outer rings and as a consequence, it is subjected to the rotation of inner ring if there are obliquities in inner and outer rings respectively, which make a vari-clearance type excitation on P_z .

(6) Mathieu type excitation⁽²⁾ in axial clearance on P_z invariably rises, but this feature may vanish with an increase in the number of balls. This excitation, therefore, must be distinguished from the others as it is characteristic of the case with a small number of balls.

(7) Besides the cases mentioned above, there may be other irregularities, with their effects becoming more complicated, but their estimation may be carried out with comparative ease by the aid of illustration, as shown in Fig. 4, equally as the effect of various parameters, i.e. the radial clearance, radial loading, dimension and/or aspect of each link and so on.

The models and their characteristics considered here may not always correspond perfectly to the practical ball bearing, but the authors believe that they will cover main characteristics of this mechanism. Since it is impossible to consider a case

without irregularities in practice or a case where all the balls may be loaded uniformly in the sense of an ideal bearing (there is even a report⁽⁴⁾ on the axial loading problem in which the experimental data are explained by calculation with an assumption of only two of balls to be loaded), the partial loading i.e. the case with a small number of balls is likely to occur. The authors are desirous of calling attention to the significance of the analysis from such viewpoint as mentioned above.

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References

- (1) H. Shimizu and H. Tamura: *Bulletin of JSME*, Vol. 9, No. 35 (1966), p.524.
- (2) A. Tamura and O. Taniguchi: *Trans. Japan Soc. Mech. Engrs.*, Vol. 26, No. 161 (1960), p.19, etc.
- (3) T. Yamamoto: *Trans. Japan Soc. Mech. Engrs.*, Vol. 26, No. 166 (1960), p.813, etc.
- (4) A. Igarashi: *Trans. Japan Soc. Mech. Engrs.*, Vol. 25, No. 155 (1959), p.675.
- (5) S. Itoo: *BEARINGU ENJINIA (Bearing Engineer, Tech. Rep. TOYO Bearing Co. LTD.)* Vol. 8, No. 1 (1959), p.1014.

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A Study of the Statics of Granular Materials*

(1st Report, Theoretical Analysis)

By Takaaki NAGAO**

The statics of granular systems of nearly spherical particles without adhesive forces was studied and a general theory applicable to many kinds of granular materials was obtained.

1. Introduction

It is industrially important to make clear the characteristics of granular materials, especially their statics. The earth-pressure theories of Rankine and Coulomb⁽¹⁾ are typical as the statics of granular materials. And Kawakita, Smith, Athy

and many other investigators proposed the formulae for compressed powder⁽²⁾. But they are more or less empirical and not satisfactory in generality. The author theoretically dealt with the system of nearly spherical granules without adhesive forces such as of water between themselves, and got a general theory fit for many kinds of granular materials.

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Symbols

$B=(B^j)$: body force per unit volume

$\Delta=\sum_j \frac{\partial(\delta u^j)}{\partial x^j}$: bulk strain-increment

$\mathcal{E}$: elastic strain energy per unit volume

h : ratio of the perpendicular displacement of contact surface to the radius of the particle

k : coefficient of friction between particles

Λ : number of contacts per unit volume

$\lambda = r^3 \Lambda$

$\delta \mathfrak{F}$: incremental frictional loss per unit volume

N : number of particles per unit volume

ν : Poisson's ratio

$\delta \omega=(\delta \omega^j)$: mean incremental rotation vector of particles

$P_{kl}=(P_{kl}^j)$: force exerted on k -particle by l -particle

$P=(P^{ij})$: (Pseudo-) stress

p : porosity

Π_1, Π_2 : relative displacements of the contact surface in the tangential and perpendicular directions

$\Pi_{2P}=\frac{\partial \Pi_2}{\partial P}, \Pi_{2Q}=\frac{\partial \Pi_2}{\partial Q}$ etc.

$\Psi = -\frac{1}{2r\Lambda}(P^{11}+P^{22}+P^{33})$, mean value of the forces which act around the particles

r : mean radius of particles

τ : probability of non-slide at a contact point

$\delta U_{kg}=(\delta u_{kg}^j)$: incremental displacement of the center of gravity of k -particle

$\delta U=(\delta u^j)$: mean incremental displacement of particles

$\hat{U}_{kl}$: displacement of k -particle as a rigid body at its contact point with l -particle

$\check{U}_{kl}$: local displacement of k -particle relative to itself as a rigid body at its contact point with l -particle

U : mean elastic strain energy stored by a particle at one of its contact points

$\delta \mathfrak{W}$: incremental work done by external forces per unit volume

x_{kg}^j : j -directional coordinate of the center of gravity of k -particle

$X_{kl}=(X_{kl}^j) = (x_{kg}^j - x_{lg}^j)$, sometimes simplified as X

$\delta X_{kl}=(\delta X_{kl}^j) = (\delta u_{kg}^j - \delta u_{lg}^j)$, sometimes simplified as δX

2. The equations of equilibrium of stresses and moments and the equation of continuity

We consider a state of piled homogeneous spherical particles. Suppose these particles exert only an elastic or plastic action on the adjacent particles in contact with themselves. x_{kg}^j ($j=1, 2, 3$; $k=1, 2, \dots, N$) represents the coordinates of the center of gravity of the k -th particle (called k -particle here after), when each particle is given its own number. We consider two particles k and l in contact. $P_{kl}=(P_{kl}^j)$ denotes the force exerted by l on k . δS_i denotes the surface perpendicular to x^i direction. We call the particles cut by this surface the particles belonging to δS_i . We write the sum of the forces which are exerted on the particles belonging to δS_i by the particles in contact with them on their positive side of x^i as follows.

$$P^{ij} \delta S_i = \sum_{k \in \delta S_i} P_{kl}^j \quad \dots\dots\dots (1)$$

Then we obtain a tensor, that is the stress

$$P=(P^{ij}) \quad \dots\dots\dots (2)$$

Let the center of gravity of k -particle be G_k , the moment around G_k be M_k , and the contact point of k and l be C_{kl} , the body force of k -particle be $B_k=(B_k^j)$, then

$$\sum_l P_{kl}^j + B_k^j = 0 \quad \dots\dots\dots (3)$$

$$M_k = \sum_l \overrightarrow{G_k C_{kl}} \times P_{kl} = 0 \quad \dots\dots\dots (4)$$

Here we define

$$\sum_{k \in \delta v} B_k^j = B^j \delta v \quad \dots\dots\dots (5)$$

When we sum Eq. (3) about 1 in δv , the forces P_{kl} and P_{lk} cancel each other and the forces which act on the surface of δv remain. As the result we obtain the stress equations

$$\left. \begin{aligned} \frac{\partial P^{11}}{\partial x^1} + \frac{\partial P^{21}}{\partial x^2} + \frac{\partial P^{31}}{\partial x^3} + B^1 &= 0, \\ \frac{\partial P^{12}}{\partial x^1} + \frac{\partial P^{22}}{\partial x^2} + \frac{\partial P^{32}}{\partial x^3} + B^2 &= 0, \\ \frac{\partial P^{13}}{\partial x^1} + \frac{\partial P^{23}}{\partial x^2} + \frac{\partial P^{33}}{\partial x^3} + B^3 &= 0, \end{aligned} \right\} \quad \dots\dots\dots (6)$$

or

$$\sum_{i=1}^3 \frac{\partial (\delta P^{ij})}{\partial x^i} + \delta B^j = 0 \quad (j=1, 2, 3) \quad \dots\dots\dots (7)$$

