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Synthesis of Spatial Four-Bar Mechanisms*

(Part II, Synthesis of Mechanisms with Two Spheric Pairs)

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In Part II, are developed the methods of dimensional synthesis of two spatial four-bar mechanisms, $A_1 \cdot 2 \cdot 1$ mechanism having two revolute and two spheric pairs and $A_1 \cdot 2 \cdot 2$ mechanism having one revolute, one prismatic and two spheric pairs: eight-precision-point synthesis of $A_1 \cdot 2 \cdot 1$ mechanism and four- and six-precision-point syntheses of $A_1 \cdot 2 \cdot 2$ mechanism are introduced.

The conditions that both mechanisms are symmetrical are considered and the syntheses to generate symmetrical functions are also introduced: the functions generated by $A_1 \cdot 2 \cdot 1$ and $A_1 \cdot 2 \cdot 2$ mechanisms may coincide with the ideal functions at ten and eight points respectively.

Examples of the syntheses are presented, which show that both mechanisms may generate much more precise functions than planar four- or six-bar mechanisms do.

1. Introduction

In Part I⁽¹⁾, 95 kinds of spatial four-bar mechanisms composed of revolute, prismatic, cylindric, spheric and sphere-groove pairs have been shown to have a single degree of freedom between the driving and driven links, and a method of synthesis with three-precision points has been presented to design $A_0 \cdot 2 \cdot 1$ mechanism to generate functions with one variable between nonparallel axes.

In Part II, are presented the methods of dimensional synthesis of two mechanisms: $A_1 \cdot 2 \cdot 1$ mechanism composed of two revolute and two spheric pairs, and $A_1 \cdot 2 \cdot 2$ mechanism composed of one revolute, one prismatic and two spheric pairs. $A_1 \cdot 2 \cdot 1$ and $A_1 \cdot 2 \cdot 2$ mechanisms may be considered to be the three-dimensional mechanisms expanded from

the planar crank-rocker and crank-slider mechanisms respectively, and they are very practical. Both mechanisms can be made symmetrical, and they may be advantageous mechanisms in case symmetrical functions are to be generated. As none of their links vary their lengths and the displacement equations are rather simple, we will have methods of synthesis with many precision points and we are able to make the mechanisms generate precise functions.

2. Synthesis of $A_1 \cdot 2 \cdot 1$ mechanism

The general form of $A_1 \cdot 2 \cdot 1$ mechanism is shown in Fig. 1. J_1 and J_4 are revolute pairs, J_2 and J_3 are spheric pairs. For this mechanism, J. Denavit and R. S. Hartenberg⁽²⁾ have already introduced a synthesis with six precision points. In this paper,

Fig. 1 The general form of $A_1 \cdot 2 \cdot 1$ mechanism

Fig. 2 $A_1 \cdot 2 \cdot 1$ mechanism

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an eight-precision-point synthesis and a ten-precision-point synthesis are presented.

Since the generality of the mechanism is not lost by the assumption that links *b* and *d* meet at right angles with the axes of pairs *J*₁ and *J*₄ respectively, let us assume $\psi_b = \psi_d = 90^\circ$. The parameters of the mechanism are determined in Fig. 2. The co-ordinate axes are taken as

X: the axis which passes through the intersecting point of axes *Y* and *Z*, and is perpendicular to them,

Y: the axis which is the common perpendicular of the axes of the pairs *J*₁ and *J*₄,

Z: the axis which is parallel to the axis of the pair *J*₄ and passes through the intersecting point of *Y*-axis and the axis of the pair *J*₁.

The co-ordinates of the pairs *J*₂ and *J*₃ are as follows.

$$J_2 \begin{cases} X_2 = l \cos \alpha + b \sin \alpha \cos \theta \\ Y_2 = b \sin \theta \\ Z_2 = l \sin \alpha - b \cos \alpha \cos \theta \end{cases} \quad (1)$$

$$J_3 \begin{cases} X_3 = d \cos \varphi \\ Y_3 = d \sin \varphi + m \\ Z_3 = n \end{cases} \quad (2)$$

The condition that the mechanism forms a closed circuit (a quadrilateral) is as follows.

$$c^2 = \overline{J_2 J_3}^2 = (X_2 - X_3)^2 + (Y_2 - Y_3)^2 + (Z_2 - Z_3)^2 \quad (3)$$

Substituting Eqs. (1) and (2) into Eq. (3), we obtain

$$\begin{aligned} b^2 - c^2 + d^2 + l^2 + m^2 + n^2 - 2ln \sin \alpha \\ - 2bd \sin \alpha \cos \theta \cos \varphi - 2bd \sin \theta \sin \varphi \\ - 2bm \sin \theta - 2ld \cos \alpha \cos \varphi + 2dm \sin \varphi \\ + 2bn \cos \alpha \cos \theta = 0 \end{aligned} \quad (4)$$

The displacement angles of the driving and driven links are related by Eq. (4).

