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Measurement of Residual Stresses in a Cylinder*

(1 st Report, Theory for a Short Specimen Cut Off from a Long Cylinder)

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The Sachs boring-out method is used for determining the axially symmetrical residual stresses in circular cylinders. However, when a specimen cut off from a long cylinder (solid or hollow) is too short, the above method will not be satisfactory, owing to the end effect of the specimen, to estimate the residual stresses in the original cylinder.

Since it is usually accepted that dimensional changes caused by any relaxation of stresses are purely elastic, the present author analyzes the whole process of measurements on purely elastic theory and gives a method determining residual stresses in the original cylinder from the experimental data for a specimen with any arbitrary length.

1. Introduction

In order to determine the triaxial state of residual stresses in a long solid or hollow cylinder which is axially symmetrical and longitudinally uniform, the Sachs boring-out method⁽¹⁾ is widely used. However in the equations derived by Sachs for determining the distributions of residual stresses from experimental data (hereinafter we call the above equations the Sachs equations), it is assumed that the length of a cylinder used for test is infinite. Therefore, in case the value of the ratio ξ of length to diameter of the specimen separated from a long cylinder is finite and especially less than 3, the distributions of residual stresses obtained by the introduction of the experimental data for the shorter specimen in the Sachs equations is not always coincident with those in the original long cylinder. The difference between these two distributions increases as the value of ξ is decreased. However, so far the dependence of the distributions of residual stresses obtained by the Sachs equations upon the value of ξ has been only experimentally investigated by Bühler⁽²⁾.

Then in this paper, we consider the case of a long solid cylinder in which the residual stresses are axially symmetrical and longitudinally uniform. We analyze the process in which the specimen with any arbitrary value of ξ is separated from

the above cylinder (we call it the separation process) and the process in which that specimen is bored-out in a succession of thin layers (we call it the boring-out process) on the fundamental assumption⁽³⁾ in mechanical method of residual stress measurement in which all processes are purely elastic. Furthermore by using the results of the above analysis, we derive a theoretically exact method in which the residual stresses in a long cylinder can be determined from the experimental data obtained with the shorter specimen separated from that cylinder.

Recently in residual stress measurements, to detect longitudinal and tangential strain components on the outer cylindrical surface of the specimen, bonded resistance gauges cemented to that surface are generally used. Therefore, by selecting the strain gauges with adequate gauge length, we may consider that the indications of those gauges are equal to the values of the strain components at mid-length of the outer cylindrical surface of the specimen. Thus, in the following analysis, we take notice of the strain components at mid-length of that surface.

2. Fundamental considerations

We consider a long solid cylinder of radius a in which the residual stresses are axially symmetrical and longitudinally uniform except for the regions near both ends of the cylinder. In the system of cylindrical coordinate (r, θ, z) where the symmetrical axis of the above cylinder is

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coincident with the z axis and the mid-point of that axis is the origin, three residual stress components are functions of r only and denoted by $\sigma_r^*(r)$, $\sigma_\theta^*(r)$ and $\sigma_z^*(r)$ respectively. These are evidently principal stresses. We now assume that all strain components are zero throughout the cylinder.

On the assumption that all deformations produced in both the separation process and the boring-out process are purely elastic, we can resolve the residual stresses into two residual stress systems

- [I] $\sigma_z^*(r)$ only exists and $\sigma_r^*(r)=\sigma_\theta^*(r)=0$,
 [II] $\sigma_r^*(r)$, $\sigma_\theta^*(r)$ exist and $\sigma_z^*(r)=0$

Now by evaluating the deformation produced in the boring-out process when each of the above stress systems exists and by adding them together, we can obtain total deformation when both stress systems simultaneously exist.

When the long cylinder is bored-out layer by layer until the radius of the bore becomes ρ ($\rho=\alpha a$, $0<\alpha<1$), we denote the values for $r=a$ of strain components ϵ_z , ϵ_θ by $\lambda(\alpha)$, $\vartheta(\alpha)$ respectively. Also, denoting the contributions of the stress system [I] to $\lambda(\alpha)$, $\vartheta(\alpha)$ by $\lambda_I(\alpha)$, $\vartheta_I(\alpha)$ respectively and those of the stress system [II] by $\lambda_{II}(\alpha)$, $\vartheta_{II}(\alpha)$ respectively, we have

$$\left. \begin{aligned} \lambda(\alpha) &= \lambda_I(\alpha) + \lambda_{II}(\alpha), \\ \vartheta(\alpha) &= \vartheta_I(\alpha) + \vartheta_{II}(\alpha) \end{aligned} \right\} \dots\dots\dots (1)$$

Then, in case the system [I] only exists, we have

$$\vartheta_I(\alpha) = -\nu \lambda_I(\alpha)$$

for $\sigma_\theta=0$ on the outer surface of cylinder, where ν is Poisson's ratio of the material of the cylinder. Similarly, in case the stress system [II] only exists, we have

$$\lambda_{II}(\alpha) = -\nu \vartheta_{II}(\alpha)$$

for $\sigma_z=0$ on the outer surface of cylinder. We now consider $A(\alpha)$, $\Theta(\alpha)$ defined by the equations

$$\left. \begin{aligned} A(\alpha) &= \lambda(\alpha) + \nu \vartheta(\alpha), \\ \Theta(\alpha) &= \vartheta(\alpha) + \nu \lambda(\alpha) \end{aligned} \right\} \dots\dots\dots (2)$$

Using Eqs. (1) and the following two equations, we find that

$$\left. \begin{aligned} A(\alpha) &= (1-\nu^2) \lambda_I(\alpha), \\ \Theta(\alpha) &= (1-\nu^2) \vartheta_{II}(\alpha) \end{aligned} \right\} \dots\dots\dots (3)$$

Hence in the case of a sufficiently long cylinder, we can resolve the deformation produced in the boring-out process into the contribution of the residual stress system [I] and that of the residual stress system [II], even though both stress systems simultaneously exist.