Fig. 1

in the expression of stress-increments. These are the same as those in the theory of elasticity⁽³⁾. The equations of moments are obtained in the same way from Eq. (4). The result is

$$P^{ij} = P^{ji} \quad (i \neq j; i, j = 1, 2, 3) \quad (8)$$

Now we consider the relation of porosity p and mean incremental displacement (δu) . We apply the equation of continuity in the dynamics of continuum to this case.

$$\begin{aligned} \rho &= \frac{\gamma}{g}(1-p) \\ \delta \rho + \operatorname{div}(\rho \delta \mathbf{u}) &= 0 \\ \therefore -\delta p + (1-p) \operatorname{div} \delta \mathbf{u} - \delta \mathbf{u} \operatorname{grad} p \\ \text{or } -\delta p + (1-p) \left\{ \frac{\partial(\delta u^1)}{\partial x^1} + \frac{\partial(\delta u^2)}{\partial x^2} + \frac{\partial(\delta u^3)}{\partial x^3} \right\} \\ &\quad - \left(\delta u^1 \frac{\partial p}{\partial x^1} + \delta u^2 \frac{\partial p}{\partial x^2} + \delta u^3 \frac{\partial p}{\partial x^3} \right) = 0 \end{aligned} \quad (9)$$

We call this the equation of continuity, too. Here

$$\delta \mathbf{u} = \frac{\sum_{k \in \partial v} r_k^3 \delta \mathbf{u}_{kg}}{\sum_{k \in \partial v} r_k^3} \quad (10)$$

3. Stress-strain relations

Consider the contact point of k - and l -particles. ζ denotes the direction passing through the centers of these two particles, and we put ξ and η directions perpendicularly with each other and the ζ direction such that $P_\eta = 0$, when $\mathbf{P}_{kl} = (P_\xi, P_\eta, P_\zeta)$. We write the local displacement of the center of this contact point relative to the k -particle as a rigid body $\tilde{\mathbf{u}}_{kl} = (u_\xi, u_\eta, u_\zeta)$ as follows.

$$u_\xi = \Pi_1(P, P_\zeta), \quad u_\eta = 0, \quad u_\zeta = \Pi_2(P, P_\zeta) \quad (11)$$

$$\begin{aligned} \frac{k^2}{1+k^2} \delta X_{kl}^1 &= (S \delta \mathbf{P}) \left(\frac{P_{kl}^1}{P^2} - \frac{X_{kl}^1}{P X} \right) + (R \delta \mathbf{P}) \left(\frac{X_{kl}^1}{X^2} - \frac{P_{kl}^1}{P X} \right) + \frac{r_k + r_l}{r_k} \\ &\quad \times \frac{X_{kl}^2 P_{kl}^3 - X_{kl}^3 P_{kl}^2}{X^2 P^2} (\Omega \delta \boldsymbol{\omega}_k - \Phi \delta \mathbf{P}) \end{aligned} \quad (18)$$

The expressions for δX_{kl}^2 and δX_{kl}^3 are obtained by the cyclic change of subscripts 1→2→3, where

$$\begin{aligned} S \delta \mathbf{P} &= \frac{r_k + r_l}{\sqrt{1+k^2}} \mathbf{P} \delta \mathbf{P} \left[\frac{1}{P} \left\{ 1 - 2 \frac{\Pi_2 \left(P, \frac{P}{\sqrt{1+k^2}} \right)}{r_k + r_l} \right\} \right. \\ &\quad \left. - \frac{2}{r_k + r_l} \left\{ \Pi_{2P} \left(P, \frac{P}{\sqrt{1+k^2}} \right) + \frac{1}{\sqrt{1+k^2}} \times \Pi_{2Q} \left(P, \frac{P}{\sqrt{1+k^2}} \right) \right\} \right] - X \delta \mathbf{P} \end{aligned} \quad (19)$$

$$R \delta \mathbf{P} = -2(r_k + r_l) \left\{ \Pi_{2P} \left(P, \frac{P}{\sqrt{1+k^2}} \right) + \frac{1}{\sqrt{1+k^2}} \Pi_{2Q} \left(P, \frac{P}{\sqrt{1+k^2}} \right) \right\} \frac{\mathbf{P} \delta \mathbf{P}}{P} \quad (20)$$

$$\Omega \delta \boldsymbol{\omega}_k = \frac{r_k}{r_k + r_l} [(\mathbf{P} X)(X \delta \boldsymbol{\omega}_k) - (\mathbf{P} \delta \boldsymbol{\omega}_k)(X^2)] \quad (21)$$

$$\Phi \delta \mathbf{P} = \frac{\sqrt{1+k^2}}{k P} \Pi_1 \left(P, \frac{P}{\sqrt{1+k^2}} \right) (X \times \mathbf{P}) \delta \mathbf{P} \quad (22)$$

and $\delta \boldsymbol{\omega}_k$ denotes the vector of incremental rotation of k -particle. At a contact point without slide, from Eq. (14),

$$\delta X_{kl}^1 = \delta \omega_{kl}^2 X_{kl}^3 - \delta \omega_{kl}^3 X_{kl}^2 - 2 \frac{X_{kl}^1}{r_k + r_l} \{ \Pi_{2P}(P, P) + \Pi_{2Q}(P, P) \} \frac{\mathbf{P} \delta \mathbf{P}}{P} \quad (23)$$

$$\text{with } P = |\mathbf{P}_{kl}| = \sqrt{P_\xi^2 + P_\zeta^2} \quad (12)$$

Now we set

$$\left. \begin{aligned} X_{kl}^j &= x_{kg}^j - x_{lg}^j \\ \delta X_{kl}^j &= \delta u_{kg}^j - \delta u_{lg}^j \\ X &= (X_{kl}^j) = X_{kl} \\ P &= (P_{kl}^j) = P_{kl} \end{aligned} \right\} \quad (13)$$

for convenience sake. $(\tilde{\mathbf{u}}_{kl}^j)$ is the expression of $\tilde{\mathbf{u}}_{kl}$ in the (x^1, x^2, x^3) coordinates. we study the relations of the incremental displacement $\delta \mathbf{u}_{kl}$ and the increment of force $\delta \mathbf{P}_{kl}$.

First we classify the contact points into two groups, the contacts where slide occurs and where slide does not occur. Suppose at a contact point with slide;

(1) The force $\mathbf{P}_{kl}$ lies on the friction cone on the contact surface,

(2) The direction of slide is on the surface formed by $\mathbf{P}_{kl}$ and $\bar{\mathbf{G}}_k \bar{\mathbf{G}}_l$,

(3) The perpendicular displacement of the contact surface is represented by u_ζ in (11).

