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A Two-Parameters Theory of Axisymmetric Compressible Flow in Axial Turbomachines*

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In this paper, a theory of a three-dimensional compressible flow in axial turbomachines with varying boundary shapes is presented, in which the assumption is made that the axial velocity profiles and the specific mass flow profiles are respectively expressible by a single parameter. For these two parameters the flow equations reduce to non-linear ordinary differential equations and can be solved numerically very easily.

Comparisons with two more accurate calculations are made to investigate the accuracy of the present theory.

1. Introduction

In present axial compressors and turbines with high pressure ratios, the flow passage area changes rapidly in the axial direction: the shape of the passage boundaries deviates largely from a cylinder. Under such conditions the effect of the boundary shape on the flow characteristics would be as important as that of the fluid compressibility.

This report gives a theory for three-dimensional compressible flows in axial turbomachines where inner and outer casings are not cylindrical. The assumptions are that the fluid is inviscid and that the flow is axisymmetric, that is, the blade rows consist of an infinite number of infinitely thin blades.

The calculation of the axisymmetric flows in turbomachines has usually been treated by the radial equilibrium theory where the streamlines are assumed to lie on cylindrical surfaces. Such calculations would be inaccurate for case where passage area variation is large. Wu and Wolfenstein⁽¹⁾, Holmquist and Rannie⁽²⁾, and Horlock⁽³⁾, have extended this radial equilibrium theory and obtained equations where the compressibility and the slope and curvature of the meridional streamlines have been taken into consideration. However, in these methods, the shape of meridional streamlines has to be assumed to solve the equations. Wu⁽⁴⁾ has given the numerical method of solution of the full basic partial differential equa-

tion of the flow through turbomachines. This method is quite general but is necessarily extremely complicated and is not considered applicable for the practical design of the machine.

The method of using the slope of the axial velocity profile has been developed recently by Whitehead and Beavers⁽⁵⁾, Howell⁽⁶⁾ and Mellor⁽⁷⁾. These methods are conceptually simple and suitable for practical purposes, but deal with only the case of the incompressible flow and of constant inner and outer casing radii.

In this report, the author gives a parameter theory for the analysis of the compressible flow in axial turbomachines with arbitrary boundary shapes. Namely, the assumption is made in the present theory that the axial velocity profiles and the specific mass flow profiles are respectively expressible by a single parameter. For these two parameters the flow equations reduce to non-linear ordinary differential equations and can be solved numerically very easily.

Comparison with exact calculations are made and it is shown that the theory is sufficiently accurate for practical purposes.

Nomenclature

r, θ, z : cylindrical coordinates
 u, v, w : velocity components in direction r, θ, z
 h, t : inner and outer casing radii
 i : static enthalpy
 m : mass flow
 n : polytropic exponent
 p : static pressure
 r_* : design radius

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- s : entropy
 $B = t^2 - h^2$
 E : constant, Eq. (31)
 G : slope of total enthalpy profile
 H : total enthalpy
 I : relative total enthalpy
 J : slope of relative total enthalpy profile
 K : constant, Eq. (42)
 M : Mach number
 $\tilde{M}$: Mach number relative to rotor
 P : slope of entropy profile
 Q : constant, Eq. (31)
 $R = t^2 + h^2$
 R_g : gas constant
 T : static temperature
 V_t : rotor tip speed
 $\beta = \frac{v - r\omega}{w}$
 $\eta = \lambda'$
 κ : ratio of specific heats
 λ : slope of specific mass flow profile
 μ : slope of axial velocity profile
 $\nu = w_*^2$
 ξ : angular momentum about z axis
 ρ : density
 ρ_0 : density at total state
 σ : area-variable, Eq. (3)
 $\phi = s/R_g$
 ψ : stream function
 ω : angular velocity of rotor
 Γ : slope of angular momentum profile
 Θ : constant, Eq. (31)

Subscripts

- $*$: $r = r_*$
 $-\infty$: $z = -\infty$
 ∞ : $z = \infty$
 i : inlet to machines
 r : rotor
 s : stator

A prime (') denotes differentiation with respect to axial coordinate.

2. Approximate representation of axial velocity profiles and specific mass flow profiles

In this paper, the radial distributions of the axial velocity w and of the specific mass flow ρw are assumed to be represented by

$$w/w_* = 1 + \mu\sigma \quad (1)$$

$$\rho w/\rho_* w_* = 1 + \lambda\sigma \quad (2)$$

where the parameters μ and λ are assumed independent of r ; σ is a non-dimensional area-variable defined as

$$\sigma = 2 \frac{r^2 - h^2}{t^2 - h^2} - 1 \quad (3)$$

where t and h are the outer and inner casing radii and functions of z for the flow passage with varying boundary shapes. The outer and inner casings of the passage are respectively represented by $\sigma=1$ and $\sigma=-1$. From Eq. (3), putting $\sigma=0$, we get

$$r_* = \left(\frac{t^2 + h^2}{2} \right)^{1/2} \quad (4)$$

where r_* is the root mean square of t and h .