2.1 Eight-precision-point synthesis

Consider the problem of designing an *A*₁·2·1 mechanism such that to the eight given positions of the driving link, there correspond the eight prescribed positions of the driven link. Namely, let us develop a method to determine the parameters *b*, *c*, *d*, *l*, *m*, *n*, α , θ_0 and φ_0 such that the displacement curve of the mechanism coincides with the ideal function at eight given points. θ_0 and φ_0 are the initial positions of the driving and driven links respectively and are determined as

$$\begin{cases} \theta = \theta_0 + \theta_x; & -\frac{\pi}{2} < \theta_0 \leq \frac{\pi}{2} \\ \varphi = \varphi_0 + \varphi_y; & -\frac{\pi}{2} < \varphi_0 \leq \frac{\pi}{2} \end{cases} \quad (5)$$

in which (θ_x, φ_y) is a point on the ideal function:

$$\begin{aligned} y = F(x); & \theta_x = \rho_\theta x, \varphi_y = \rho_\varphi y \\ (\rho_\theta \text{ and } \rho_\varphi \text{ are constants}) & \quad (6) \end{aligned}$$

$$\text{or } \varphi_y = f(\theta_x) \quad (7)$$

Substituting Eq. (5) into Eq. (4), we obtain the following equation, by which θ_x and φ_y are related with the parameters of the mechanism.

$$\begin{aligned} k_0 + k_1 \sin \theta_x + k_2 \cos \theta_x + k_3 \sin \varphi_y + k_4 \cos \varphi_y \\ + k_5 \sin \theta_x \cos \varphi_y + k_6 \cos \theta_x \sin \varphi_y \\ + k_7 \sin \theta_x \sin \varphi_y + k_8 \cos \theta_x \cos \varphi_y = 0 \end{aligned} \quad (8)$$

where

$$\begin{aligned} k_0 &= b^2 - c^2 + d^2 + 1, \\ k_1 &= 2b(-m \cos \theta_0 - n \cos \alpha \sin \theta_0), \\ k_2 &= 2b(-m \sin \theta_0 + n \cos \alpha \cos \theta_0), \\ k_3 &= 2d(m \cos \varphi_0 + l \cos \alpha \sin \varphi_0), \\ k_4 &= 2d(m \sin \varphi_0 - l \cos \alpha \cos \varphi_0), \\ k_5 &= 2bd(\sin \alpha \sin \theta_0 \cos \varphi_0 - \cos \theta_0 \sin \varphi_0), \\ k_6 &= 2bd(\sin \alpha \cos \theta_0 \sin \varphi_0 - \sin \theta_0 \cos \varphi_0), \\ k_7 &= -2bd(\sin \alpha \sin \theta_0 \sin \varphi_0 + \cos \theta_0 \cos \varphi_0), \\ k_8 &= -2bd(\sin \alpha \cos \theta_0 \cos \varphi_0 + \sin \theta_0 \sin \varphi_0) \end{aligned} \quad (9)$$

in which $a \equiv 1$, i.e., *a* is the unit of length.

Using Eqs. (8) and (9), the following method of synthesis is obtained.

Synthesis procedure: Determine eight precision points $(\theta_{xj}, \varphi_{yj})$ and substitute them into Eq. (10). Then solve the simultaneous equations, Eq. (11), for $k_1, k_2, \dots, k_8$. The design parameters will be determined by calculating Eqs. (12), (13) and (14) in this order.

In case an ideal function is given, not in the form of Eq. (7), but in the form of Eq. (6), the scale factors, ρ_θ and ρ_φ , must be chosen as follows. Namely, if the ideal function is given in the interval, $x_s \leq x \leq x_f$ and $y_s \leq y \leq y_f$ and is intended to be generated in the range, $\Delta\theta_x$ and $\Delta\varphi_y$, the scale factors are

$$\rho_\theta = \frac{\Delta\theta_x}{x_f - x_s} \text{ and } \rho_\varphi = \frac{\Delta\varphi_y}{y_f - y_s}$$

where $\Delta\theta_x$ and $\Delta\varphi_y$ are the ranges of variation of the displacements of links *b* and *d* respectively.