At a contact point without slide, when $\delta \mathbf{u}_{kl}$ and $\delta \mathbf{u}_{lk}$ denote the displacements of k - and l -particles at the contact point,

$$\delta \mathbf{u}_{kl} = \delta \mathbf{u}_{lk} \quad (14)$$

holds. At a contact point with slide, from the suppositions (1), (2) and (3),

$$\frac{P X}{|\mathbf{P}| |\mathbf{X}|} = \frac{1}{\sqrt{1+k^2}} \quad (15)$$

$$\bar{\mathbf{G}}_k \bar{\mathbf{G}}_l (\mathbf{P} \times \delta \mathbf{u}_{kl}) = 0 \quad (16)$$

$$|\mathbf{X}| = (r_k + r_l) - 2 \Pi_2 \left(P, \frac{P}{\sqrt{1+k^2}} \right) \quad (17)$$

We solve these equations with respect to δX_{kl}^j , and obtain

where we take

$$\left. \begin{aligned} \delta\omega_{kl} &\equiv \frac{r_k \delta\omega_k^j + r_l \delta\omega_l^j}{r_k + r_l} \\ \Pi_{2P} &\equiv \frac{\partial \Pi_2(P, Q)}{\partial P}, \quad \Pi_{2Q} \equiv \frac{\partial \Pi_2(P, Q)}{\partial Q} \end{aligned} \right\} \dots (24)$$

Here we have obtained the expressions of (δX_{kl}^j) at contact points with and without slide. By averaging them, the stress-strain relations will be derived. The way of averaging is as follows.

(1) We suppose the correlation between X_{kl} and P_{kl} is negligible and they are averaged separately.

(2) We take the value of a function corresponding to the mean value of any variable instead of the mean value of the function.

Let, $E^i [\quad]$ stand for the mean value of any variable at the surface δS_i . When we consider the particles in contact with the particles belonging to S_i on their positive side, the number of contacts

per unit area is rA from the stochastic consideration. Therefore

$$E^i[P_{kl}^j] = \frac{P_{ij}^j}{rA} \dots (25)$$

Now we may write

$$\delta X_{kl}^j = \sum_i \frac{\partial(\delta u^j)}{\partial x^i} X_{kl}^j \dots (26)$$

Let δX_{kls}^j and δX_{kln}^j denote the relative displacements between the particles at contact points with and without slide, and P_s^{ij} and P_n^{ij} denote the parts of the stress supported by the contact points with and without slide respectively. Then we suppose

$$\left. \begin{aligned} \delta X_{kl}^j &= \tau \delta X_{kls}^j + (1-\tau) \delta X_{kln}^j \\ P_s^{ij} &= (1-\tau) P_{ij}^j, \quad P_n^{ij} = \tau P_{ij}^j \end{aligned} \right\} \dots (27)$$

holds. Using Eqs. (25) and (26), we average Eqs. (18) and (23) separately, compose them as in Eq. (27), and obtain the stress-strain relations.

$$\begin{aligned} -\frac{2k^2}{3(1+k^2)} \frac{\partial(\delta u^1)}{\partial x^1} &= \frac{rA(1-\tau)}{\sqrt{1+k^2}} \frac{\sum_j P_{11}^j P_{1j}^j \delta\left(\frac{P_{1j}^j}{rA}\right)}{[\sum_j (P_{1j}^j)^2]^{3/2}} + \frac{2}{3} \frac{rA(1-\tau)}{1+k^2} \frac{\sum_j P_{1j}^j \delta\left(\frac{P_{1j}^j}{rA}\right)}{\sum_j (P_{1j}^j)^2} \\ &+ \frac{2}{3} \frac{rA(1-\tau) P_{11}^{11}}{\sum_j (P_{1j}^j)^2} \delta\left(\frac{P_{11}^{11}}{rA}\right) + \frac{rA(1-\tau)}{2\sqrt{1+k^2}} \frac{\delta\left(\frac{P_{11}^{11}}{rA}\right)}{\sqrt{\sum_j (P_{1j}^j)^2}} + \frac{2k^2(1-\tau)}{3(1+k^2)r} \frac{\sum_j P_{1j}^j \delta\left(\frac{P_{1j}^j}{rA}\right)}{\sqrt{\sum_j (P_{1j}^j)^2}} \\ &\times \left\{ \Pi_{2P} \left(\frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA}, \frac{1}{\sqrt{1+k^2}} \frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA} \right) + \Pi_{2Q} \left(\frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA}, \frac{1}{\sqrt{1+k^2}} \frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA} \right) \right\} \\ &- (1-\tau) \frac{\sqrt{1+k^2}}{2k} \frac{(rA)^2}{[\sum_j (P_{1j}^j)^2]^2} \Pi_1 \left(\frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA}, \frac{1}{\sqrt{1+k^2}} \frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA} \right) \\ &\times \left[\{ (P_{12}^{12})^2 + (P_{13}^{12})^2 \} \delta\left(\frac{P_{11}^{11}}{rA}\right) - P_{11}^{11} P_{12}^{12} \delta\left(\frac{P_{12}^{12}}{rA}\right) - P_{11}^{11} P_{13}^{12} \delta\left(\frac{P_{13}^{12}}{rA}\right) \right] + \frac{(1-\tau)}{4\sqrt{1+k^2}} \frac{P_{13}^{12} \delta\omega^2 - P_{12}^{12} \delta\omega^3}{\sqrt{\sum_j (P_{1j}^j)^2}} \\ &+ \frac{2\tau}{3r} \frac{k^2}{1+k^2} \frac{\sum_j P_{1j}^j \delta\left(\frac{P_{1j}^j}{rA}\right)}{\sqrt{\sum_j (P_{1j}^j)^2}} \left\{ \Pi_{2P} \left(\frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA}, \frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA} \right) + \Pi_{2Q} \left(\frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA}, \frac{\sqrt{\sum_j (P_{1j}^j)^2}}{rA} \right) \right\} \end{aligned} \dots (28)$$

$$\begin{aligned} -\frac{2k^2}{3(1+k^2)} \frac{\partial(\delta u^1)}{\partial x^2} &= \frac{rA(1-\tau)}{\sqrt{1+k^2}} \frac{\sum_j P_{21}^j P_{2j}^j \delta\left(\frac{P_{2j}^j}{rA}\right)}{[\sum_j (P_{2j}^j)^2]^{3/2}} + \frac{2rA(1-\tau)}{3 \sum_j (P_{2j}^j)^2} P_{21}^{21} \delta\left(\frac{P_{22}^{22}}{rA}\right) \\ &+ \frac{rA(1-\tau)}{4\sqrt{1+k^2}} \frac{1}{\sqrt{\sum_j (P_{2j}^j)^2}} \delta\left(\frac{P_{21}^{21}}{rA}\right) - (1-\tau) \frac{\sqrt{1+k^2}}{2k} \frac{(rA)^2}{[\sum_j (P_{2j}^j)^2]^2} \Pi_1 \left(\frac{\sqrt{\sum_j (P_{2j}^j)^2}}{rA}, \frac{1}{\sqrt{1+k^2}} \frac{\sqrt{\sum_j (P_{2j}^j)^2}}{rA} \right) \\ &\times \left[\{ 2(P_{23}^{23})^2 + (P_{22}^{22})^2 \} \delta\left(\frac{P_{21}^{21}}{rA}\right) - P_{21}^{21} P_{22}^{22} \delta\left(\frac{P_{22}^{22}}{rA}\right) - 2P_{21}^{21} P_{23}^{23} \delta\left(\frac{P_{23}^{23}}{rA}\right) \right] \\ &+ \frac{2}{3} \frac{(1-\tau) P_{23}^{23} \sum_j P_{2j}^j \delta\omega^j}{\sum_j (P_{2j}^j)^2} + \frac{(1-\tau)}{4\sqrt{1+k^2}} \frac{2P_{23}^{23} \delta\omega^2 - P_{22}^{22} \delta\omega^3}{\sqrt{\sum_j (P_{2j}^j)^2}} + \frac{2k^2}{3(1+k^2)} \tau \delta\omega^3 \dots (29) \\ -\frac{2k^2}{3(1+k^2)} \frac{\partial(\delta u^1)}{\partial x^3} &= \frac{rA(1-\tau)}{\sqrt{1+k^2}} \frac{\sum_j P_{31}^j P_{3j}^j \delta\left(\frac{P_{3j}^j}{rA}\right)}{[\sum_j (P_{3j}^j)^2]^{3/2}} + \frac{2}{3} \frac{rA(1-\tau)}{\sum_j (P_{3j}^j)^2} P_{31}^{31} \delta\left(\frac{P_{33}^{33}}{rA}\right) \\ &+ \frac{rA(1-\tau)}{4\sqrt{1+k^2}} \frac{1}{\sqrt{\sum_j (P_{3j}^j)^2}} \delta\left(\frac{P_{31}^{31}}{rA}\right) - (1-\tau) \frac{\sqrt{1+k^2}}{2k} \frac{(rA)^2}{[\sum_j (P_{3j}^j)^2]^2} \Pi_1 \left(\frac{\sqrt{\sum_j (P_{3j}^j)^2}}{rA}, \frac{1}{\sqrt{1+k^2}} \frac{\sqrt{\sum_j (P_{3j}^j)^2}}{rA} \right) \end{aligned}$$