From Eqs. (1) and (2) the axial velocity profiles and the specific mass flow profiles which are now assumed to be parabolic are controlled by the parameters μ and λ , but these families of profiles are considered to be sufficiently similar to those used in practice in turbomachines.

Using the continuity equation a stream function ψ is defined as

$$u = -\frac{1}{\rho r} \frac{\partial \psi}{\partial z}, \quad w = \frac{1}{\rho r} \frac{\partial \psi}{\partial r} \quad (5)$$

Substituting for ρw from Eq. (2) and integrating with respect to r , the second of Eq. (5) gives

$$\psi = \frac{1}{4} \rho_* w_* (t^2 - h^2) \left\{ \sigma + 1 + \frac{\lambda}{2} (\sigma^2 - 1) \right\} \quad (6)$$

so that the mass flow through the flow passage is given by

$$m = 2\pi \left[\psi \right]_{\sigma=-1}^{\sigma=1} = \pi \rho_* w_* (t^2 - h^2) \quad (7)$$

and this is found to be independent of the para-

Fig. 1 Coordinate system and velocity components

Fig. 2 Radial displacement of streamlines at r_*

meters μ and λ .

Since differentiation of Eqs. (1) and (2), using Eq. (3), gives at $r=r_*$

$$\mu = \frac{(t^2 - h^2)}{4r_* w_*} \left(\frac{\partial w}{\partial r} \right)_* \quad (8)$$

$$\lambda = \frac{(t^2 - h^2)}{4r_* \rho_* w_*} \left\{ \frac{\partial(\rho w)}{\partial r} \right\}_* \quad (9)$$

μ and λ are respectively the radial slope of the axial velocity profile and of the specific mass flow profile at $r=r_*$. λ is also related directly to the radial displacement of the meridional streamlines as follows: referring to Fig. 2, consider a streamline which lies at the point A on the radius r_{*1} at an axial station ① where the axial velocity profile which is specified by λ passes through the point B displaced in the radial direction by $\delta\sigma$ from r_{*2} at a station ② where λ has changed to $\lambda + \delta\lambda$. If the axial distance δz between ① and ② is taken sufficiently small, we may put along a streamline

$$\left(\frac{\partial\psi}{\partial z} \right)_* \delta z + \left(\frac{\partial\psi}{\partial\sigma} \right)_* \delta\sigma = 0 \quad (10)$$

Substituting Eq. (6) into Eq. (10) and taking $\rho_* w_* (t^2 - h^2) = \text{const.}$ into account from Eq. (7), an expression for the relation of the radial displacement of the streamline at $\sigma=0$ with the increase of λ is obtained as

$$\delta\lambda = 2\delta\sigma \quad (11)$$

The radial velocity u and the density ρ are also written with μ and λ . Putting ψ of Eq. (6) into the first of Eq. (5), we get

$$u/w_* = -\frac{B}{4r} (1 - \mu\sigma) \left\{ \frac{(\sigma^2 - 1)}{2(1 + \lambda\sigma)} \right\} \lambda' - \frac{1}{B} (R' + \sigma B') \quad (12)$$

and from Eqs. (1) and (2)

$$w_* \left(\frac{\partial w}{\partial\sigma} \right)_* = w_*^2 \mu$$

$$\frac{w_*}{r_*} \left(\frac{\partial u}{\partial z} \right)_* = \frac{w_*^2}{2R} \left[\frac{B}{2} \lambda'' + R'' - (2B' + R'\lambda) \frac{R'}{B_*} + \left\{ \frac{1}{w_*} \frac{\partial w_*}{\partial z} + \frac{B'}{B} + \frac{R'}{B} (\lambda - \mu) \right\} \left(\frac{B}{2} \lambda' + R' \right) \right]$$

Eq. (17) gives at r_*

$$\begin{aligned} \nu\mu - \frac{B}{8R} \left[\frac{B}{2} \lambda'' + R'' - (2B' + R'\lambda) \frac{R'}{B} + \left\{ \frac{\nu'}{2\nu} + \frac{B'}{B} + \frac{R'}{B} (\lambda - \mu) \right\} \left(\frac{B}{2} \lambda' + R' \right) \right] \nu \\ = \left(\frac{\partial H}{\partial\sigma} \right)_* - \frac{\xi_*}{2R} \left(\frac{\partial \xi}{\partial\sigma} \right)_* - T_* \left(\frac{\partial s}{\partial\sigma} \right)_* \quad (18) \end{aligned}$$

where $\xi \equiv rv$, $\nu \equiv w_*^2$

Assuming the flow is thermally and calorically perfect, and using the well-known relations

$$p = \rho R_g T \quad (19)$$

$$di = \frac{R_g \kappa}{\kappa - 1} dT \quad (20)$$

Eq. (16) gives

$$d\phi = \frac{1}{\kappa - 1} \frac{dT}{T} - \frac{d\rho}{\rho} \quad (21)$$

where $\phi \equiv s/R_g$.