We next consider the case of a short specimen. We first study the separation process in which the specimen of length $2l$ is separated from the mid-region of the long solid cylinder having only a residual stress system [II]. As the result of the separation both end planes become free from stress,

but the distributions of residual stresses in the specimen suffer no change, for no residual stresses acted on those planes before separation. Hence, the specimen is not deformed by the separation. Consequently, to obtain the changes of stresses and strains produced in the subsequent boring-out process is equivalent to obtaining the stresses and strains in a hollow cylinder when that cylinder with no longitudinal constraint is submitted to uniform pressure on the inner surface. These distributions of stresses and strains are independent of the length of cylinder. As stated above, in the long cylinder having only a stress system [II], we can get exact distributions of the residual stresses by the Sachs equations regardless of the value of the ratio ξ of the length $2l$ to the diameter $2a$ of the specimen separated from that cylinder.

Similarly we consider the separation process in which the specimen of length $2l$ is separated from the long solid cylinder having only the residual stress system [I]. When both end planes of the specimen become free from stress by the separation, the residual stress $\sigma_z^*(r)$ which acted on those planes before the separation is relaxed. As shown in Fig. 1, this separation process is mechanically equivalent to application of the surface forces on the planes $z=\pm l$ which are equal in the magnitude and opposite in the direction to the residual stress $\sigma_z^*(r)$. As the result of the separation, the distributions of residual stresses in the specimen separated undergo great change in the vicinity of both end planes, and this specimen is nonuniformly deformed in longitudinal and radial directions. The end effect caused by the separation is less negligible even in the mid-region of the specimen, as the value of ξ is decreased. Then we denote the values for $r=a$, $z=0$ of the strain components ϵ_z , ϵ_θ by $\lambda_I(0; \xi)$, $\vartheta_I(0; \xi)$ respectively. Furthermore, since the residual stresses in the specimen separated by the length $2l$ are also longitudinally nonuniform, we should expect that when this specimen is bored-out layer by layer until the radius of the bore becomes ρ ($\rho=\alpha a$, $0<\alpha<1$), the values for $r=a$, $z=0$ of the strain components ϵ_z , ϵ_θ differ from $\lambda_I(\alpha)$, $\vartheta_I(\alpha)$ of the preceding chapter respectively, and then we denote them by $\lambda_I(\alpha) + \lambda_I(\alpha; \xi)$, $\vartheta_I(\alpha) + \vartheta_I(\alpha; \xi)$ respectively.

Fig. 1 Mechanical model of the separation process

Ultimately, when the specimen of length $2l$ is separated from the long solid cylinder of the radius a which has residual stress systems [I] and [II] and thereafter bored out layer by layer until the radius of the bore becomes ρ , denoting the values for $r=a$, $z=0$ of the strain components ε_z , ε_θ of the specimen by $\lambda(\alpha; \xi)$, $\vartheta(\alpha; \xi)$ respectively, we have

$$\left. \begin{aligned} \lambda(\alpha; \xi) &= \lambda_I(\alpha) + \lambda_{II}(\alpha; \xi) + \lambda_{III}(\alpha), \\ \vartheta(\alpha; \xi) &= \vartheta_I(\alpha) + \vartheta_{II}(\alpha; \xi) + \vartheta_{III}(\alpha) \end{aligned} \right\} \dots\dots(4)$$

In general, $\vartheta_I(\alpha; \xi) = -\nu \lambda_I(\alpha; \xi)$

Therefore considering $\lambda(\alpha; \xi)$, $\theta(\alpha; \xi)$ as defined by the equations

$$\left. \begin{aligned} \lambda(\alpha; \xi) &= \lambda(\alpha; \xi) + \nu \vartheta(\alpha; \xi) \\ \theta(\alpha; \xi) &= \vartheta(\alpha; \xi) + \nu \lambda(\alpha; \xi) \end{aligned} \right\} \dots\dots\dots(5)$$

and using Eqs. (3) and (4), we have

$$\left. \begin{aligned} \lambda(\alpha; \xi) &= \lambda(\alpha) + \{\lambda_{II}(\alpha; \xi) + \nu \vartheta_{II}(\alpha; \xi)\}, \\ \theta(\alpha; \xi) &= \theta(\alpha) + \{\vartheta_{II}(\alpha; \xi) + \nu \lambda_{II}(\alpha; \xi)\} \end{aligned} \right\} \dots\dots\dots(6)$$

$\lambda(\alpha; \xi)$, $\theta(\alpha; \xi)$, being functions of α , are the only experimental data obtained with the specimen of length $2l$. It can be seen from Eqs. (6) that the contributions of the residual stress system [II] to $\lambda(\alpha; \xi)$, $\theta(\alpha; \xi)$ are constant in spite of the value of ξ , but in general the contributions from the residual stress system [I] depends on the value of ξ . Hence in order to find $\lambda(\alpha)$, $\theta(\alpha)$ from $\lambda(\alpha; \xi)$, $\theta(\alpha; \xi)$ respectively, it is necessary and satisfactory to analyze the separation process and the boring-out process in the long solid cylinder having only a residual stress system [I], that is, we obtain $\lambda_I(0; \xi)$, $\vartheta_I(0; \xi)$, $\lambda_I(\alpha; \xi)$ and $\vartheta_I(\alpha; \xi)$.