$$\begin{aligned} & \times \left[\{2(P^{32})^2 + (P^{33})^2\} \delta \left(\frac{P^{31}}{rA} \right) - 2P^{31}P^{32} \delta \left(\frac{P^{32}}{rA} \right) - P^{31}P^{33} \delta \left(\frac{P^{33}}{rA} \right) \right] \\ & + \frac{2}{3} \frac{(1-\tau)P^{32} \sum_j P^{3j} \delta \omega^j}{\sum_j (P^{3j})^2} + \frac{(1-\tau)}{4\sqrt{1+k^2}} \frac{P^{33} \delta \omega^2 - 2P^{32} \delta \omega^3}{\sqrt{\sum_j (P^{3j})^2}} - \frac{2k^2}{3(1+k^2)} \tau \delta \omega^2 \dots \dots \dots (30) \end{aligned}$$

The expressions for $\frac{\partial(\delta u^2)}{\partial x^j}$, $\frac{\partial(\delta u^3)}{\partial x^j}$ are obtained by the cyclic change of subscripts 1→2→3.

4. Energy conservation

$\mathcal{E}$ denotes the elastic strain energy per unit volume. Let $\delta \mathfrak{L}$ be the frictional loss, $\delta \mathfrak{W}$ be the work done by the external forces, and $\delta \mathfrak{Y}$ be the plastic work done, then

$$\delta \mathfrak{W} = \delta \mathfrak{L} + \delta \mathfrak{Y} + \delta \mathcal{E} + \mathcal{E} \Delta \dots \dots \dots (31)$$

holds, where $\mathcal{E} \Delta$ means the change of the elastic strain energy due to the volumetric change of the system. $\delta \mathcal{E}$ is the energy change per unit volume. We evaluate every term in Eq. (31), and obtain the equation of energy conservation. As the interaction is not yet determined here, the final result will be described later.

5. Geometrical relation

We study the relation of porosity p and the number of contacts A . We connect the centers of particles in contact with each other through their contact points, and call them links. We obtain closed loops composed of some vertices, or the centers of particles, and some links. Some of them have some joint parts. We connect both ends of these joint parts together and call them imaginary links. Furthermore by appropriately connecting the vertices of these loops, we make sets of triangles. Then the space is divided into polyhedrons covered by triangles. Again by appropriately connecting the vertices of the polyhedrons by imaginary links, they can be divided into tetrahedrons. Here we set the symbols of the numbers of vertices N , (real) links A , imaginary links X , triangles T , and

tetrahedrons Z per unit volume. Then by the Euler's theorem in the topological geometry⁽⁴⁾,

$$N - (A + X) + T - Z = 0 \dots \dots \dots (32)$$

And

$$T = 2Z \dots \dots \dots (33)$$

$$Z = A + X - N \dots \dots \dots (34)$$

For simplicity we suppose the real link length is l and uniform and imaginary link length l' . We classify these tetrahedrons into 12 kinds, and name them $\tau_{\mu\nu}$ as noted in Fig. 3. Then we find the probability $p_{\mu\nu}$ that the tetrahedron $\tau_{\mu\nu}$ is formed as follows. The probability that one edge of a tetrahedron is an imaginary link is

$$x = \frac{X}{A + X} \dots \dots \dots (35)$$

and that for a real link is

$$1 - x = \frac{A}{A + X} \dots \dots \dots (36)$$

Therefore taking the configuration in the space into consideration, we find the probability of $\tau_{\mu\nu}$ that is

$$p_{\mu\nu} = \frac{12}{n_{\mu\nu}} x^\mu (1-x)^{6-\mu} \dots \dots \dots (37)$$

where $n_{\mu\nu}$ denotes how many times $\tau_{\mu\nu}$ takes the same configuration when it is turned once in the space, and is noted in Fig. 3. We can evaluate the volume and the sum of four vertex-angles of every $\tau_{\mu\nu}$ and then their means $\bar{V}$ and $\bar{\Omega}$ using the weight $p_{\mu\nu}$. Putting

$$\frac{l' - l}{l} = \eta \dots \dots \dots (38)$$

we obtain as the result

$$\bar{V} = \frac{l^3}{6\sqrt{2}} [1 + v(\eta, x)] \dots \dots \dots (39)$$

$$\bar{\Omega} = 12 \arccos \frac{1}{3} - 4\pi + \omega(\eta, x) \dots \dots \dots (40)$$

where

Fig. 2

Fig. 3

$$v(\eta, x) \cong 3x\eta + \eta^2 x \left(-\frac{21}{4} + \frac{33}{4}x - \frac{17}{12}x^2 + \frac{17}{12}x^3 \right) + \eta^3 x \left(-\frac{27}{8} + \frac{39}{8}x - \frac{1}{2}x^2 \right) \\ + \eta^4 x \left(-\frac{135}{64} - \frac{1523}{64}x + \frac{177}{8}x^2 + \frac{1023}{16}x^3 - 90x^4 + 30x^5 \right) + O(\eta^5) \quad (41)$$

$$\omega(\eta, x) \cong -\frac{4}{\sqrt{2}}\eta^2 x(1-x) + \frac{\eta^3}{\sqrt{2}}x(1-x) \left(-\frac{1111}{54} + \frac{538}{9}x + \frac{2191}{54}x^2 - \frac{12715}{27}x^3 + 438x^4 \right) \\ + \frac{\eta^4}{\sqrt{2}}x(1-x) \left(-\frac{127}{108} + \frac{1189}{9}x - \frac{2427341}{1728}x^2 + \frac{1698221}{864}x^3 + \frac{20155}{576}x^4 \right) + O(\eta^5) \quad (42)$$

At the same time

$$\frac{V}{Z} = \bar{V} \quad (43)$$

$$\frac{4\pi N}{Z} = \bar{Q} \quad (44)$$

hold, where V denotes the unit volume. For η and x , the following approximation was obtained, supposing most of polyhedrons have only 0, 1 or 2 imaginary links.