$$\begin{aligned} a_{j1} &= \sin \theta_{xj}, & a_{j2} &= \cos \theta_{xj}, \\ a_{j3} &= \sin \varphi_{yj}, & a_{j4} &= \cos \varphi_{yj}, \\ a_{j5} &= \sin \theta_{xj} \cos \varphi_{yj}, & a_{j6} &= \cos \theta_{xj} \sin \varphi_{yj}, \\ a_{j7} &= \sin \theta_{xj} \sin \varphi_{yj}, & a_{j8} &= \cos \theta_{xj} \cos \varphi_{yj}, \end{aligned} \quad j=1, 2, 3, \dots, 8 \quad (10)$$

$$a_{j1}K_1 + a_{j2}K_2 + a_{j3}K_3 + a_{j4}K_4 + a_{j5}K_5 + a_{j6}K_6 + a_{j7}K_7 + a_{j8}K_8 = -1 \quad (11)$$

$$A_1 = K_6/K_5, \quad A_2 = K_7/K_5, \quad A_3 = K_8/K_5 \quad (12)$$

$$\begin{aligned} \theta_0 &= \frac{1}{2} \tan^{-1} \frac{2(A_3 + A_1 A_2)}{1 - A_1^2 + A_2^2 - A_3^2}, \\ \varphi_0 &= \frac{1}{2} \sin^{-1} \left\{ \frac{A_2 + A_1 A_3}{A_3 + A_1 A_2} \sin 2\theta_0 \right\}, \\ \alpha &= \sin^{-1} \left\{ \frac{\sin \theta_0 \cos \varphi_0 - A_1 \cos \theta_0 \sin \varphi_0}{\cos \theta_0 \sin \varphi_0 - A_1 \sin \theta_0 \cos \varphi_0} \right\}, \\ &-\frac{\pi}{2} < \theta_0, \varphi_0, \alpha \leq \frac{\pi}{2} \end{aligned} \quad (13)$$

$$\left. \begin{aligned}
 B_1 &= -\frac{1}{2}(K_1 \cos \theta_0 + K_2 \sin \theta_0), \\
 B_2 &= \frac{1}{2 \cos \alpha}(-K_1 \sin \theta_0 + K_2 \cos \theta_0), \\
 B_3 &= \frac{1}{2}(K_3 \cos \varphi_0 + K_4 \sin \varphi_0), \\
 B_4 &= \frac{1}{2 \cos \alpha}(K_3 \sin \varphi_0 - K_4 \cos \varphi_0), \\
 B_5 &= \frac{K_5}{2} \frac{1}{\sin \alpha \sin \theta_0 \cos \varphi_0 - \cos \theta_0 \sin \varphi_0}, \\
 b &= B_5 \left(B_3^2 + B_4^2 + \frac{B_2^2 B_3^2}{B_1^2} - \frac{2 B_2 B_3 B_4}{B_1} \sin \alpha \right)^{-1/2}, \\
 d &= \frac{B_3}{B_1} b, \quad k_0 = \frac{bd}{B_5}, \\
 c &= \sqrt{1 + b^2 + d^2 - k_0}, \\
 l &= \frac{k_0}{d} B_4, \quad m = \frac{k_0}{b} B_1, \quad n = \frac{k_0}{b} B^2
 \end{aligned} \right\} \dots\dots\dots (14)$$

2.2 Synthesis to generate symmetrical functions (Ten-precision-point synthesis)

In case the ideal function $\varphi_y = f(\theta_x)$ is symmetrical with respect to an axis parallel to φ_y -axis, $\theta_x = \Theta$, (Fig. 3), i.e., when it has the property of $f(\Theta + \theta_s) = f(\Theta - \theta_s)$ (15) the generated function will be made to coincide with the ideal function at ten points by the following method of synthesis.

Consider the case that the displacement equation, Eq. (4), has the same value of φ for

Fig. 3 Symmetrical function

Fig. 4 The symmetrical mechanism

$\theta = \theta_0 + \Theta + \theta_s$ and $\theta' = \theta_0 + \Theta - \theta_s$ (16) then, the following equation holds.

$$\{d \sin \alpha \sin (\theta_0 + \Theta)\} \cos \varphi + \{d \cos (\theta_0 + \Theta) + m \cos (\theta_0 + \Theta)\} \sin \varphi + \{dn \cos \alpha \sin (\theta_0 + \Theta) + m \cos (\theta_0 + \Theta)\} = 0 \dots\dots\dots (17)$$

The condition for Eq. (17) to hold regardless of the value of φ is

$$\left. \begin{aligned}
 d \sin \alpha \sin (\theta_0 + \Theta) &= 0 \\
 d \cos (\theta_0 + \Theta) &= 0 \\
 dn \cos \alpha \sin (\theta_0 + \Theta) + m \cos (\theta_0 + \Theta) &= 0
 \end{aligned} \right\} \dots\dots\dots (18)$$