3. Analysis of separation process

Consider a sufficiently long solid cylinder of the radius a which has only a residual stress system [I], i.e. $\sigma_z^*(r)$ except for the regions near both ends. Since the residual stress $\sigma_z^*(r)$ is a function of r only, it can be expanded in a series of the form

$$\sigma_z^*(r) = \sum_{m=1}^{\infty} \sigma_{zm}^*(r) = \sum_{m=1}^{\infty} A_m J_0(p_m r) \dots\dots\dots(7)$$

where A_m is now unknown coefficient and $p_m a$ is the m -th positive root of the equation $J_1(p_m a) = 0$.

Separating the specimen of length $2l$ from the mid-region of the long cylinder, both end faces $z = \pm l$ of the specimen become free from stress. Therefore, as discussed in the preceding chapter, the state of residual stresses in the separated specimen of length $2l$ and radius a is equal to the sum of the longitudinally uniform residual stress

$$\begin{aligned} \phi(r, z; 0, \xi) &= \sum_{i=1}^{\infty} \frac{b_i \sin k_i z}{I_1(k_i a)} \left[I_1(k_i a) k_i r I_1(k_i r) - \{2(1-\nu) I_1(k_i a) + k_i a I_0(k_i a)\} I_0(k_i r) \right] \\ &+ \sum_{m=1}^{\infty} \frac{\beta_m J_0(p_m r)}{\sinh p_m l} \left[\sinh p_m l p_m z \cosh p_m z - \{p_m l \cosh p_m l + 2\nu \sinh p_m l\} \sinh p_m z \right] \dots\dots\dots(13) \end{aligned}$$

$\sigma_z^*(r)$ in the original long cylinder and the elastic stresses which satisfies the boundary conditions

$$\left. \begin{aligned} \text{for } z = \pm l \quad \sigma_z &= -\sigma_z^*(r), \quad \tau_{rz} = 0, \\ \text{for } r = a \quad \sigma_r &= \tau_{rz} = 0 \end{aligned} \right\} \dots\dots\dots(8)$$

In general, the problem⁽⁴⁾ of determining the distributions of stresses in a solid of revolution can be reduced to one of finding the solution of the following biharmonic equation satisfying the appropriate boundary conditions:

$$\nabla^4 \phi \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)^2 \phi = 0 \dots\dots\dots(9)$$

where ϕ is a stress function. Provided that the stress function ϕ satisfies Eq. (9), stress and strain components are expressed in the forms

$$\left. \begin{aligned} \sigma_r &= \frac{\partial}{\partial z} \left\{ \nu \nabla^2 \phi - \frac{\partial^2 \phi}{\partial r^2} \right\}, \\ \sigma_\theta &= \frac{\partial}{\partial z} \left\{ \nu \nabla^2 \phi - \frac{1}{r} \frac{\partial \phi}{\partial r} \right\}, \\ \sigma_z &= \frac{\partial}{\partial z} \left\{ (2-\nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\}, \\ \tau_{rz} &= \frac{\partial}{\partial r} \left\{ (1-\nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} \end{aligned} \right\} \dots\dots\dots(10)$$

$$\left. \begin{aligned} \varepsilon_r &= -\frac{1+\nu}{E} \frac{\partial}{\partial z} \left(\frac{\partial^2 \phi}{\partial r^2} \right), \\ \varepsilon_\theta &= -\frac{1+\nu}{E} \frac{\partial}{\partial z} \left(\frac{1}{r} \frac{\partial \phi}{\partial r} \right), \\ \varepsilon_z &= \frac{1+\nu}{E} \frac{\partial}{\partial z} \left\{ 2(1-\nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} \end{aligned} \right\} \dots\dots\dots(11)$$

where E is Young's modulus of the material of the cylinder.

Therefore, to obtain the elastic stress distributions in the cylinder of length $2l$ and radius a which satisfy the boundary conditions (8) is reduced to obtaining the solution of Eq. (9) which satisfies the boundary conditions

$$\left. \begin{aligned} \text{for } z = \pm l \\ \frac{\partial}{\partial z} \left\{ (2-\nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} &= -\sigma_z^*(r), \\ \frac{\partial}{\partial r} \left\{ (1-\nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} &= 0, \\ \text{for } r = a \\ \frac{\partial}{\partial z} \left\{ \nu \nabla^2 \phi - \frac{\partial^2 \phi}{\partial r^2} \right\} &= 0, \\ \frac{\partial}{\partial r} \left\{ (1-\nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} &= 0 \end{aligned} \right\} \dots\dots\dots(12)$$

Now, the general method for solving such a boundary value problem in a finite cylinder has been obtained independently by G. Picket⁽⁵⁾ and H. Saito⁽⁶⁾. Following the above method and denoting stress function satisfying the boundary conditions (12) by $\phi(r, z; 0, \xi)$, we have

where $I_0(k_i r)$, $I_1(k_i r)$ are the 1st kind deformed cylindrical functions, $k_i l$ is the i -th positive root of the equation $\sinh k_i l = 0$ and also coefficients b_i and β_m , being functions of parameter ξ , must satisfy the pair of series relations

$$\left. \begin{aligned} & (-1)^i b_i k_i^3 k_i a I_1(k_i a) \left[1 + \frac{2(1-\nu)}{(k_i a)^2} - \left\{ \frac{I_0(k_i a)}{I_1(k_i a)} \right\}^2 \right] \\ & = \sum_{m=1}^{\infty} \beta_m p_m^3 J_0(p_m a) \frac{4(k_i a)^2 p_m a \sinh p_m l}{\xi \{ (p_m a)^2 + (k_i a)^2 \}^2}, \\ & \beta_m p_m^3 J_0(p_m a) (\cosh p_m l + p_m l / \sinh p_m l) + A_m J_0(p_m a) \\ & = \sum_{i=1}^{\infty} (-1)^{i'+1} b_{i'} (k_{i'}^3 k_{i'} a) I_1(k_{i'} a) \frac{4(p_m a)^2}{\{ (p_m a)^2 + (k_{i'} a)^2 \}^2} \end{aligned} \right\} \dots\dots\dots (14)$$

Therefore, when the value of ξ is given, b_i and β_m can be expressed in terms of A_m by solving the above simultaneous equations (14).

