case deviates from the simple shearing flow, but as $v_\theta \gg v_r$, the deviation may be small. Hence, it is assumed that the Weissenberg's assumption is useful. Therefore, Eq. (36) becomes

$$-\frac{\rho v_\theta^2}{r} = -\frac{\partial p}{\partial r} + \frac{\partial P_{zz}}{\partial r} - \frac{P_{\theta\theta} - P_{rr}}{r} + \frac{\partial \tau_{rz}}{\partial z} \dots (40)$$

And the shearing stress $\tau_{\theta z}$ is

$$\tau_{\theta z} = \mu_{\text{psu}}^{1/n} \left(\frac{\partial v_{\theta}}{\partial z} \right) \left\{ 4 \left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{\partial v_{\theta}}{\partial z} \right)^2 + \left(\frac{\partial v_r}{\partial z} \right)^2 \right\}^{1-n/2n}$$

However, since the gap is sufficiently small compared with the rotating disk diameter and $v_{\theta} \gg v_r$, the following relation may be obtained

$$\left(\frac{\partial v_{\theta}}{\partial z} \right)^2 \gg \left(\frac{\partial v_r}{\partial z} \right)^2 > \left(\frac{\partial v_r}{\partial r} \right)^2$$

Hence

$$\tau_{\theta z}^n = \mu_{\text{psu}} \left(\frac{\partial v_{\theta}}{\partial z} \right)$$

Eq. (16) may also hold approximately for the present problem. Then, Eq. (40) is obtained in the following form by the same procedure as Eq. (20).

$$\frac{\partial p}{\partial r} = \frac{\partial P_{zz}}{\partial r} - \frac{\mu_{\text{psu}}^{1/n} \omega^{2/n-m} r^{2/n-m-1}}{G' H^{2/n-m}} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\omega^2 \rho r}{3} \quad (41)$$

The shearing stress τ_{rz} for the power law fluid in Eq. (41) is expressed as follows:

$$\begin{aligned} \tau_{rz} = \mu_{\text{psu}}^{1/n} & \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \left[2 \left\{ \left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_{\theta}}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left(\frac{\partial v_{\theta}}{\partial z} \right)^2 \right\} \right. \\ & + \left\{ r \frac{\partial}{\partial r} \left(\frac{v_{\theta}}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right\}^2 + \left\{ \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_{\theta}}{\partial z} \right\}^2 \\ & \left. + \left\{ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right\}^2 \right]^{(1-n)/2n} \end{aligned}$$

This general form can be written using Eqs. (34) and (17) as follows:

$$\tau_{rz} = \mu_{\text{psu}}^{1/n} \left(\frac{\partial v_r}{\partial z} \right) \left(\frac{\partial v_{\theta}}{\partial z} \right)^{(1-n)/n} \quad (42)$$

Then, the frictional term $\partial \tau_{rz} / \partial z$ in Eq. (41) is approximately analyzed with the following assumption.

$$\frac{\partial \tau_{rz}}{\partial z} = f(r) \quad (43)$$

From the boundary conditions

$$v_r = 0 \text{ at } z=0, z=H \quad (44)$$

and Eq. (7) the velocity component of r -direction is obtained as follows:

$$v_r = \frac{1}{2\mu_{\text{psu}}^{1/n} \left(-\frac{\omega r}{H} \right)^{(1-n)/n}} f(r) (z^2 - Hz)$$

Hence

$$Q = \int_0^H 2\pi r v_r dz = \frac{\pi r H^3}{6\mu_{\text{psu}}^{1/n} \left(\frac{\omega r}{H} \right)^{(1-n)/n}} f(r)$$

Thus, from the above two expressions we have

$$v_r = \frac{3Q}{\pi r H^3} (z^2 - Hz)$$

Then, the frictional term is expressed using Eq. (42) as follows:

$$\frac{\partial \tau_{rz}}{\partial z} = \frac{6\mu_{\text{psu}}^{1/n} \gamma^{(1-n)/n}}{\pi H^3 r} Q \quad (45)$$

Substituting Eqs. (45) and (18) into Eq. (41), the following equation is obtained

$$\begin{aligned} \frac{\partial p}{\partial r} = \frac{\partial P_{zz}}{\partial r} - \frac{\mu_{\text{psu}}^{1/n} \omega^{2/n-m} r^{2/n-m-1}}{G' H^{2/n-m}} \\ - \frac{6\mu_{\text{psu}}^{1/n} \omega^{(1-n)/n} \gamma^{(1-2n)/n}}{\pi H^{(2n+1)/n}} Q + \frac{\omega^2 \rho r}{3} \quad (46) \end{aligned}$$