Since the length of the driven link is not zero, i.e., $d \neq 0$, from Eq. (18) we have

$$\sin \alpha = 0, \quad \cos (\theta_0 + \Theta) = 0, \quad n = 0$$

or

$$\alpha = 0, \quad n = 0, \quad \theta_0 = \pm \frac{\pi}{2} - \Theta \dots\dots\dots (19)$$

We know from the above that the generated function $\varphi = g(\theta)$ is symmetrical with respect to $\theta = \pm \pi/2$, i.e.,

$$g\left(\pm \frac{\pi}{2} + \theta\right) = g\left(\pm \frac{\pi}{2} - \theta\right)$$

when the driving shaft I (the axis of the pair J_1) and the driven shaft II (the axis of the pair J_4) are perpendicular and the pair J_4 is on Y-axis, the common perpendicular of the shafts I and II. This relation is illustrated in Fig. 4.

Substituting the symmetrical condition, Eq. (19), into the displacement equation, Eq. (4), we have the following method of synthesis.

Fig. 5 The general form of A₁.2.2 mechanism

Fig. 6 A₁.2.2 mechanism

Synthesis procedure: The method to determine the parameters φ_0 , l , b , c and d under the condition that $\alpha=0$, $n=0$, $\theta_0=\pi/2-\Theta$ and m is given arbitrarily.

(i) Substitute five precision points (θ_j, φ_{yj}) into the next equation, and solve the obtained simultaneous equations for $k_1, k_2, \dots, k_5$.

$$k_1 \sin \theta_j + k_2 \sin \varphi_{yj} + k_3 \cos \varphi_{yj} + k_4 \sin \theta_j \cos \varphi_{yj} + k_5 = \sin \theta_j \sin \varphi_{yj} \dots (20)$$

where

$$\theta_j = \frac{\pi}{2} + \theta_{\xi j}; \quad \varphi_{yj} = f(\Theta + \theta_{\xi j}) \dots (21)$$

$$j=1, 2, 3, 4, 5$$

(ii) Substitute the values of $k_1, k_2, \dots, k_5$ into the next equations and calculate them for the unknown parameters.

$$\left. \begin{aligned} \varphi_0 &= -\tan^{-1} k_4, & d &= \frac{-m}{k_1 \cos \varphi_0}, \\ l &= \frac{k_2 k_4 + k_3}{k_3 k_4 - k_2} m, & b &= \frac{-l k_4 + m}{k_2}, \\ c &= (b^2 + d^2 + l^2 + m^2 - 2bd k_5 \cos \varphi_0)^{1/2} \end{aligned} \right\} \dots (22)$$

3. Synthesis of $A_1 \cdot 2 \cdot 2$ mechanism

The general form of $A_1 \cdot 2 \cdot 2$ mechanism is illustrated in Fig. 5. J_1 is a revolute pair, J_2 and J_3 are spheric pairs and J_4 is a prismatic pair. Similar to $A_1 \cdot 2 \cdot 1$ mechanism, this mechanism does not lose the generality if the angles ψ_b and ψ_d are assumed to be 90° . Let us name the parameters of the mechanism as shown in Fig. 6 under the condition $\psi_b = \psi_d = 90^\circ$. The co-ordinate axes are determined in the same manner as in Fig. 2.

Comparing Figs. 2 and 6, we know that the displacement equation of $A_1 \cdot 2 \cdot 2$ mechanism may be obtained by substituting

$$\varphi = \frac{\pi}{2} \text{ and } n = s \quad (s \text{ is a variable})$$

into Eq. (4), thus

$$b^2 - c^2 + (d+m)^2 + l^2 - 2b(d+m) \sin \theta - 2s(l \sin \alpha - b \cos \alpha \cos \theta) + s^2 = 0 \dots (23)$$

3.1 Four-precision-point synthesis

Rearranging Eq. (23), we have

$$k_1 + k_2 \sin \theta + k_3 s + k_4 s \cos \theta = -s^2 \dots (24)$$

where

$$\left. \begin{aligned} k_1 &= b^2 - c^2 + (d+m)^2 + l^2 \\ k_2 &= -2b(d+m) \\ k_3 &= -2l \sin \alpha \\ k_4 &= 2b \cos \alpha \end{aligned} \right\} \dots (25)$$

Using Eqs. (24) and (25), the following method of synthesis is obtained.

Synthesis procedure: The method to determine the design parameters b , c , $d+m$ and l under the condition of s_0 , θ_0 and α being given.