Knowing the stress function $\phi(r, z; 0, \xi)$, the sum of the stress components derived from this stress function and the residual stress $\sigma_z^*(r)$ which is longitudinally uniform are equal to the residual stresses in the specimen of length $2l$ separated from the long cylinder, namely

$$\left. \begin{aligned} \sigma_r(r, z; 0, \xi) &= \sum_{i=1}^{\infty} \frac{b_i k_i^3 \cos k_i z}{I_1(k_i a)} \left[\{ I_1(k_i a) + k_i a I_0(k_i a) \} I_0(k_i r) \right. \\ &\quad \left. - I_1(k_i a) k_i r I_1(k_i r) - \{ 2(1-\nu) I_1(k_i a) + k_i a I_0(k_i a) \} \frac{I_1(k_i r)}{k_i r} \right] \\ &\quad + \sum_{m=1}^{\infty} \frac{\beta_m p_m^3 J_0(p_m r)}{\sinh p_m l} \left[\{ \sinh p_m l - p_m l \cosh p_m l \} \cosh p_m z + \sinh p_m l p_m z \sinh p_m z \right] \\ &\quad - \sum_{m=1}^{\infty} \frac{\beta_m p_m^3 J_1(p_m r)}{\sinh p_m l p_m r} \left[\{ (1-2\nu) \sinh p_m l - p_m l \cosh p_m l \} \cosh p_m z + \sinh p_m l p_m z \sinh p_m z \right], \\ \sigma_z(r, z; 0, \xi) &= \sum_{i=1}^{\infty} \frac{b_i k_i^3 \cos k_i z}{I_1(k_i a)} \left[\{ 2(1-\nu) I_1(k_i a) + k_i a I_0(k_i a) \} \frac{I_1(k_i r)}{k_i r} - (1-2\nu) I_1(k_i a) I_0(k_i r) \right] \\ &\quad + \sum_{m=1}^{\infty} \frac{\beta_m p_m^3 J_0(p_m r)}{\sinh p_m l} 2\nu \sinh p_m l \cosh p_m z \\ &\quad + \sum_{m=1}^{\infty} \frac{\beta_m p_m^3}{\sinh p_m l} \frac{J_1(p_m r)}{p_m r} \left[\{ (1-2\nu) \sinh p_m l \right. \\ &\quad \left. - p_m l \cosh p_m l \} \cosh p_m z + \sinh p_m l p_m z \sinh p_m z \right], \\ \sigma_z(r, z; 0, \xi) &= \sum_{i=1}^{\infty} \frac{b_i k_i^3 \cos k_i z}{I_1(k_i a)} \left[\{ 2I_1(k_i a) - k_i a I_0(k_i a) \} I_0(k_i r) + I_1(k_i a) k_i r I_1(k_i r) \right] \\ &\quad + \sum_{m=1}^{\infty} \left[A_m J_0(p_m r) + \frac{\beta_m p_m^3 J_0(p_m r)}{\sinh p_m l} \{ (\sinh p_m l \right. \\ &\quad \left. + p_m l \cosh p_m l) \cosh p_m z - \sinh p_m l p_m z \sinh p_m z \} \right], \\ \tau_{rz}(r, z; 0, \xi) &= \sum_{i=1}^{\infty} \frac{b_i k_i^3 \sin k_i z}{I_1(k_i a)} \{ I_1(k_i a) k_i r I_0(k_i r) - k_i a I_0(k_i a) I_1(k_i r) \} \\ &\quad + \sum_{m=1}^{\infty} \frac{\beta_m p_m^3 J_1(p_m r)}{\sinh p_m l} \{ \sinh p_m l p_m z \cosh p_m z - p_m l \cosh p_m l \sinh p_m z \} \end{aligned} \right\} \dots\dots\dots (15)$$

Furthermore, substituting Eq. (13) into Eqs. (11), the values for $r=a$, $z=0$ of strain components ϵ_z , ϵ_r are obtained, and now denoting them by $\lambda_I(0; \xi)$, $\vartheta_I(0; \xi)$ respectively, we have

$$\left. \begin{aligned} \lambda_I(0; \xi) &= \frac{1+\nu}{E} \sum_{i=1}^{\infty} b_i k_i^3 k_i a I_1(k_i a) \left[1 + \frac{2(1-\nu)}{k_i a} \frac{I_0(k_i a)}{I_1(k_i a)} - \left\{ \frac{I_0(k_i a)}{I_1(k_i a)} \right\}^2 \right] \\ &\quad + \frac{1+\nu}{E} \sum_{m=1}^{\infty} \beta_m p_m^3 J_0(p_m a) \{ (1-2\nu) + p_m l \coth p_m l \}, \\ \vartheta_I(0; \xi) &= \frac{1+\nu}{E} \sum_{i=1}^{\infty} b_i k_i^3 \frac{2(1-\nu) I_1(k_i a)}{k_i a} \end{aligned} \right\} \dots\dots\dots (16)$$

Eqs. (16) are also written in the form

$$\left. \begin{aligned} \lambda_I(0; \xi) &= \sum_{m=1}^{\infty} \lambda_{Im}(0; \xi) = \sum_{m=1}^{\infty} C_m(0; \xi) A_m, \\ \vartheta_I(0; \xi) &= \sum_{m=1}^{\infty} \vartheta_{Im}(0; \xi) = \sum_{m=1}^{\infty} D_m(0; \xi) A_m \end{aligned} \right\} \dots\dots\dots (17)$$

where $C_m(0; \xi)$, $D_m(0; \xi)$ are constants, being determined by specifying the value of parameter ξ .