Furthermore, if a polytropic process is assumed for the flow through blade rows, we can put

$$\rho/\rho_* = \frac{\lambda}{\mu} + \frac{(1 - \lambda/\mu)}{(1 + \mu\sigma)} \quad (13)$$

where $R \equiv t^2 + h^2$, $B \equiv t^2 - h^2$ and in the following, a prime denotes differentiation with respect to z .

By using the above relations, flow in passages with arbitrary boundary shapes is expressed with two parameters μ and λ , so that the problem of finding the flow in an axial turbomachine is reduced to finding the variations of μ and λ along the axial direction.

3. The differential equations for λ and μ

There exerts no external force on the fluid in a duct, and we may assume the blades exert no force on the fluid in the radial direction, so that the radial equation of motion is written as

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \quad (14)$$

By making use of the total enthalpy

$$H = i + \frac{1}{2} (u^2 + v^2 + w^2) \quad (15)$$

and the entropy

$$Tds = di - \frac{dp}{\rho} \quad (16)$$

Eq. (14) gives

$$\frac{v}{r} \frac{\partial(rv)}{\partial r} - w \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) = \frac{\partial H}{\partial r} - T \frac{\partial s}{\partial r} \quad (17)$$

Owing to the approximated expressions Eqs. (1) and (2), the equation of motion can not be satisfied at every radial station. It is possible to satisfy exactly this equation at one radius, and we may choose r_* as this design radius. Changing r to σ and using the relations obtained from Eqs. (1), (2), (5) and (6)

$$\frac{d\rho}{\rho} = \frac{1}{(n-1)} \frac{dT}{T} \quad (22)$$

therefore we obtain from Eqs. (21) and (22)

$$d\phi = \frac{n-\kappa}{\kappa-1} \frac{d\rho}{\rho} \quad (23)$$

and from Eqs. (15), (20) and (22)

$$d\left(H - \frac{u^2}{2} - \frac{v^2}{2} - \frac{w^2}{2}\right) = (n-1) \left(H - \frac{u^2}{2} - \frac{v^2}{2} - \frac{w^2}{2}\right) \frac{d\rho}{\rho} \quad (24)$$

Eqs. (23) and (24) are another two relations which decide the flow parameters.

From these equations, we obtain the relations at r_* in the same manner as for Eq. (17). Using the following relations obtained from Eqs. (1), (2) and (7)

$$\frac{1}{\rho_*} \left(\frac{\partial \rho}{\partial \sigma} \right)_* = \lambda - \mu, \quad \frac{1}{\rho_*} \left(\frac{\partial \rho}{\partial z} \right)_* = -\frac{\nu'}{2\nu} - \frac{B'}{B} - \frac{R'}{B} (\lambda - \mu)$$

Eq. (23) gives at r_*

$$\left(\frac{\partial \phi}{\partial \sigma} \right)_* = \frac{(\kappa-n)}{(\kappa-1)} (\mu - \lambda) \quad (25)$$

$$\left(\frac{\partial \phi}{\partial z} \right)_* = \frac{\partial \phi_*}{\partial z} - \frac{R'}{B} \left(\frac{\partial \phi}{\partial \sigma} \right)_* = \frac{(\kappa-n)}{(\kappa-1)} \left\{ \frac{\nu'}{2\nu} + \frac{B'}{B} + \frac{R'}{B} (\lambda - \mu) \right\} \quad (26)$$