(i) Solve the following simultaneous equations

for k_1, k_2, k_3 and k_4 .

$$k_1 + k_2 \sin \theta_j + k_3 s_j + k_4 s_j \cos \theta_j = -s_j^2 \dots (26)$$

where

$$\left. \begin{aligned} \theta_j &= \theta_0 + \theta_{xj}; & s_j &= s_0 + s_{yj} \\ s_{yj} &= f(\theta_{xj}), & j &= 1, 2, 3, 4 \end{aligned} \right\} \dots (27)$$

(ii) Substitute the obtained values of k_1, k_2, k_3 and k_4 into the following equations, and calculate them for the unknown parameters.

$$\left. \begin{aligned} b &= \frac{k_4}{2 \cos \alpha}, & d+m &= \frac{-k_2}{2b}, \\ l &= \frac{-k_3}{2 \sin \alpha}, \\ c &= \{b^2 + (d+m)^2 + l^2 - k_1\}^{1/2} \end{aligned} \right\} \dots (28)$$

3.2 Six-precision-point synthesis

Consider the case that initial positions of the links b and d , θ_0 and s_0 , are also taken as design parameters.

Substituting

$$\theta = \theta_0 + \theta_x, \quad s = s_0 + s_y \dots (29)$$

into Eq. (23), we obtain

$$k_1 + k_2 \sin \theta_x + k_3 \cos \theta_x + k_4 s_y + k_5 s_y \sin \theta_x + k_6 s_y \cos \theta_x = -s_y^2 \dots (30)$$

where

$$\left. \begin{aligned} k_1 &= b^2 - c^2 + (d+m)^2 + l^2 + s_0^2 - 2ls_0 \sin \alpha, \\ k_2 &= -2b(d+m) \cos \theta_0 - 2bs_0 \cos \alpha \sin \theta_0, \\ k_3 &= 2bs_0 \cos \alpha \cos \theta_0 - 2b(d+m) \sin \theta_0, \\ k_4 &= 2s_0 - 2l \sin \alpha, & k_5 &= -2b \cos \alpha \sin \theta_0, \\ k_6 &= 2b \cos \alpha \cos \theta_0 \end{aligned} \right\} \dots (31)$$

Using Eqs. (20) and (31), the following method of synthesis is obtained.

Synthesis procedure: The method to determine the parameters b , c , $d+m$, l , θ_0 and s_0 . α is given arbitrarily.

(i) Substitute six precision points (θ_{xj}, s_{yj}) into the following equation and solve the obtained simultaneous equations for $k_1, k_2, \dots, k_6$.

$$k_1 + k_2 \sin \theta_{xj} + k_3 \cos \theta_{xj} + k_4 s_{yj} + k_5 s_{yj} \sin \theta_{xj} + k_6 s_{yj} \cos \theta_{xj} = -s_{yj}^2 \dots (32)$$

$$j=1, 2, \dots, 6$$

Fig. 7 Symmetrical function

(ii) Substitute the solutions of Eq. (32) into the following equations and calculate them for the unknown parameters.

$$\left. \begin{aligned} \theta_0 &= -\tan^{-1} \frac{k_5}{k_6}; \quad -\frac{\pi}{2} < \theta_0 \leq \frac{\pi}{2}, \\ b &= \frac{-k_5}{2 \cos \alpha \sin \theta_0}, \quad s_0 = \frac{k_2 k_5 + k_3 k_6}{k_5^2 + k_6^2}, \\ l &= \frac{2s_0 - k_4}{2 \sin \alpha}, \quad d+m = \frac{(k_3 - k_6 s_0) \cos \alpha}{k_5}, \\ c &= \{b^2 + (d+m)^2 + l^2 + s_0^2 - 2ls_0 \sin \alpha - k_1\}^{1/2} \end{aligned} \right\} \dots (33)$$

3.3 Synthesis to generate symmetrical functions (Eight-precision-point synthesis)

Similar to $A_1 \cdot 2 \cdot 1$ mechanism, we can make $A_1 \cdot 2 \cdot 2$ mechanism generate such a symmetrical function as shown in Fig. 7.

In case that the displacement equation, Eq. (23), has the same value of φ for

$\theta = \theta_0 + \Theta + \theta_\varepsilon$ and $\theta' = \theta_0 + \Theta - \theta_\varepsilon$ (34)
the following equation holds.

$$\{(d+m) \cos(\theta_0 + \Theta) + s_y \cos \alpha \sin(\theta_0 + \Theta)\} b \sin \theta_\varepsilon = 0 \dots (35)$$

The condition for Eq. (35) to hold regardless of the values of θ_ε and s_y in a certain range is

$$d+m=0 \text{ and } \sin(\theta_0 + \Theta) = 0$$

or

$$d+m=0 \quad \theta_0 = -\Theta, \quad \pi - \Theta \dots (36)$$

in which b is regarded not as zero because of b being the length of the crank.