4. Boring-out process of short specimen

4.1 Principle of analysis

Consider a specimen of length $2l$ and radius a which was separated from the mid-region of the long cylinder having only a residual stress system $[I]$. When the above specimen is bored-out in a succession of thin layers from the solid section until the radius of the bore becomes ρ , it satisfies the boundary conditions for $z = \pm l$, $\sigma_z = \tau_{rz} = 0$, for $r = a$ and $r = \rho$, $\sigma_r = \tau_{rz} = 0$ (18)
The hollow cylinder of length $2l$ and inner radius ρ which was separated from the long hollow cylinder of inner radius ρ previously bored-out also satisfies the boundary conditions (18). On the assumption that all processes are purely elastic, it can be seen that the above two hollow cylinders of length $2l$ and inner radius ρ have the same distributions of stresses and strains. We therefore may obtain the stresses and strains in the specimen of length $2l$ and inner radius ρ on the assumption that it was separated from the long hollow cylinder of inner radius ρ previously bored-out.

4.2 Boring-out process of long cylinder

Considering the equilibrium of the hatched portion in Fig. 2 of the long solid cylinder of radius a which has only a residual stress $\sigma_z^*(r)$, we have

$$\int_0^L 2\pi\rho\tau_{rz}(\rho, z)dz = \int_0^\rho 2\pi r\sigma_z^*(r)dr \quad \text{.....(19)}$$

where L is the half-length of the cylinder, $\tau_{rz}(\rho, z)$ is shearing stress acting on the cylindrical surface $r = \rho$ in the regions near both end planes of the cylinder. If the bore of the radius ρ is bored-out in the above cylinder, the shearing stress $\tau_{rz}(\rho, z)$ will be relaxed. The stresses and strains caused by relaxation are equivalent to those caused by the normal forces which are uniformly acting on both end planes and equal to Eq. (19) in magnitude, in the region apart from both ends by St. Venant's principle. Therefore denoting the residual stress in the mid-region of the hollow cylinder by $\sigma_z^*(r; \alpha)$, we have

$$\begin{aligned} \sigma_z^*(r; \alpha) &= \sum_{m=1}^{\infty} \sigma_{zm}^*(r; \alpha) = \sigma_z^*(r) + \frac{1}{\pi(a^2 - \rho^2)} \int_0^\rho 2\pi r\sigma_z^*(r)dr \\ &= \sum_{m=1}^{\infty} A_m \left\{ J_0(p_m r) + \frac{2\alpha p_m a J_1(\alpha p_m a)}{(p_m a)^2 (1 - \alpha^2)} \right\} \quad (\alpha a \leq r \leq a) \quad \text{.....(20)} \end{aligned}$$

For the values of the strain components ϵ_z , ϵ_θ at the outer surface of the hollow cylinder which are equal to $\lambda_I(\alpha)$, $\vartheta_I(\alpha)$ respectively, we have

$$\lambda_I(\alpha) = \sum_{m=1}^{\infty} \lambda_{Im}(\alpha) = \sum_{m=1}^{\infty} \frac{2A_m \alpha p_m a J_1(\alpha p_m a)}{E(p_m a)^2 (1 - \alpha^2)}, \quad \vartheta_I(\alpha) = \sum_{m=1}^{\infty} \vartheta_{Im}(\alpha) = -\nu \lambda_I(\alpha) \quad \text{.....(21)}$$

4.3 Separation process of long hollow cylinder

Consider a long hollow cylinder of inner radius ρ and outer radius a which has only a residual stress $\sigma_z^*(r; \alpha)$ except for the region near both ends. In the same way of discussion as in chapter 3, the state of the residual stresses of the specimen of length $2l$ separated from the above long hollow cylinder is equal to the sum of the residual stress $\sigma_{zm}^*(r)$ being longitudinally uniform and the elastic stresses satisfying the boundary conditions

$$\left. \begin{aligned} \text{for } z = \pm l \quad & \sigma_z = -\sigma_z^*(r; \alpha), \quad \tau_{rz} = 0, \\ \text{for } r = a, \quad r = \rho \quad & \sigma_r = \tau_{rz} = 0 \end{aligned} \right\} \quad \text{.....(22)}$$

To obtain the latter stresses is equivalent to getting the stress function $\phi(r, z; \alpha, \xi)$, the solution of Eq. (9) satisfying the boundary conditions

$$\left. \begin{aligned} \text{for } z = \pm l \quad & \frac{\partial}{\partial z} \left\{ (2 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} = -\sigma_z^*(r; \alpha), \quad \frac{\partial}{\partial r} \left\{ (1 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} = 0, \\ \text{for } r = a \text{ and } r = \rho \quad & \frac{\partial}{\partial z} \left\{ \nu \nabla^2 \phi - \frac{\partial^2 \phi}{\partial r^2} \right\} = 0, \quad \frac{\partial}{\partial r} \left\{ (1 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} = 0 \end{aligned} \right\} \quad \text{.....(23)}$$