The entropy change in the machine is determined by these two equations and this affects the velocity and density of the flow through the entropy term in the right-hand side of Eq. (18). Similarly, using Eq. (12), Eq. (24) gives at r_*

$$(n-1)(\lambda - \mu) \left[H_* - \frac{\xi_*^2}{R} - \frac{\nu}{2} \left\{ 1 + \frac{1}{8R} \left(\frac{B}{2} \lambda' + R' \right)^2 \right\} \right] = \left(\frac{\partial H}{\partial \sigma} \right)_* - \nu \mu - \frac{\nu}{8R} \left(R' + \frac{B}{2} \lambda' \right) \left\{ B' + R' \lambda - \left(\lambda - \mu + \frac{B}{2R} \right) \left(R' + \frac{B}{2} \lambda' \right) \right\} - \frac{2}{R} \xi_* \left(\frac{\partial \xi}{\partial \sigma} \right)_* + \frac{B}{R^2} \xi_*^2 \quad (27)$$

$$(n-1) \left\{ \frac{\nu'}{2\nu} + \frac{B'}{B} + \frac{R'}{B} (\lambda - \mu) \right\} \left[H_* - \frac{\xi_*^2}{R} - \frac{\nu}{2} \left\{ 1 + \frac{1}{8R} \left(R' + \frac{B}{2} \lambda' \right)^2 \right\} \right] = -H'_* + \frac{R'}{B} \left(\frac{\partial H}{\partial \sigma} \right)_* + \frac{\nu'}{2} - \frac{R'}{B} \nu \mu - \frac{2}{R} \xi_* \left\{ \xi'_* - \frac{R'}{B} \left(\frac{\partial \xi}{\partial \sigma} \right)_* \right\} + \frac{\nu}{8R} \left(R' + \frac{B}{2} \lambda' \right) \left[R'' + \frac{B}{2} \lambda'' - (2B' + R' \lambda) \frac{R'}{B} + \left\{ \frac{\nu'}{2\nu} + \frac{B'}{B} + (\lambda - \mu) \frac{R'}{B} \right\} \left(\frac{B}{2} \lambda' + R' \right) \right] \quad (28)$$

From the above obtained Eqs. (18), (25), (26), (27) and (28), the five unknowns λ , μ , ν , ϕ_* , $\left(\frac{\partial \phi}{\partial \sigma} \right)_*$ (H_* and ξ_* will be given in the following chapters) are determined. These are independent of r , so that the equations are the ordinary differential equations and can be solved easily by a numerical method. In the following chapters, these equations are rearranged and more simplified for practical uses.

4. Flow outside of blade rows

If the flow is assumed to be steady and adiabatic, the angular momentum ξ , the total enthalpy H and the entropy ϕ are constant along streamlines in the space outside of the blade rows. Referring to Fig. 2, the streamlines AB and CD pass respectively through the points A and D at r_* of the axial positions ① and ②. If the axial distance δz is taken to be sufficiently small the radial displacement of the streamline $\delta \sigma$ becomes also small and ξ , H and ϕ at the point A may be written as

$$g_A = g_D + \left(\frac{\partial g}{\partial \sigma} \right)_* \delta \sigma \quad (29)$$

where $g(\sigma, z)$ is an arbitrary function which is constant along streamlines. The radial distance between AB and CD is small, and therefore, from Eq. (5), the difference of the streamfunction $\delta \psi$ between AB and CD is

$$\delta \psi \approx \frac{B}{4} \rho_* \omega_* \delta \sigma$$

Since $\delta \psi$ is constant along the streamline and $B \rho_* \omega_*$ is invariable from Eq. (7), $\delta \sigma$ becomes also constant from ① to ②. Eq. (29), using Eq. (11) yields

$$\left\{ g + \frac{1}{2} \left(\frac{\partial g}{\partial \rho} \right)_* \lambda \right\}_A = \left\{ g + \frac{1}{2} \left(\frac{\partial g}{\partial \sigma} \right)_* \lambda \right\}_D$$

so that at r_*

$$g_* + \frac{1}{2} \left(\frac{\partial g}{\partial \sigma} \right)_* \lambda = \text{const.} \quad (30)$$

Replacing g by H , ϕ and ξ , we obtain

$$\left. \begin{aligned} H_* + \frac{1}{2} G \lambda &= \text{const.} \equiv E \\ \phi_* + \frac{1}{2} P \lambda &= \text{const.} \equiv Q \\ \xi_* + \frac{1}{2} \Gamma \lambda &= \text{const.} \equiv \Theta \end{aligned} \right\} \quad (31)$$

where

$$\left(\frac{\partial H}{\partial \sigma}\right)_* = \text{const.} \equiv G, \quad \left(\frac{\partial \phi}{\partial \sigma}\right)_* = \text{const.} \equiv P, \quad \left(\frac{\partial \xi}{\partial \sigma}\right)_* = \text{const.} \equiv \Gamma \quad (32)$$