From the above, we know that the generated function $s=g(\theta)$ is symmetrical with respect to $\theta=0, \pi$, i.e.,

$$g(\theta) = g(-\theta), \quad g(\pi + \theta) = g(\pi - \theta)$$

when the driving shaft and the locus of the pair J_3 are on the same plane. The symmetrical mechanism is illustrated in Fig. 8.

Substituting Eq. (36) into the displacement equation, Eq. (23), we obtain

$$k_1 + k_2 \cos \theta_\varepsilon + k_3 s_y \cos \theta_\varepsilon + k_4 s_y = -s_y^2 \dots (37)$$

where

$$\left. \begin{aligned} k_1 &= b^2 - c^2 + l^2 - 2ls_0 \sin \alpha + s_0^2, \\ k_2 &= 2bs_0 \cos \alpha, \\ k_3 &= 2b \cos \alpha, \\ k_4 &= 2(s_0 - l \sin \alpha) \end{aligned} \right\} \dots (38)$$

Using Eqs. (37) and (38), the following method of synthesis is obtained.

Synthesis procedure: The method to determine the design parameters s_0 , b , c and l under the condition that $\theta_0 = -\Theta$, $d+m=0$ and α is given arbitrarily.

(i) Substitute four precision points $(\theta_{\varepsilon j}, s_{y j})$ into the following equation and solve the obtained simultaneous equations for k_1 , k_2 , k_3 and k_4 .

$$k_1 + k_2 \cos \theta_{\varepsilon j} + k_3 s_{y j} \cos \theta_{\varepsilon j} + k_4 s_{y j} = -s_{y j}^2$$

$$j=1, 2, 3, 4 \dots (39)$$

(ii) Substitute the solution of Eq. (39) into the following equations, and calculate them for the unknown parameters.

$$\left. \begin{aligned} s_0 &= \frac{k_2}{k_3}, \quad l = \frac{2s_0 - k_4}{2 \sin \alpha}, \quad b = \frac{k_3}{2 \cos \alpha}, \\ c &= (b^2 + l^2 - 2ls_0 \sin \alpha + s_0^2 - k_1)^{1/2} \end{aligned} \right\} \dots (40)$$

4. Example

Link mechanisms are frequently driven by rotating shafts of motors, internal combustion engines, etc. So, let us put $0^\circ \leq \theta_x \leq 360^\circ$ in the following examples in order to make the driving link rotate completely. Therefore, all of the ideal functions are periodical functions with the period of 360° .

4.1 Examples of synthesis of $A_1 \cdot 2 \cdot 1$ mechanism

Example (1)

An example of the eight-precision-point synthesis

Ideal function: $\varphi_y = 30^\circ \sin \theta_x$

Taking the precision points as shown in Table 1, the following mechanism is obtained.

$$\theta_0 = -0.063 \, 053^\circ, \quad b = 0.177 \, 099,$$

$$\varphi_0 = 88.574 \, 3^\circ, \quad c = 1.015 \, 50,$$

$$\alpha = 0.012 \, 902^\circ, \quad d = 0.004 \, 313,$$

$$l = 0.999 \, 964,$$

Table 1 The precision points of example (1)

j	θ_{xj}°	φ_{yj}°	j	θ_{xj}°	φ_{yj}°
1	0	0	5	180	0
2	60	25.980 8	6	240	-25.980 8
3	90	30	7	270	-30
4	120	25.980 8	8	330	-15

Table 2 The precision points of example (2)

j	θ_{xj}°	φ_{yj}°	j	θ_{xj}°	φ_{yj}°
1	0	0	5	180	-24.356 96
2	45	19.790 03	6	225	-36.763 79
3	90	12.178 48	7	270	-39.580 07
4	135	-6.469 82	8	315	-28.695 55

Fig. 8 Symmetrical mechanism

$$m=0.008\ 441,$$

$$n=-0.000\ 014 \quad (a \equiv 1)$$

The maximum error of the function generated by the resulting mechanism is about $0.29'$, which corresponds to 0.008 per cent of the range of variation of φ_y . The error is very small, though the ratio of the lengths of links c and d is unpractical.

From the fact that the resulting mechanism satisfies the condition of Eq.(19), i.e., $\alpha \neq 0$, $n \neq 0$ and $\theta_0 \neq 0$ (in this example, $\theta_0 = \pi/2$), we know that the generated function is almost symmetrical with respect to $\theta = 90^\circ$.