Thus we find that

$$\begin{aligned} \phi(r, z; \alpha, \xi) &= \sum_{i=1}^{\infty} \sin k_i z \{ c_i I_0(k_i r) \\ &+ d_i k_i r I_1(k_i r) + e_i K_0(k_i r) + f_i k_i r K_1(k_i r) \} \\ &+ \sum_{n=1}^{\infty} \frac{\tilde{r}_n}{\sinh q_n l} \{ \sinh q_n l q_n z \cosh q_n z \\ &- (q_n l \cosh q_n l + 2\nu \sinh q_n l) \sinh q_n z \} \\ &\times \{ J_0(q_n r) Y_1(q_n a) - Y_0(q_n r) J_1(q_n a) \} \quad \text{.....(24)} \end{aligned}$$

where $Y_0(q_n r)$, $Y_1(q_n r)$ are the second kind cylin-

Fig. 2 Equilibrium of the residual stresses

drical functions and $K_0(k_j r)$, $K_1(k_j r)$ are the second kind deformed cylindrical functions and $k_j l$ is the j -th positive root of the equation $\sinh k_j l = 0$ and $q_n a$, the function of the parameter α , is the n -th positive root of the equation

$$J_1(\alpha q_n a) Y_1(q_n a) - Y_1(\alpha q_n a) J_1(q_n a) = 0 \quad \dots\dots\dots (25)$$

Furthermore, c_j , d_j , e_j , f_j and τ_n , being the functions of the parameters ξ and α , must satisfy the group of series relations

$$\left. \begin{aligned} & c_j I_1(k_j a) + d_j \{2(1-\nu) I_1(k_j a) + k_j a I_0(k_j a)\} - e_j K_1(k_j a) \\ & \quad + f_j \{2(1-\nu) K_1(k_j a) - k_j a K_0(k_j a)\} = 0, \\ & c_j I_1(\alpha k_j a) + d_j \{2(1-\nu) I_1(\alpha k_j a) + \alpha k_j a K_0(\alpha k_j a)\} \\ & \quad - e_j K_1(\alpha k_j a) + f_j \{2(1-\nu) K_1(\alpha k_j a) - \alpha k_j a K_0(\alpha k_j a)\} = 0, \\ & c_j k_j^3 \left\{ I_0(k_j a) - \frac{I_1(k_j a)}{k_j a} \right\} + d_j k_j^3 \{(1-2\nu) I_0(k_j a) + k_j a I_1(k_j a)\} + e_j k_j^3 \left\{ K_0(k_j a) + \frac{K_1(k_j a)}{k_j a} \right\} \\ & \quad - f_j k_j^3 \{(1-2\nu) K_0(k_j a) - k_j a K_1(k_j a)\} = (-1)^{j+1} \sum_{n=1}^{\infty} \frac{8\tau_n q_n^3}{\pi \xi} \frac{(k_j a)^2 \sinh q_n l}{\{(q_n a)^2 + (k_j a)^2\}^2}, \\ & c_j k_j^3 \left\{ I_0(\alpha k_j a) - \frac{I_1(\alpha k_j a)}{\alpha k_j a} \right\} + d_j k_j^3 \{(1-2\nu) I_0(\alpha k_j a) + \alpha k_j a I_1(\alpha k_j a)\} \\ & \quad + e_j k_j^3 \left\{ K_0(\alpha k_j a) + \frac{K_1(\alpha k_j a)}{\alpha k_j a} \right\} - f_j k_j^3 \{(1-2\nu) K_0(\alpha k_j a) - \alpha k_j a K_1(\alpha k_j a)\} \\ & = (-1)^{j+1} \sum_{n=1}^{\infty} \frac{8\tau_n q_n^3}{\pi \xi \alpha} \frac{J_1(q_n a)}{J_1(\alpha q_n a)} \frac{(k_j a)^2 \sinh q_n l}{\{(q_n a)^2 + (k_j a)^2\}^2}, \\ & \tau_n q_n^3 \frac{\cosh q_n l \sinh q_n l + q_n l}{\sinh q_n l} \frac{[J_1(\alpha q_n a)]^2 - [J_1(q_n a)]^2}{\pi J_1(\alpha q_n a)} + q_n a J_1(q_n a) \sum_{m=1}^{\infty} \frac{A_m p_m a J_1(\alpha p_m a)}{(p_m a)^2 - (q_n a)^2} \\ & = \sum_{j=1}^{\infty} (-1)^j \frac{2(q_n a)^3 k_j a}{\{(p_m a)^2 + (q_n a)^2\}^2} \left[d_j k_j^3 \{J_1(q_n a) I_1(\alpha k_j a) \right. \\ & \quad \left. - J_1(\alpha q_n a) I_1(k_j a)\} + f_j k_j^3 \{J_1(q_n a) K_1(\alpha k_j a) - J_1(\alpha q_n a) K_1(k_j a)\} \right] \end{aligned} \right\} \dots\dots\dots (26)$$

Solving the above simultaneous equation (26) for the given values of ξ and α , c_j , d_j , e_j , f_j and τ_n can be expressed in terms of A_m .