$$\eta = \eta_0 + \eta_1 h + x(\eta_2 + \eta_3 h) + O(x^2) \quad (45)$$

$$\left. \begin{aligned} \eta_0 &= 0.480361, & \eta_2 &= 0.20721, \\ \eta_1 &= 0.355553, & \eta_3 &= -9.763415, \\ h &= 1 - \frac{l}{2r} \end{aligned} \right\} \quad (46)$$

From Eqs. (34), (39), (40), (43), (44) and (45), we find the relation of p and

$$1-p = \lambda[F(\lambda) + G(\lambda)h], \quad \lambda = r^3 \Lambda \quad (47)$$

$$F(\lambda) = \frac{3.01 - 1.323\lambda}{2.308 - 0.347\lambda} - 0.557 \times \frac{2.505 - 1.768\lambda}{4.813 - 2.12\lambda} \quad (48)$$

$$G(\lambda) = \frac{-7.34 + 18.65\lambda - 5.28\lambda^2}{(2.308 - 0.347\lambda)^2} - \frac{2.84 + 0.956\lambda}{4.813 - 2.12\lambda} \\ + \frac{(1.395 - 0.985\lambda)(4.68 + 1.570\lambda)}{(4.813 - 2.12\lambda)^2} \quad (49)$$

$G(\lambda)$ fluctuates widely depending upon how to cut the small terms. The change of $1-p$ for $h=0$ with respect to λ is shown in Fig. 4.

On the same graph are shown the data on the ideal case of the piled spheres of uniform scale and the calculated value from the measured data in Smith's experiments⁽⁵⁾, and we find them rather different from what we obtained. This seems to be caused by the inaccuracy of the approximation of η . The cross point of the graph corresponds to the closest packing of spheres.

Here we consider the meaning of $G(\lambda)$. Suppose the state is $\lambda = \lambda_0$ and $p = p_0$ before loading, and $1-p_0 = \lambda_0 F(\lambda_0)$ (50)

If the volume changes from V to V' after loading, the number of contacts in the unit volume increases. This increase is due to the three facts that:

- (1) The contacts in V are squeezed into V' .
- (2) They come to be packed closer (the configuration changes).
- (3) The particles are pressed and some particles newly get into contact.

Then we put spheres of radius $r(1-h)$ in the place

of particles of radius r in V' in consideration.

The number of contacts in such a system is equal to what remains after we take off the increase of contacts due to the case 3 from Λ . Then

$$\lambda' \equiv r^3(1-h)^3 \Lambda' \equiv r^3(1-3h)\Lambda' \quad (51)$$

And we suppose we can write

$$\Lambda' = (1-\vartheta h)\Lambda \quad (52)$$

where ϑ means the influence of the increase due to the case 3. Then

$$\lambda' = r^3(1-3h)(1-\vartheta h)\Lambda \cong \lambda(1-3h-\vartheta h) \quad (53)$$

and the porosity p' is given as follows.

$$\left. \begin{aligned} (1-p')V' &= \frac{4}{3}\pi r^3(1-h)^3 N \\ (1-p)V &= \frac{4}{3}\pi r^3 N \end{aligned} \right\} \quad (54)$$

$$1-p' = (1-3h)(1-p) \quad (55)$$

Supposing here

$$1-p' = \lambda' F(\lambda') \quad (56)$$

and substituting Eqs. (53) and (55) into Eq. (56), we obtain

$$1-p = \lambda[F(\lambda) - \{\vartheta F(\lambda) + (3+\vartheta)\lambda F'(\lambda)\}h] \quad (57)$$

We compare this with Eq. (47) and find

$$G(\lambda) = -\{\vartheta F(\lambda) + (3+\vartheta)\lambda F'(\lambda)\} \quad (58)$$

This means that the influence of the case 3 is the big factor determining $G(\lambda)$.

6. Analysis of interactions

At the beginning of this study, it was supposed that every particle was an elastic perfect sphere.

Fig. 4 Porosity and number of contacts

But when we compared the theoretical results with the results of compression tests of actual granular materials, the theoretical value of load was about 50~500 times the measured value for the identical strain value. Furthermore according to the analysis for the elastic interaction, the stress was in proportion to the 1.5th power of the strain, whereas it was mostly the 2nd or 3rd power according to the experimental results. We thought this was caused by a big influence of the structure of the surface of particle, and observed it by a microscope, and found that the surface was covered with asperities.

Accordingly the author devised two models of a particle which seems to be fit to an actual granule. These are round particles with regular asperities around themselves. In this case, we obtain many kinds of interactions combining the various cases that the deformations of the inner parts and asperities of particles are elastic, plastic or negligible. The author considers that these interactions explain the results of actual compression tests very well. Analysis is as follows.

6.1 The case of elastic contact of spheres

Consider the elastic contact of two perfect spheres. For this problem we have Hertz's solution⁽⁶⁾, but this is the case where the force acts on the surface of contact perpendicularly. The solution was given by Mindlin⁽⁷⁾ for the case where there was the tangential component of the force. And Johnson⁽⁸⁾ carried out the experiments according to this solution and the result almost completely supported it. According to this result, the relation of w , u , the perpendicular and tangential components of the displacement of the center of contact surface, and P_z , P_x , the perpendicular and tangential components of the force at the contact surface, is given as follows:

$$\left. \begin{aligned} w &= B_1 P_z^{2/3}, \\ u &= A_1 k P_z^{2/3} \left[1 - \left(1 - \frac{P_x}{k P_z} \right)^{2/3} \right] \end{aligned} \right\} \dots\dots\dots (59)$$

where

$$B_1 = \frac{1}{2} \sqrt[3]{\frac{9\pi^2 \mu^2}{8r}}, \quad A_1 = \frac{2-\nu}{2(1-\nu)} B_1, \quad \mu = \frac{2(1-\nu^2)}{\pi E} \dots\dots\dots (60)$$

and ν is the Poisson's ratio and E is the Young's modulus. Here we supposed the radii and the physical properties of the two spheres were identical.

6.2 The case of plastic contact of spheres

We suppose the contact surface is uniformly subjected to the yielding pressure p_m in the case of plastic contact of spheres. This is supported by experiments according to Bowden and Tabor⁽⁹⁾.

First we consider the case of the perpendicularly

pressed surface. P means the load, and ρ_0 the radius of contact surface, r the radius of the particle, and w the perpendicular displacement of the center of surface, then the relation of w and P is

$$w = r - \sqrt{r^2 - \rho_0^2} \approx \frac{\rho_0^2}{2r} \dots\dots\dots (61)$$

$$\pi \rho_0^2 p_m = P,$$

$$\therefore w = \frac{P}{2\pi r p_m} \dots\dots\dots (62)$$

6.3 The case of the Model I-Plastic asperities

Consider the contact of particles as in Fig. 6. That is a perfect sphere with uniform asperities around itself. We call this Model I. We take the distance ρ from the center O on the x - y plane (the plane which contains the contact surface). n denotes the number of asperities per unit area, and P denotes the load perpendicularly acting on this surface. Let $W(\rho)$ denote the force acting on one asperity at the distance ρ , and ρ_0 denote the apparent radius of the contact surface.