Example (2) An example of the eight-precision-point synthesis

Ideal function:

$$\begin{cases} y_1 = x(x-1)(x-2) & (0 \leq x \leq 1) \\ y_2 = \frac{1}{4}(x+1)(x-1)(x-3) & (1 \leq x \leq 3) \end{cases}$$

In order to make the driving link rotate completely and to make the follower oscillate in the range of 60° , i.e., $\Delta\theta_x = 360^\circ$ and $\Delta\varphi_y = 60^\circ$, the scale factors are taken as

$$\rho_\theta = \frac{\Delta\theta_x}{x_f - x_s} = \frac{360^\circ}{3-0} = 120^\circ$$

$$\rho_\varphi = \frac{\Delta\varphi_y}{y_{\max} - y_{\min}} = \frac{60^\circ}{\frac{2\sqrt{3}}{9} - \left(-\frac{4\sqrt{3}}{9}\right)} = 51.961\ 524^\circ$$

Table 3 The precise points of example (3)

j	θ_j°	$\varphi_{y,j}^\circ$	j	θ_j°	$\varphi_{y,j}^\circ$
1	-67.5	-27.716 385	3	22.5	11.480 502
2	-22.5	-11.480 502	4	67.5	27.716 385

Fig. 9 The error of the generated function of Example (2)

Fig. 10 The error of the generated function of Example (3)

Taking the precision points as shown in Table 2, the following mechanism is obtained.

$$\begin{aligned} \theta_0 &= 63.971\ 297^\circ, & b &= 0.373\ 507, \\ \varphi_0 &= -25.254\ 509^\circ & c &= 0.554\ 924, \\ \alpha &= -32.108\ 513^\circ, & d &= 0.856\ 205, \\ l &= 0.536\ 862, \\ m &= 0.867\ 945, \\ n &= -0.085\ 659 & (a \equiv 1) \end{aligned}$$

The error, i.e., the difference between the generated function φ_{yA} and the ideal φ_y is illustrated in Fig. 9. The maximum error is $31.8'$ which corresponds to 0.9 per cent of the range of variation of φ_y . The error is larger than that of example (1), however the ratios of the lengths of links are practical values.

Example (3) An example of the synthesis to generate symmetrical functions

Since the ideal function of example (1) is symmetrical with respect to $\theta_x = \theta = 90^\circ$, the method of synthesis described in section 2.2 will be applied.

Under the condition that $\theta_0 = 0$, $\alpha = 0$, $n = 0$ and $d/m = -1.2$, taking the precision points as shown in Table 3, the following mechanism is obtained.

$$\begin{aligned} b &= 1.342\ 571, & l &= 5.476\ 322, \\ c &= 4.830\ 025, & m &= -1, \\ d &= 1.2, & \varphi_0 &= -24.988\ 15^\circ \end{aligned}$$

The error of the generated function φ_{yA} is illustrated in Fig. 10. The maximum error is $3.5'$,

Table 4 The precision points of example (4)

j	$\theta_{\varepsilon j}^\circ$	$s_{y,j}$ cm	j	$\theta_{\varepsilon j}^\circ$	$s_{y,j}$ cm
1	22.5	6.432 292	3	112.5	0.807 292
2	67.5	4.557 292	4	157.5	-2.864 583

Fig. 11 The experimental equipment of $A_1 \cdot 2.1$ mechanism

Fig. 12 The error of the generated function of Example (4)

which corresponds to 0.1 per cent of the range of variation of φ_y . Though the error is larger than that of example (1), it is much smaller than those of the functions generated by planar link mechanisms. If the precision points were taken different from the values in Table 3, maximum error would not decrease largely, because the maximum errors in each interval between precision points are nearly equal.

The ratio c/d of the resulting mechanism of example (1) is too large to be practicable, however the mechanism obtained in this example may be used in practice though the maximum value of the pressure angle is fairly large.

In this example, the ratio d/m is given prior to the synthesis and eight-precision-point synthesis is made in order to obtain a mechanism suitable for static and dynamic experiments. In Fig. 11 is illustrated the experimental equipment with $m = -50.00$ mm. The maximum error of the function generated by the equipment is about $23'$ and backlash of the driven link is $\pm 3'$.