Knowing the stress function $\phi(r, z; \alpha, \xi)$, the residual stresses in the specimen of length $2l$, inner radius ρ and outer radius a are equal to the sum of the stress components derived from the stress function $\phi(r, z; \alpha, \xi)$ and residual stress $\sigma_z^*(r; \alpha)$ which is longitudinally uniform, namely

$$\left. \begin{aligned} \sigma_r(r, z; \alpha, \xi) = & - \sum_{j=1}^{\infty} k_j^3 \cos k_j z \left[c_j \left\{ I_0(k_j r) - \frac{I_1(k_j r)}{k_j r} \right\} + d_j \{(1-2\nu) I_0(k_j r) + k_j r I_1(k_j r)\} \right] \\ & - \sum_{j=1}^{\infty} k_j^3 \cos k_j z \left[e_j \left\{ K_0(k_j r) + \frac{K_1(k_j r)}{k_j r} \right\} - f_j \{(1-2\nu) K_0(k_j r) - k_j r K_1(k_j r)\} \right] \\ & + \sum_{n=1}^{\infty} \frac{\tau_n q_n^3}{\sinh q_n l} \{ (\sinh q_n l - q_n l \cosh q_n l) \cosh q_n z \\ & + \sinh q_n l q_n z \sinh q_n z \} \{ J_0(q_n r) Y_1(q_n a) - Y_0(q_n r) J_1(q_n a) \} \\ & - \sum_{n=1}^{\infty} \frac{\tau_n q_n^3}{\sinh q_n l} \{ (1-2\nu) \sinh q_n l - q_n l \cosh q_n l \} \cosh q_n z \\ & + \sinh q_n l q_n z \sinh q_n z \left[\frac{1}{q_n r} \{ J_1(q_n r) Y_1(q_n a) - Y_1(q_n r) J_1(q_n a) \} \right], \\ \sigma_\theta(r, z; \alpha, \xi) = & - \sum_{j=1}^{\infty} k_j^3 \cos k_j z \left\{ c_j \frac{I_1(k_j r)}{k_j r} + (1-2\nu) d_j I_0(k_j r) \right. \\ & \left. - e_j \frac{K_1(k_j r)}{k_j r} - (1-2\nu) f_j K_0(k_j r) \right\} \\ & + \sum_{n=1}^{\infty} \tau_n q_n^3 2\nu \cosh q_n z \{ J_0(q_n r) Y_1(q_n a) - Y_0(q_n r) J_1(q_n a) \} \\ & + \sum_{n=1}^{\infty} \frac{\tau_n q_n^3}{\sinh q_n l} \left[\{ (1-2\nu) \sinh q_n l - q_n l \cosh q_n l \} \cosh q_n z \right. \\ & \left. + \sinh q_n l q_n z \sinh q_n z \right] \frac{1}{q_n r} \{ J_1(q_n r) Y_1(q_n a) - Y_1(q_n r) J_1(q_n a) \}, \\ \sigma_z(r, z; \alpha, \xi) = & \sigma_z^*(r; \alpha) \\ & + \sum_{j=1}^{\infty} k_j^3 \cos k_j z \left[c_j I_0(k_j r) + d_j \{2(2-\nu) I_0(k_j r) + k_j r I_1(k_j r)\} \right] \end{aligned} \right\} \dots\dots\dots (27)$$

$$\begin{aligned}
& + \sum_{j=1}^{\infty} k_j^3 \cos k_j z \left[e_j K_0(k_j r) - f_j \{2(2-\nu)K_0(k_j r) - k_j r K_1(k_j r)\} \right] \\
& + \sum_{n=1}^{\infty} \frac{\gamma_n q_n^3}{\sinh q_n l} \{ \sinh q_n l + q_n l \cosh q_n l \} \cosh q_n z \\
& - \sinh q_n l q_n z \sinh q_n z \{ J_0(q_n r) Y_1(q_n a) - Y_0(q_n r) J_1(q_n a) \}, \\
\tau_{rz}(r, z; \alpha, \xi) = & \sum_{j=1}^{\infty} k_j^3 \sin k_j z \left[c_j I_1(k_j r) + d_j \{2(1-\nu)I_1(k_j r) + k_j r I_0(k_j r)\} \right] \\
& - \sum_{j=1}^{\infty} k_j^3 \sin k_j z \left[e_j K_1(k_j r) - f_j \{2(1-\nu)K_1(k_j r) - k_j r K_0(k_j r)\} \right] \\
& + \sum_{n=1}^{\infty} \frac{\gamma_n q_n^3}{\sinh q_n l} \{ \sinh q_n l q_n z \cosh q_n z - q_n l \cosh q_n l \sinh q_n z \} \\
& \times \{ J_1(q_n r) Y_1(q_n a) - Y_1(q_n r) J_1(q_n a) \}
\end{aligned}$$

Furthermore the values for $r=a$, $z=0$ of the strain components derived from the stress function $\phi(r, z; \alpha, \xi)$ are equal to $\lambda_I(\alpha; \xi)$, $\vartheta_I(\alpha; \xi)$ respectively, namely

$$\begin{aligned}
\lambda_I(\alpha; \xi) = & \frac{1+\nu}{E} \sum_{j=1}^{\infty} k_j^3 \left[c_j I_0(k_j a) + d_j \{4(1-\nu)I_0(k_j a) \right. \\
& \left. + k_j a I_1(k_j a)\} + e_j K_0(k_j a) - f_j \{4(1-\nu)K_0(k_j a) - k_j a K_1(k_j a)\} \right] \\
& - \frac{1+\nu}{E} \sum_{n=1}^{\infty} \frac{2\gamma_n q_n^3}{\pi q_n a \sinh q_n l} \{ (1-2\nu) \sinh q_n l + q_n l \cosh q_n l \}, \\
\vartheta_I(\alpha; \xi) = & \frac{1+\nu}{E} \sum_{j=1}^{\infty} k_j^3 \left\{ c_j \frac{I_1(k_j a)}{k_j a} + d_j I_0(k_j a) - e_j \frac{K_1(k_j a)}{k_j a} - f_j K_0(k_j a) \right\}
\end{aligned} \quad \dots\dots\dots(28)$$

Eqs. (28) can be rewritten in the form

$$\begin{aligned}
\lambda_I(\alpha; \xi) = & \sum_{m=1}^{\infty} \lambda_{Im}(\alpha; \xi) = \sum_{m=1}^{\infty} C_m(\alpha, \xi) A_m, \\
\vartheta_I(\alpha; \xi) = & \sum_{m=1}^{\infty} \vartheta_{Im}(\alpha; \xi) = \sum_{m=1}^{\infty} D_m(\alpha; \xi) A_m
\end{aligned} \quad \dots\dots\dots(29)$$

where $C_m(\alpha; \xi)$, $D_m(\alpha; \xi)$ are functions of α determined by the specification of the value of the parameter ξ .