Substituting Eqs. (31) and (32) in to Eq. (18), and writing T_* as

$$T_* = \frac{(\kappa-1)}{R_g \kappa} \left(H_* - \frac{u_*^2}{2} - \frac{v_*^2}{2} - \frac{w_*^2}{2} \right)$$

yields

$$\eta' = \frac{16R}{B^2} \left[\left\{ \frac{2\Gamma}{R} \left(\Theta - \frac{1}{2} \Gamma \lambda \right) - G \right\} \frac{1}{\nu} - \frac{B}{8R} \left[R'' - (2B' + R'\lambda) \frac{R'}{B} \right. \right. \\ \left. \left. + \left\{ \frac{\nu'}{2\nu} + \frac{B'}{B} + \frac{R'}{B} (\lambda - \mu) \right\} \left(\frac{B}{2} \eta + R' \right) \right] + \mu + \frac{(\kappa-1)}{\kappa(n-1)} (2U_1 + 1) P \right] \quad (33)$$

$$\text{where } \eta \equiv \lambda' \quad (34)$$

Similarly, substituting Eqs. (31) and (32) in to Eq. (28) and eliminating η' from Eq. (33), we obtain

$$\nu' = \frac{V_1}{U_1} \quad (35)$$

$$U_1 = \frac{(n-1)}{2\nu} \left[E - \frac{1}{2} G \lambda - \frac{1}{R} \left(\Theta - \frac{1}{2} \Gamma \lambda \right)^2 - \frac{\nu}{2} \left\{ 1 + \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right)^2 \right\} \right] - \frac{1}{2}$$

$$V_1 = \frac{1}{2} \nu \mu \eta + \frac{\nu(2U_1+1)}{B} \left\{ \frac{(\kappa-1)}{\kappa(n-1)} \left(\frac{B}{2} \eta + R' \right) P - R' (\lambda - \mu) - B' \right\}$$

and, in like manner, from Eq. (27)

$$\mu = \frac{Y_1}{X_1} \quad (36)$$

$$X_1 = \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right)^2 - 2U_1$$

$$Y_1 = \left\{ G - \frac{2\Gamma}{R} \left(\Theta - \frac{1}{2} \Gamma \lambda \right) + \frac{B}{R^2} \left(\Theta - \frac{1}{2} \Gamma \lambda \right)^2 \right\} \frac{1}{\nu} \\ - \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right) \left\{ B' + R' \lambda - \left(\lambda + \frac{B}{2R} \right) \left(R' + \frac{B}{2} \eta \right) \right\} - \lambda (2U_1 + 1)$$

Using Eqs. (33) to (36), the variations of $\lambda(z)$, $\mu(z)$, $\nu(z)$ and $\eta(z)$ are determined. ϕ_* is given by the relation (31). R and B in these equations are given by the wall shapes of the flow passage and the constants G , P , Γ , E , Q , Θ are determined by boundary conditions given usually at the inlet and outlet of the passage.

If the radial gradient of the entropy, $\left(\frac{\partial \phi}{\partial \sigma}\right)_*$, is neglected as permitted in practice for usual machines, Eqs. (33) and (34) are more simplified as follows, putting $P=0$,

$$\eta' = \frac{16R}{B^2} \left[\left\{ \frac{2\Gamma}{R} \left(\Theta - \frac{1}{2} \Gamma \lambda \right) - G \right\} \frac{1}{\nu} - \frac{B}{8R} \left[R'' - (2B' + R'\lambda) \frac{R'}{B} + \left\{ \frac{\nu'}{2\nu} + \frac{B'}{B} + \frac{R'}{B} (\lambda - \mu) \right\} \left(\frac{B}{2} \eta + R' \right) \right] + \mu \right] \quad (37)$$

$$\nu' = \frac{V_2}{U_1} \quad (38)$$

$$U_1 = \frac{(n-1)}{2\nu} \left[E - \frac{1}{2} G \lambda - \frac{1}{R} \left(\Theta - \frac{1}{2} \Gamma \lambda \right)^2 - \frac{\nu}{2} \left\{ 1 + \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right)^2 \right\} \right] - \frac{1}{2}$$

$$V_2 = \frac{1}{2} \eta \nu \mu - \frac{\nu}{B} \left\{ B' + R' (\lambda - \mu) \right\} (2U_1 + 1)$$

The solution thus obtained must satisfy the condition of continuity, that is, the velocity normal to the inner and outer wall has to vanish. From Eqs. (1) and (12) we get

$$(u/w)_{\sigma=1} = t', \quad (u/w)_{\sigma=-1} = h'$$

and so we find this condition is satisfied independently of the flow parameters in the present theory.