In order to make the dimensionless form of Eq. (46), the dimensionless quantities are defined as follows:

$$\left. \begin{aligned} r' &= \frac{r}{H}, \quad R_e^* = \frac{\rho \omega^{(2n-1)/n} H^2}{\mu_{\text{psu}}^{1/n}}, \\ Q' &= \frac{Q}{\omega H^3}, \quad N_e = \frac{\mu_{\text{psu}}^{1/n} \omega^{1/n-m}}{G'}, \\ \tau_{zz}' &= \frac{\tau_{zz}}{\mu_{\text{psu}}^{1/n} \omega^{1/n}} = \frac{-p + P_{zz}}{\mu_{\text{psu}}^{1/n} \omega^{1/n}} = Z \end{aligned} \right\} \quad (47)$$

Substituting Eq. (47) into Eq. (46), the following expression is obtained

$$\frac{\partial Z}{\partial r'} = -N_e r'^{2/n-m-1} - \frac{6Q'}{\pi} r'^{(1-2n)/n} + \frac{R_e^*}{3} r' \quad (48)$$

Integrating Eq. (48) with respect to r' , we have

$$\begin{aligned} Z = & -\frac{n}{(2-nm)} N_e r'^{2/n-m} \\ & - \frac{6n}{\pi(n-1)} \frac{Q'}{r'^{(n-1)/n}} + \frac{R_e^* r'^2}{6} + C \quad (49) \end{aligned}$$

where C is an integral constant. Therefore, putting $Z=Z_0$ at $r'=0$ and $Q'=0$, C is found, and then the following equation is obtained

$$\begin{aligned} Z_0 - Z = & \frac{n}{(2-nm)} N_e r'^{2/n-m} \\ & + \frac{6n}{\pi(n-1)} \frac{Q'}{r'^{(n-1)/n}} - \frac{R_e^* r'^2}{6} \quad (50) \end{aligned}$$

Thus, the pressure distributions for the "flow" developed due to the normal stress effect in the elastodynamic pump are given by Eq. (50).

6. Comparison of the experimental results with the analytical results

$$\text{The plots of } (Z_0 - Z) - \frac{6nQ'}{\pi(n-1)r'^{(n-1)/n}} + \frac{R_e^*}{6} r'^2$$

vs. r' are shown in Fig. 8 (a), (b) and (c). Here, the straight line in Fig. (8) is obtained from experimental results of "no-flow". The line agrees with experimental results, and this verifies the correctness of Eq. (50). Other experimental results which are not shown in Fig. (8) also are the same. Thus, Eq. (50) approximately is proved by the experiments.

7. Conclusions

1. The pressure distributions due to the normal stress effect are measured by the Green-smith-Rivlin's parallel plate type rheogoniometer and in the elastodynamic pump containing a dis-

(a)

(b) Material No. 2

(c) Material No. 3

Fig. 8 Comparison of the analytical predictions and the experimental data (material No. 2)

charge flow.

2. The flow patterns in the elastodynamic pump are analyzed using the viscoelastic fluid model combining the Rivlin's theory with the power law for the non-Newtonian viscosity, and the 3-constants Oldroyd model respectively. Especially analytical results for the discharge flow with some assumption seem to be valuable, because this problem has not been treated before.

3. These analytical results are compared with experimental results and the following conclusions are obtained.

i) For the "no-flow". The analytical results of the combination model are in good agreement with experimental results. And the Oldroyd model is not suitable for the normal stress effect in a high range of shear rates. Thus, it is shown that in the working range of the elastodynamic pump, the former model may be more convenient for practical purposes.

ii) For the "flow". Approximate analytical results using the combination model are in good agreement with experimental results. Because of the shearing stress due to the radial flow, the pressure is lower than that in the case of "no-flow". The authors wish to express their thanks to Professor T. Itaya of Tokyo Institute of Technology, and Professor S. Iwanami of Tokyo Metropolitan University for their guidance in the course of the study. The authors would like to express their appreciation also to Professor R. B. Bird of University of Wisconsin for his kind advice, and also wish to acknowledge the financial support provided by National Science Foundation Grant G-11 996 which Professor R.B. Bird received. Moreover, the authors express their cordial thanks to K. Ishii and others of Japan Steel Works for their cooperation in the preparation of the experimental apparatus.

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