Then

$$P = \int_0^{\rho_0} W(\rho) n 2\pi \rho d\rho \dots\dots\dots (63)$$

Let $w(\rho)$ be the displacement of the asperity at the distance ρ , then from a geometrical consideration,

Fig. 5 Contact of spheres

Fig. 6 Model I

$$w(\rho) = w - \frac{\rho^2}{2r}, \quad w = w(0) \dots \dots \dots (64)$$

We consider the case where the asperities are completely plastic and the inner parts of particles are elastic. The local (plastic) displacement of asperities $\hat{w}(\rho)$ is from Eq. (62)

$$\hat{w}(\rho) = \frac{W(\rho)}{2\pi R p_m} \dots \dots \dots (65)$$

where R is the radius of curvature of asperities. The elastic displacement of the inner part of particle $\tilde{w}(\rho)$ corresponds to the pressure $nW(\rho)$ acting on its surface and is given by the theory of elastic contact as follows⁽¹⁰⁾

$$\tilde{w}(\rho) = \frac{1-\nu^2}{\pi E} \int_S \frac{nW(\rho')}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \theta')}} \times \rho' d\rho' d\theta' \dots \dots \dots (66)$$

Here (ρ, θ) and (ρ', θ') means the points taken in the contact surface S . Here.

$$w(\rho) = w(\rho) + \tilde{w}(\rho) \dots \dots \dots (67)$$

so that we obtain the integral equation, substituting Eqs. (65), (66) and (64) into Eq. (67).

$$\begin{aligned} & \frac{W(\rho)}{2\pi R p_m} \\ & + \frac{n(1-\nu^2)}{\pi E} \int_S \frac{W(\rho')}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \theta')}} \\ & \times \rho' d\rho' d\theta' = w - \frac{\rho^2}{2r} \dots \dots \dots (68) \end{aligned}$$

This determines $W(\rho)$. In the case of the negligible elastic deformation of the inner part of particle, we can neglect the integral term of Eq. (68) and obtain

$$W(\rho) = 2\pi R p_m \left(w - \frac{\rho^2}{2r} \right) \dots \dots \dots (69)$$

Substituting this into Eq. (63) and after integration, we use

$$W(\rho_0) = 0 \quad \text{or} \quad w = \frac{\rho_0^2}{2r} \dots \dots \dots (70)$$

and obtain

Fig. 7 Model II

$$P = \frac{1}{2r} \pi^2 n p_m R \rho_0^4 \dots \dots \dots (71)$$

From Eqs. (70) and (71),

$$w = \sqrt{\frac{P}{2\pi^2 n p_m R r}} \dots \dots \dots (72)$$

6.4 Model II - Plastic small asperities, and negligible deformations of the big asperities and the inner parts of particles

Model II is shown in Fig. 7. In this case the interaction can be also similarly analysed. The relation of the displacement w and the load P is given by the formula

$$w = \frac{1}{2r} \left(\frac{6r^2 P}{\pi^2 n n' R R' p_m} \right)^{1/3} \dots \dots \dots (73)$$

where R and R' are the radii of curvature of the small and big asperities and n and n' are the numbers of them per unit area.

The author analysed and obtained many other interactions which are not mentioned here because of limited pages.

6.5 Summary of the interactions

The important interactions of them are expressed as follows.

$$w = b_m P^{1/m} \dots \dots \dots (74)$$

Here the values of m and b_m are shown in Table 1. In the above we discussed the cases of perpendicularly pressed contact surface.

We can extend them to the cases where the contact surface is pressed by an oblique force, with the supposition that in the case of the plastic contact of spheres⁽¹¹⁾,

$$p^2 + \alpha S^2 = p_m^2 \quad (\alpha: \text{const.}) \dots \dots \dots (75)$$

$$\frac{du}{dw} = \frac{S}{p} \dots \dots \dots (76)$$

hold, where the compressive stress is p and the shearing stress is S over the surface. The interactions obtained after the extension are as follows. In the case of plastic asperities ($m=1, 2, 3$),

Table 1

Model	Property	m	b_m	Remarks
Sphere	Elastic	$\frac{3}{2}$	$\frac{1}{2} \sqrt{\frac{9\pi^2 \mu^2}{8r}}$	$a_m = \frac{2-\nu}{2(1-\nu)} b_m$
	Plastic	1	$\frac{1}{2\pi r p_m}$	$\mu = \frac{2(1-\nu^2)}{\pi E}$
Model I	Elastic asperities	$\frac{5}{2}$	$\frac{1}{2r} \left(\frac{15\mu r}{2n \left(\frac{8R}{r} \right)^{1/2}} \right)^{2/5}$	Inner deformations neglected
	Plastic asperities	2	$\left(\frac{1}{2\pi^2 n p_m R r} \right)^{1/2}$	
Model II	Plastic small asperities	3	$\frac{1}{2r} \left(\frac{6r^2}{\pi^2 n n' R R' p_m} \right)^{1/3}$	Deformations of large asperities and inner parts neglected

$$\left. \begin{aligned} u_i &= b_m \sqrt{P^2 - P_i^2} P_i^{(1/m)-1} \left(1 + \alpha \frac{P^2 - P_i^2}{P_i^2} \right)^{1/2m} \\ \alpha_i &= b_m P_i^{1/m} \left(1 + \alpha \frac{P^2 - P_i^2}{P_i^2} \right)^{1/2m} \end{aligned} \right\} \quad (77)$$

In the case of elastic asperities $\left(m = \frac{3}{2}, \frac{5}{2} \right)$,

$$\left. \begin{aligned} u &= a_m k P_i^{1/m} \left[1 - \left(1 - \frac{\sqrt{P^2 - P_i^2}}{k P_i} \right)^{1/m} \right] \\ u &= b_m P_i^{1/m} \end{aligned} \right\} \dots\dots\dots (78)$$

where

$$a_m = \frac{2-\nu}{2(1-\nu)} b_m$$

and the symbols are as explained in chapter 3.

$$\begin{aligned} \frac{\partial(\delta u^1)}{\partial x^1} &= -\frac{3\sqrt{1+k^2}}{2k^2} r\Lambda(1-\tau) \frac{\sum_j P^{11} P^{1j} \delta\left(\frac{P^{1j}}{r\Lambda}\right)}{\{\sum_j (P^{1j})^2\}^{3/2}} - \frac{r\Lambda(1-\tau)}{k^2} \frac{\sum_j P^{1j} \delta\left(\frac{P^{1j}}{r\Lambda}\right)}{\sum_j (P^{1j})^2} \\ &\quad - \frac{1+k^2}{k^2} \frac{r\Lambda(1-\tau) P^{11}}{\sum_j (P^{1j})^2} \delta\left(\frac{P^{11}}{r\Lambda}\right) - \frac{3\sqrt{1+k^2}}{4k^2} r\Lambda(1-\tau) \frac{\delta\left(\frac{P^{11}}{r\Lambda}\right)}{\sqrt{\sum_j (P^{1j})^2}} \\ &\quad - \frac{\tau}{r} \frac{b_m}{m} (r\Lambda)^{1-(1/m)} \{\sum_j (P^{1j})^2\}^{(1/2m)-1} \sum_j P^{1j} \delta\left(\frac{P^{1j}}{r\Lambda}\right) \dots\dots\dots (79) \end{aligned}$$