5. Flow through a single blade row

5.1 Flow equations in a blade row

In order to apply the equations obtained in chapter 3 to the flow, in a blade row, it is necessary to specify the variation of H and ξ . From the wellknown Euler's equation

$$dH = \omega d\xi \quad (39)$$

$$I \equiv H - \omega \xi \quad (40)$$

is invariable on the same streamline in a blade row. Therefore, we can apply the relation (30) to I , so

that putting

$$\left(\frac{\partial I}{\partial \sigma}\right)_* = \text{const.} \equiv J \quad (41)$$

we obtain

$$I_* + \frac{1}{2} J \lambda = \text{const.} \equiv K \quad (42)$$

In the following, the flow equations for a rotor row are treated and these equations apply also to a stator row, if the angular velocity ω is put equal to zero. Using Eqs. (40) to (42) and substituting Eq. (25) in to Eq. (18), we obtain

$$\eta' = \frac{16R}{B^2} \left[\left\{ \frac{2}{R} \xi_* - \omega \left(\frac{\partial \xi}{\partial \sigma} \right)_* - J \right\} \frac{1}{\nu} - \frac{B}{8R} \left[R'' - (2B' + R'\lambda) \frac{R'}{B} + \left\{ \frac{\nu'}{2\nu} + \frac{B'}{B} + \frac{R'}{B} (\lambda - \mu) \right\} \left(\frac{B}{2} \eta + R' \right) \right] + \mu + \frac{(\kappa - n)}{\kappa(n-1)} (2U_2 + 1)(\mu - \lambda) \right] \quad (43)$$

where

$$\eta = \lambda' \quad (34)$$

In like manner, from Eqs. (28) and (43),

$$\nu' = \frac{V_3}{U_2} \quad (44)$$

$$U_2 = \frac{(n-1)}{2\nu} \left[K - \frac{1}{2} J \lambda - \frac{\xi_*^2}{R} + \omega \xi_* - \frac{\nu}{2} \left\{ 1 + \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right)^2 \right\} \right] - \frac{1}{2}$$

$$V_3 = \left(\frac{2}{R} \xi_* - \omega \right) \left\{ \xi_* + \frac{1}{2} \eta \left(\frac{\partial \xi}{\partial \sigma} \right)_* \right\} + \frac{1}{2} \eta \nu \mu + \frac{\nu(2U_2 + 1)}{B} \left[\left\{ \frac{(\kappa - n)}{\kappa(n-1)} \left(\frac{B}{2} \eta + R' \right) + R' \right\} (\mu - \lambda) - B' \right]$$

and from Eq. (27)

$$\mu = \frac{Y_2}{X_2} \quad (45)$$

$$X_2 = \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right)^2 - 2U_2$$

$$Y_2 = \left\{ J - \left(\frac{2}{R} \xi_* - \omega \left(\frac{\partial \xi}{\partial \sigma} \right)_* + \frac{B}{R^2} \xi_*^2 \right) \frac{1}{\nu} - \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right) \left\{ B' + R'\lambda - \left(\lambda + \frac{B}{2R} \right) \left(R' + \frac{B}{2R} \eta \right) \right\} - \lambda(2U_2 + 1) \right\}$$

From Eq. (26), the entropy ϕ_* is given by

$$\phi'_* = \frac{(\kappa - n)}{(n-1)} \left(\frac{\nu'}{2\nu} + \frac{B'}{B} \right) \quad (46)$$

Since ξ_* and $\left(\frac{\partial \xi}{\partial \sigma} \right)_*$ in Eqs. (43) to (45) are specified by the load distribution or by the geometrical shape of the blade, the variation of the flow parameters in the blade row can be obtained from these equations. If we may neglect $\left(\frac{\partial \phi}{\partial \sigma} \right)_*$, Eqs. (43) and (44) are simplified to

$$\eta' = \frac{16R}{B^2} \left[\left\{ \frac{2}{R} \xi_* - \omega \left(\frac{\partial \xi}{\partial \sigma} \right)_* - J \right\} \frac{1}{\nu} - \frac{B}{8R} \left[R'' - (2B' + R'\lambda) \frac{R'}{B} + \left\{ \frac{\nu'}{2\nu} + \frac{B'}{B} + \frac{R'}{B} (\lambda - \mu) \right\} \left(\frac{B}{2} \eta + R' \right) \right] + \mu \right] \quad (47)$$

$$\nu' = \frac{V_4}{U_2} \quad (48)$$

$$U_2 = \frac{(n-1)}{2\nu} \left[K - \frac{1}{2} J \lambda - \frac{\xi_*^2}{R} + \omega \xi_* - \frac{\nu}{2} \left\{ 1 + \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right)^2 \right\} \right] - \frac{1}{2}$$

$$V_4 = \left(\frac{2}{R} \xi_* - \omega \right) \left\{ \xi_* + \frac{1}{2} \eta \left(\frac{\partial \xi}{\partial \sigma} \right)_* \right\} + \frac{1}{2} \eta \nu \mu + \frac{\nu(2U_2 + 1)}{B} \left\{ R'(\mu - \lambda) - B' \right\}$$

When the variation of the relative flow angle to the blade is given as in the so-called direct problem, ξ_* , $\left(\frac{\partial \xi}{\partial \sigma} \right)_*$ and ξ'_* are written as

$$\beta \equiv \frac{v - r\omega}{w} \quad (49)$$

where β is the tangent of the relative flow angle to the blade from z axis.