4.2 Example of synthesis of $A_1 \cdot 2 \cdot 2$ mechanism

Example (4) An example of the synthesis to generate symmetrical functions

$$\text{Ideal function: } \begin{cases} y_1 = -x(x-2) & (0 \leq x \leq 2) \\ y_2 = 2(x-2)(x-3) & (2 < x \leq 3) \end{cases}$$

In order to make the driving link rotate completely and to make the stroke of the follower 10 cm, i.e., $\Delta\theta_x = 360^\circ$ and $\Delta s_y = 10$ cm, the scale factors are taken as

$$\rho_\theta = \frac{\Delta\theta_x}{x_f - x_s} = \frac{360^\circ}{3 - 0} = 120^\circ$$

$$\rho_s = \frac{\Delta s_y}{y_{\max} - y_{\min}} = \frac{10}{1 - \left(-\frac{1}{2}\right)} = \frac{20}{3} \text{ cm}$$

Since the ideal function is symmetrical with respect to $\theta_x = \Theta = 120^\circ$, the method of synthesis described in section 3.3 will be applied.

Taking precision points as shown in Table 4, the following mechanism is obtained under the

Table 5 The accuracy of the functions generated by some mechanisms

Mechanism	A	Maximum error ($\Delta\varphi_y$) _{max}	$\frac{(\Delta\varphi_y)_{\max}}{2A} \times 100\%$
Planar 4-bar mechanism	45°	9°	10
A type planar 6-bar mechanism	45°	0.53°	0.6
B type planar 6-bar mechanism	45°	3.4°	3.8
$A_1 \cdot 2 \cdot 1$ spatial 4-bar mechanism	30°	$3.5'$ ($0.29'$)	0.1 (0.008)

condition that

$$d+m=0, \theta_0 = -\Theta = -120^\circ \text{ and } \alpha = 30^\circ.$$

$$b = -9.686156, \quad l = 19.032086,$$

$$c = 29.672555, \quad s_0 = -16.046286 \text{ (unit: cm)}$$

The error of the generated function s_{yA} is illustrated in Fig. 12. The maximum error is 0.18 cm, which corresponds to 1.8 per cent of the stroke of the slider. Fig. 13 shows the equipment which is made for static and dynamic experiments. The maximum error of the function generated by the equipment is 0.12 cm and the backlash of the slider is ± 0.025 cm.

For the ideal function of example (4), the six-precision-point synthesis was also applied with the precision points taken according to Chebychev spacing⁽⁴⁾. However the resulting mechanism was not practicable because the maximum pressure angle was almost 90° .

5. Comparison with planar mechanisms with respect to the preciseness of the generated functions

In Table 5, are listed the maximum errors of the functions generated by some kinds of mechanisms, which are obtained as generators of $\varphi = A \sin \theta_x$. The planar four-bar mechanism is synthesized with four precision points taken according to Chebychev spacing. A-type six-bar mechanism⁽²⁾ is the result of ten-point approximation, and B-type mechanism is obtained by applying the four-precision-point synthesis of planar four-bar mechanism. $A_1 \cdot 2 \cdot 1$ mechanism is the result of example (3) [the value in the parentheses is the result of example (1)]. If five-precision-point synthesis⁽³⁾ is applied to planar four-bar and B-type six-bar mechanisms, a little more precise functions may be obtained.

From Table 5 we know that $A_1 \cdot 2 \cdot 1$ mechanism generates much more precise functions than planar mechanisms do, though Table 5 is only an example of sine function.

6. Conclusion

Methods of synthesis of two spatial four-bar

Fig. 13 The experimental equipment of $A_1 \cdot 2 \cdot 2$ mechanism

mechanisms are developed to generate functions with one variable between the driving and driven links. Namely, eight-precision-point synthesis is introduced for $A_1\cdot 2\cdot 1$ mechanism with two revolute and two spheric pairs, and four- and six-precision-point syntheses for $A_1\cdot 2\cdot 2$ mechanism with one revolute, one prismatic and two spheric pairs.

The conditions that both mechanisms are symmetrical are examined, and the methods of synthesis to generate symmetrical functions are presented; the functions generated by $A_1\cdot 2\cdot 1$ and $A_1\cdot 2\cdot 2$ mechanisms may coincide with the ideal functions at ten and eight points respectively. In case such symmetrical functions as parabolic function, sine function, etc. are to be generated, these methods may be profitable.

The equations of the synthesis are presented in the form convenient for the programming of electronic digital computers; manual calculation is possible, but six- or eight-precision-point syntheses may take time because numbers with many figures

must be used in order to get the solutions of simultaneous equations with six or eight unknowns with moderate preciseness.

Examples of the eight-precision-point synthesis were calculated by an electronic digital computer (IBM 7090), and other examples were calculated manually with an electric computer. Examples show that the error of the functions generated by $A_1\cdot 2\cdot 1$ and $A_1\cdot 2\cdot 2$ mechanisms, is much smaller than that by planar four- or six-bar mechanisms, and that both spatial mechanisms may be adopted as precise function generators.

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