$$\begin{aligned} \frac{\partial(\delta u^1)}{\partial x^2} &= -\frac{3\sqrt{1+k^2}}{2k^2} r\Lambda(1-\tau) \frac{\sum_j P^{21} P^{2j} \delta\left(\frac{P^{2j}}{r\Lambda}\right)}{\{\sum_j (P^{2j})^2\}^{3/2}} - \frac{1+k^2}{k^2} \frac{r\Lambda(1-\tau)}{\sum_j (P^{2j})^2} P^{21} \delta\left(\frac{P^{22}}{r\Lambda}\right) \\ &\quad - \frac{3\sqrt{1+k^2}}{8k^2} r\Lambda(1-\tau) \frac{\delta\left(\frac{P^{21}}{r\Lambda}\right)}{\sqrt{\sum_j (P^{2j})^2}} - \tau \delta\omega^3 \dots\dots\dots (80) \end{aligned}$$

$$\begin{aligned} \frac{\partial(\delta u^1)}{\partial x^3} &= -\frac{3\sqrt{1+k^2}}{2k^2} r\Lambda(1-\tau) \frac{\sum_j P^{31} P^{3j} \delta\left(\frac{P^{3j}}{r\Lambda}\right)}{\{\sum_j (P^{3j})^2\}^{3/2}} \\ &\quad - \frac{1+k^2}{k^2} \frac{r\Lambda(1-\tau)}{\sum_j (P^{3j})^2} P^{31} \delta\left(\frac{P^{33}}{r\Lambda}\right) \\ &\quad - \frac{3\sqrt{1+k^2}}{8k^2} r\Lambda(1-\tau) \frac{\delta\left(\frac{P^{31}}{r\Lambda}\right)}{\sqrt{\sum_j (P^{3j})^2}} + \tau \delta\omega^2 \dots\dots\dots (81) \end{aligned}$$

The expressions of δu^2 and δu^3 are obtained by the cyclic changes of subscripts 1→2→3.

7.2 Geometrical relations

We obtain the result by taking

$$h = \frac{b_m}{r} \Psi^{1/m} \dots\dots\dots (82)$$

in Eq. (47), where Ψ means the average of the absolute value of the forces acting around one particle and is expressed as follows.

$$\Psi = -\frac{1}{2r\Lambda} (P^{11} + P^{22} + P^{33}) \dots\dots\dots (83)$$

7.3 Energy conservation

In the case of the elastic interactions $\left(m = \frac{3}{2}, \frac{5}{2} \right)$, evaluating every term in (31), we obtain for $|1-\tau| \ll 1$,

7. Substitution of the interactions into the general relations

We will substitute these interactions into the general relations.

7.1 The stress-strain relations

We can neglect the terms of products of $(1-\tau)$ and Π_1 , Π_{2P} etc. in Eqs. (28)~(30). Because if $(1-\tau) \gg h$, then these terms are very small in comparison with other terms as $\Pi_1/r \sim h \ll 1$, and if $(1-\tau) \sim h$, then $(1-\tau)\Pi_1/r \sim h^2$ and it is also negligible. Therefore the results are as follows.

$$\frac{\delta \Lambda}{\Lambda} + \Lambda - \beta_1(1-\tau) \frac{\delta \Psi}{\Psi} - \beta_2 \delta \tau = 0 \dots\dots\dots (84)$$

where

$$\left. \begin{aligned} \beta_1 &= \frac{a_m}{\pi_2 b_m} k^2 \left\{ (1-\pi_1) \frac{m+1}{m} - 1 \right\} \\ \beta_2 &= k^2 \frac{\pi_1}{\pi_2} \frac{a_m}{b_m} \quad (\text{cf. Table 1}) \end{aligned} \right\} \dots\dots\dots (85)$$

$$\left. \begin{aligned} \text{and for } m &= \frac{3}{2}; \pi_1 = \frac{7}{4}, \pi_2 = \frac{2}{5}, \\ \text{for } m &= \frac{5}{2}; \pi_1 = \frac{5}{4}, \pi_2 = \frac{2}{7} \end{aligned} \right\} \dots\dots\dots (86)$$

These π_1 and π_2 are defined as follows. When U means the elastic strain energy stored by one particle at a contact point,

$$U = \pi_1 \sqrt{P^2 - P_i^2} \Pi_1(P, P_i) + \pi_2 P_i \Pi_2(P, P_i) \dots\dots\dots (87)$$

where Π_1 & Π_2 are defined in Eq. (11)

In the case of plastic asperities,

$$\left. \begin{aligned} \mathfrak{E} &= 2\Lambda U, \\ \delta \mathfrak{E} &= 2\delta \Lambda U + 2\Lambda \frac{\partial U}{\partial \Psi} \delta \Psi + 2\Lambda \frac{\partial U}{\partial \tau} \delta \tau \end{aligned} \right\} \dots\dots\dots (88)$$

Here the problem is the amount of the stored elastic strain energy out of the work done by the external forces. In the analysis of the plastic interactions mentioned above, we cut off the elastic deformation. Therefore

$$\delta \mathfrak{B} = \delta \mathfrak{Z} + \delta \mathfrak{Y} \dots\dots\dots (89)$$

But actually a portion of $\delta\mathfrak{S}$ changes to the elastic strain energy. We suppose it is $2A\frac{\partial U}{\partial \Psi}\delta\Psi$ (the increment of the elastic strain energy in the case of no change of the probability of slide at every contact point), then Eq. (89) means

$$\delta\mathfrak{S} = \delta\mathfrak{S} + \delta\mathfrak{S} + 2A\frac{\partial U}{\partial \Psi}\delta\Psi \quad \dots\dots\dots(90)$$

We substitute Eqs. (88) and (90) into Eq. (31) and obtain

$$\frac{\delta A}{A} + A + \frac{1}{U} \frac{\partial U}{\partial \tau} \delta\tau = 0 \quad \dots\dots\dots(91)$$

Putting

$$\beta_1 = 0, \quad \beta_2 = -\frac{1}{U} \frac{\partial U}{\partial \tau} \quad (\text{for plastic interactions}) \quad \dots\dots\dots(92)$$

Eq. (91) takes the identical form with Eq. (84).

This means that we may use the same equation (84) for both elastic and plastic asperities except the expression of β_1 & β_2 . β_2 of Eq. (92) is from the calculation of U ,

$$\beta_2 = \frac{1 + \varepsilon_0 k^2}{(1 + \alpha k^2) \frac{2m - 2n + 1}{4m}} - 1, \quad \varepsilon_0 = \frac{2 - \nu}{2(1 - \nu)},$$

$$[\text{for } \alpha \text{ cf. Eq. (75)}] \quad \dots\dots\dots(93)$$

where n is a constant depending upon what parts store mainly the elastic strain energy, of the small asperities, the fig ones and the inner parts of particles.

8. Summary

In the above were derived the equations which represent the characteristics of the system of nearly spherical particles in statics in the case of no adhesive force such as water. As a result the correlations of variables and the concrete interactions were made clear.

Furthermore the formula for compression in a cylinder was derived. Actual compression tests

were done and found to support the theoretical conclusions. These results will be explained in the 2nd report. In this theory we did not deal with those important cases of unloading and reloading.