Substituting β into Eqs. (43) to (45) and omitting the gradient of entropy, the following equations are obtained.

$$\eta' = \frac{16R}{B^2} \left[\frac{B}{2R} \beta_*^2 + \beta_* \left(\frac{\partial \beta}{\partial \sigma} \right)_* + (1 + \beta_*^2) \mu + \frac{B}{2} \omega \beta_* \sqrt{\frac{2}{R\nu}} - \frac{B}{8R} \left[R'' - (2B' + R'\lambda) \frac{R'}{B} \right. \right. \\ \left. \left. + \left\{ \frac{\nu'}{2\nu} + \frac{B'}{B} + \frac{R'}{B} (\lambda - \mu) \right\} \left(\frac{B}{2} \eta + R' \right) \right] - \frac{J}{\nu} \right] \dots\dots\dots (50)$$

$$V' = \frac{V_5}{U_3} \dots\dots\dots (51)$$

$$U_3 = \frac{(n-1)}{2\nu} \left[K - \frac{1}{2} J \lambda + \frac{R}{4} \omega^2 - \frac{\nu}{2} \left\{ 1 + \beta_*^2 + \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right)^2 \right\} - \frac{1}{2} (1 + \beta_*^2) \right]$$

$$V_5 = \frac{1}{2} \left(R' + \frac{B}{2} \eta \right) \left(\frac{1}{R} \beta_*^2 \nu + \omega \beta_* \sqrt{\frac{2}{R\nu}} \right) + \left\{ \frac{\beta_*}{2} \left(\frac{\partial \beta}{\partial \sigma} \right)_* \eta + \beta_* \beta'_* \right\} \nu \\ + \frac{(1 + \beta_*^2)}{2} \eta \nu \mu + \frac{\nu}{B} (2U_3 + 1 + \beta_*^2) \{ R' (\mu - \lambda) - \beta' \}$$

$$\mu = \frac{Y_3}{X_3} \dots\dots\dots (52)$$

$$X_3 = \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right)^2 - 2U_3$$

$$Y_3 = \left(J + \frac{B}{4} \omega^2 \right) \frac{1}{\nu} - \beta_* \left(\frac{\partial \beta}{\partial \sigma} \right)_* - \frac{1}{8R} \left(R' + \frac{B}{2} \eta \right) \left\{ B' + R'\lambda - \left(\lambda + \frac{B}{2R} \right) \left(R' + \frac{B}{2} \eta \right) \right\} - \lambda (2U_3 + 1 + \beta_*^2)$$

We can use Eqs. (50) to (52) for the direct problem.

5.2 Boundary conditions

The solutions for the flow in a blade row are to be joined to those for the duct flow obtained in chapter 4 by the condition that the axial, radial velocities and the density are continuous at the boundaries, and at these stations, denoted by 1, 2, 3 and 4, in Fig. 3

$$\lambda_1 = \lambda_2, \mu_1 = \mu_2, \nu_1 = \nu_2, \eta_1 = \eta_2 \dots\dots\dots (53)$$

$$\lambda_3 = \lambda_4, \mu_3 = \mu_4, \nu_3 = \nu_4, \eta_3 = \eta_4 \dots\dots\dots (54)$$

At the exit of the blade, the tangential velocity and the total enthalpy are continuous, so that

$$\left. \begin{aligned} \Gamma_4 &= \left(\frac{\partial \xi}{\partial \sigma} \right)_{*3} \\ \Theta_4 &= \xi_{*3} + \frac{1}{2} \left(\frac{\partial \xi}{\partial \sigma} \right)_{*3} \lambda_3 \\ G_4 &= J_3 + \omega \left(\frac{\partial \xi}{\partial \sigma} \right)_{*3} \\ E_4 &= K_3 + \omega \left\{ \xi_{*3} + \frac{1}{2} \left(\frac{\partial \xi}{\partial \sigma} \right)_{*3} \lambda_3 \right\} \end{aligned} \right\} \dots\dots\dots (55)$$

and since, at the design point, continuity of the tangential velocity and of the total enthalpy may be also assumed at the inlet of the blade, we get