5. Method for determining the residual stresses

Substituting Eqs. (17), (21) and (29) into Eq. (6), we have

$$A(\alpha; \xi) = \frac{1-\nu^2}{E} \sum_{m'=1}^{\infty} \frac{2A_{m'} \alpha p_{m'} a J_1(\alpha p_{m'} a)}{(p_{m'} a)^2 (1-\alpha^2)} + \sum_{m'=1}^{\infty} \{C_{m'}(\alpha; \xi) + \nu D_{m'}(\alpha; \xi)\} A_{m'} \quad \dots\dots\dots(30)$$

$$\Theta(\alpha; \xi) = \Theta(\alpha) + \sum_{m=1}^{\infty} \{C_m(\alpha; \xi) + \nu D_m(\alpha; \xi)\} A_m \quad \dots\dots\dots(31)$$

$A(\alpha; \xi)$, $\Theta(\alpha; \xi)$ are known by substitution of the quantities obtained from the experiments for the specimen of any arbitrary ξ , i.e., $\lambda(\alpha; \xi)$, $\vartheta(\alpha; \xi)$ into Eqs. (5). Therefore multiplying both sides of Eq. (30) by $(1-\alpha^2)J_1(\alpha p_m a)$ and integrating from 0 to 1, we have the simultaneous linear equations

$$\begin{aligned}
A_m + \frac{E}{1-\nu^2} \frac{p_m a}{[J_0(p_m a)]^2} \sum_{m'=1}^{\infty} A_{m'} \int_0^1 \{C_{m'}(\alpha; \xi) + \nu D_{m'}(\alpha; \xi)\} J_1(\alpha p_m a) d\alpha \\
= \frac{E}{1-\nu^2} \frac{p_m a}{[J_0(p_m a)]^2} \int_0^1 (1-\alpha^2) A(\alpha; \xi) J_1(\alpha p_m a) d\alpha \quad m=1, 2, \dots\dots \quad \dots\dots\dots(32)
\end{aligned}$$

By solving Eqs. (32), therefore we can determine A_m . Substituting A_m into Eqs. (7), we can obtain the residual stress $\sigma_z^*(r)$ in the original long cylinder. Furthermore, by substitution of A_m into Eq. (31), we can get $\Theta(\alpha)$. If $\Theta(\alpha)$ is known, we can obtain the residual stresses $\sigma_r^*(r)$, $\sigma_\theta^*(r)$ from the Sachs equations.

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Measurement of Residual Stresses in a Cylinder*

(2nd Report, Evaluation and Experiment on a Short Specimen Cut Off from a Long Cylinder)

By Haruo KAWAGOE**

On the basis of the analysis of his previous report, the author evaluates deformations in various specimens cut off from a long cylinder in which a given longitudinal residual stress exists. According to the present results, it appears that if $\xi > 2 \sim 2.25$ (ξ is the ratio of length to diameter of a specimen), we can use the equations introduced by Sachs with safety for determining the residual stresses, and if $\xi < 2$, the distributions of the residual stresses computed from the Sachs equations are somewhat different from those in the original cylinder except for its outer half portion.

Furthermore, the author shows that the agreement between the calculated result and the experimental is good, and proposes a method for determining approximately the distributions of residual stresses in the original long cylinder from the data with a short specimen ($\xi > 1.25$).

1. Introduction

In the previous report⁽¹⁾, we derived the theoretically exact method for determining the residual stresses in a long cylinder from the experimental data obtained with the short specimen separated from that cylinder. However the above method is very laborious to perform and in effect not applicable. Therefore in this paper we calculate numerically $\lambda_{Im}(0; \xi)$, $\vartheta_{Im}(0; \xi)$, $\lambda_{Im}(\alpha; \xi)$ and $\vartheta_{Im}(\alpha; \xi)$ in the previous report though they are approximate. And then we get the limit value of ξ , the ratio of length to diameter of the separated specimen, and show that for ξ beyond that value we can determine the residual stresses in the long cylinder by substituting the experimental data obtained with the separated specimen into the Sachs equations. Furthermore, we derive a practical method for determining the residual stresses in the long cylinder

from the experimental data obtained with a specimen of which ξ is less than that limit value.

The notations used in the previous report are also useful in the present report.

2. Numerical calculation

2.1 The separation process of the solid cylinder

Consider a sufficiently long solid cylinder of radius a having only the residual stress

$$\sigma_{zm}^*(r) = A_m J_0(p_m r) \quad m=1, 2, \dots \quad (1)$$

As stated in chapter 3 of the previous report, $\lambda_{Im}(0; \xi)$, $\vartheta_{Im}(0; \xi)$ respectively are the values for $r=a$, $z=0$ of the strain components ϵ_z , ϵ_θ of the specimen of length $2l$ ($\xi=l/a$) separated from the above cylinder which are derived from the stress function $\phi_m(r, z; 0, \xi)$, i.e., the solution of the equation

$$\nabla^4 \phi = 0 \quad (2)$$

satisfying the boundary conditions

for $z = \pm l$

$$\left. \begin{aligned} \frac{\partial}{\partial z} \left\{ (2-\nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} &= -\sigma_{zm}^*(r), \\ \phi &= 0 \end{aligned} \right\}$$

* Received 1st June, 1966.

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