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References

- (1) Terzaghi: *Theoretical Soil Mechanics*, (1963), John Wiley & Sons.
- (2) K. Kawakita and Y. Tsutsumi: *Japanese Jour. Appl. Phys.*, Vol. 4, No. 1 (1965), p.56.
- (3) A. E. H. Love: *A Treatise on the Mathematical Theory of Elasticity*, (1944), p.85, Dover.
- (4) P. Alexandrov: translated into Japanese by Sampei & Chiba, *Topological Geometry I*, (1957), p.114, Kyoritsu.
- (5) W. O. Smith et al.: *Phys. Rev.*, Vol. 34, No.1271 (1929).
- (6) S. Timoshenko and J. N. Goodier: *Theory of Elasticity*, (1951), p.372, McGraw-Hill.
- (7) R. D. Mindlin *Jour. Appl. Mech.*, Vol. 16, No. 3 (1949), p.259.
- (8) K. L. Johnson: *Proc. Roy. Soc. Lond.*, Ser. A, Vol. 230 (1955), p.531.
- (9) F. P. Bowden and D. Tabor: *The Friction and Lubrication of Solids*, Part I, (1961), Oxford.
- (10) L. A. Gahlin: translated into Japanese by Sato, *The Theory of Elastic Contact*, (1958), p.124.
- (11) F. P. Bowden and D. Tabor: *The Friction and Lubrication of Solids*, Part II, (1964), p.55, Oxford.

Discussion

Y. YUASA: In this study the problems concerning the statics of granular materials are considered in a purely analytical way and the fundamental equations for the system of granular materials are derived from the micro- and macroscopic point of view and they are seemingly interesting. We pay our respects to the author for his attempt at such a new theoretical development.

(1) The equality (55) has no factor of V/V' on its right side, whereas there must be V/V' according to the derivation from the two equalities in (54). Is it permissible to treat as $V \rightleftharpoons V'$ when the volume V changes into V' by the action of force?

(2) In the analysis of interactions between the particles in the case of the elastic contact of

spheres mentioned in the section 6.1, when we see

$A_1 = \frac{2 - \nu}{2(1 - \nu)} B_1$ in Eq. (60) and $a_m = \frac{2 - \nu}{2(1 - \nu)} b_m$ in the remark column of the Table 1, we must think the author set $A_1 = a_m$ and $B_1 = b_m$. Is it right?

And if $B_1 = b_m$, then clearly $b_m = \frac{1}{2} \sqrt{\frac{9\pi^2 \mu^2}{8r}}$ from

Eq. (60), whereas according to the calculation from

Eq. (74) we obtain $b_m = \frac{w}{P^{1/3}} = \frac{B_1 P_z^{2/3}}{P^{2/3}}$. Consequently is it permissible to set $P_z = P$?

(3) In the section 7.1 where the interactions determined in the chapter 6 are substituted into the stress-strain relations (28) ~ (30) obtained in the chapter 3, the terms of the product of $(1 - \tau)$ and Π_1, Π_{2P} etc. are negligible for the reason

mentioned in this paper. We wish to have an explanation for the relation between the last term on the right side of Eq. (79) derived from Eq. (28) and the last two terms of Eq. (28).

T. WAKABAYASHI: This theory is analytical and different from the traditional ones. Therefore many factors are inevitably introduced and the arguments are advanced in the way of frontal attack, so that it is a little difficult for experimentalists to understand, but it seems to be significant. The significance of the theory like this consists rather in making clear the physical meanings of parameters used in the traditional more or less empirical formulae than in deriving directly the formulae for the practical use. In view of the above this report No. 1 does not tell any other parameters than the porosity. We expect the explanations in 2nd Report. The following are the points difficult to understand:

(4) The meaning of Q in Eq. (24).

(5) No correlation between X_{kl} and P_{kl} in the supposition (1).

(6) Where and how does the irreversibility between stress and strain appear in Eq. (79)?

K. SAKAGUCHI: (7) This study bases on the premise of the isotropy, that is, the contact points of particles are uniformly distributed in all directions. For the natural piling or the deformation in a small amount, this supposition seems to hold with sufficient accuracy, but is this theory applicable to the case of large deformations like these experiments which were done to show the propriety of this theory?

Author's closure

(1) As for V in (54), this is an error and it is not V but V' . This means that when the volume changes from V to V' , the porosity changes from p_0 to p , and when the radii of particles r in V' are changed for $r(1-h)$, the porosity becomes p' .

(2) $A_1=a_m$, $B_1=b_m$, $P=P_z$ or not?

Exactly it is. B_1 and A_1 in Eq. (60) are regarded as $b_{3/2}$ and $a_{3/2}$ in Table 1. And $P=P_z$ means the force acting on the contact surface perpendicularly.

(3) The relation between the last terms of Eqs. (28) and (79). The expressions of Π_1 and Π_2 will be found by the comparison of Eqs. (77) and (78) with Eq. (11). By the use of the expressions in Eq. (24), the last term of Eq. (28) is easily derived and obtained as follows. According to Eq. (77),

$$\begin{aligned}\Pi_2(P_1, P_c) &= b_m P_c^{1/m} \left(1 + \alpha \frac{P^2 - P_c^2}{P_c^2} \right)^{1/2m} \\ \Pi_{2P} &= \frac{\partial \Pi_2}{\partial P} = b_m P_c^{1/m} \left(1 + \alpha \frac{P^2 - P_c^2}{P_c^2} \right)^{(1/2m)-1} \alpha \frac{2P}{P_c^2}\end{aligned}$$

$$\begin{aligned}\Pi_{2Q} &= \frac{\partial \Pi_2}{\partial P_c} = b_m P_c^{(1/m)-1} \left(1 + \alpha \frac{P^2 - P_c^2}{P_c^2} \right)^{1/2m} \\ &\quad - 2b_m P_c^{1/m} \left(1 + \alpha \frac{P^2 - P_c^2}{P_c^2} \right)^{(1/2m)-1} \alpha \frac{P^2}{P_c^3}\end{aligned}$$

Here we put $P=P_c$, then

$$\left[\frac{\partial \Pi_2}{\partial P} + \frac{\partial \Pi_2}{\partial P_c} \right]_{P=P_c} = b_m P^{(1/m)-1}$$

we obtain the similar equality by the use of Eq. (78). Using this and taking

$$P = \frac{\sqrt{\sum_j (P^{1j})^2}}{rA}$$

we obtain the last term of Eq. (79) from the last term of Eq. (28) at once.

(4) Q is only a symbol used instead of P_c in $\Pi_2(P, P_c)$ etc.

(5) This is not a well-founded supposition. Actually there may be any correlation, but that will make it complicated to average the equations. This supposition is only for simplicity. Therefore formally the existence of correlation must be judged on experimental evidences.

(6) (79) does not contain the irreversibility. As mentioned in the summary, this theory does not discuss the unloading and reloading. We must discuss the mechanism of unloading in order to argue the irreversibility. In this case the recovery of elastic deformations of particles and the relative displacements among them at unloading will come into discussion.

(7) The supposition of isotropy is questionable in some respects, nevertheless it seems to be the natural one at present. It has the purposes to make much of the generality and to avoid the complexity involved with the consideration of anisotropy. But this is a mere supposition and can be asserted only on experimental evidences.

In the case of large deformations and nonuniform stresses, the particles naturally will not keep their original shapes and deform so much that anisotropy will be caused. Therefore it seems to be possible to make the equations with anisotropy. But the author considers as follows.

Anisotropy does not only make the equations complicated but also its influence is secondary, whereas the influences of the other properties shown by averages are direct and primary, for example, the influence of the spherical property of particles on the stress distribution or displacements, or the influence of the coefficient of friction. Therefore the aim of this paper is to discuss first the characteristics due to their primary properties.

In a word, for the case of large deformations, anisotropy has not a decisive, but only a secondary influence.