Fig. 3 Flow through a single blade row

$$\left. \begin{aligned} \left(\frac{\partial \xi}{\partial \sigma} \right)_{*2} &= \Gamma_1 \\ \xi_{*2} &= \Theta_1 - \frac{1}{2} \Gamma_1 \lambda_1 \\ J_2 &= G_1 - \omega \Gamma_1 \\ K_2 &= E_1 - \omega \Theta_1 \end{aligned} \right\} \dots\dots\dots (56)$$

Furthermore, for the direct problem, using β of Eq. (49), Eq. (55) gives at the exit

$$\left. \begin{aligned} \Gamma_4 &= \beta_{*3} \sqrt{\frac{R}{2\nu_3}} \left\{ \frac{B}{2R} + \frac{1}{\beta_{*3}} \left(\frac{\partial \beta}{\partial \sigma} \right)_{*3} + \mu_3 \right\} + \frac{B}{2} \omega \\ \Theta_4 &= \beta_{*3} \sqrt{\frac{R}{2\nu_3}} + \frac{R}{2} \omega + \frac{1}{2} \Gamma_4 \lambda_3 \\ G_4 &= J_3 + \omega \beta_{*3} \sqrt{\frac{R}{2\nu_3}} \left\{ \frac{B}{2R} + \frac{1}{\beta_{*3}} \left(\frac{\partial \beta}{\partial \sigma} \right)_{*3} + \mu_3 \right\} \\ &\quad + \frac{B}{2} \omega^2 \\ E_4 &= K_3 - \frac{1}{2} J_3 \lambda_3 + \omega \beta_{*3} \sqrt{\frac{R}{2\nu_3}} + \frac{R}{2} \omega^2 + \frac{1}{2} G_4 \lambda_3 \end{aligned} \right\} \dots\dots\dots (57)$$

Similarly, Eq. (56) gives at the inlet

$$\left. \begin{aligned} \beta_{*2} &= \sqrt{\frac{2}{R\nu_1}} \left(\Theta_1 - \frac{R}{2} \omega - \frac{1}{2} \Gamma_1 \lambda_1 \right) \\ \left(\frac{\partial \beta}{\partial \sigma} \right)_{*2} &= \sqrt{\frac{2}{R\nu_1}} \left(\Gamma_1 - \frac{B}{2} \omega \right) - \beta_{*2} \left(\frac{B}{2R} + \mu_1 \right) \\ J_2 &= G_1 - \omega \Gamma_1 \\ K_2 &= E_1 + \frac{1}{2} J_1 \lambda_1 - \omega \beta_{*2} \sqrt{\frac{R}{2\nu_1}} - \frac{R}{2} \omega^2 - \frac{1}{2} G_1 \lambda_1 \end{aligned} \right\} \dots\dots\dots (58)$$

Thus, as found from the above relations, the values of ξ_* , $(\partial \xi / \partial \sigma)_*$ or β_* , $(\partial \beta / \partial \sigma)_*$ and constants J and K at the inlet of blades are determined by the flow conditions at the duct-exit. However, we can specify any variation of ξ_* and $(\partial \xi / \partial \sigma)_*$ for the inverse problem and of β_* and $(\partial \beta / \partial \sigma)_*$ for the direct problem within the blade row. By

using these relations, we can integrate the flow equations obtained in chapters 4 and 5 through the machine.

5.3 Numerical method of solution

The flow equations are first order non-linear ordinary differential equations for η , λ and ν . Since it will be impossible to obtain the exact analytical solutions of these equations, we must use a numerical method of solution. If one boundary value for each of the unknowns η , λ and ν is given by the flow conditions at the inlet of the machine (Fig. 3), it is easy to solve these equations numerically. However in practice, it is impossible to prescribe $\eta_i \equiv \lambda_i$ at the beginning of calculation, so that this must be determined by the other additive conditions. In the case where the cylindrical duct with inner and outer radii is set at the inlet and outlet of the machine, we may use the following simple radial equilibrium condition.

$$z = \pm \infty: \eta = 0 \dots \dots \dots (59)$$

Further, since in most cases the uniform flow without swirl flows into machines, the radial and axial gradients of density are usually small at the station sufficiently upstream of the blade row. Therefore, assuming that the gradient of density is equal to zero at i (Fig. 3) and putting $\mu = \lambda$, $R' = B' = 0$, we obtain from Eq. (37)

$$\eta_i = \sqrt{\frac{16R}{B^2\nu_i} \left(\nu_i - \frac{\Gamma_i^2}{R} \right) \left\{ \lambda_j + \frac{(2\Gamma_i\Theta_i/R - G_i)}{(\nu_i - \Gamma_i^2/R)} \right\}} \dots \dots \dots (60)$$

This gives the relation between η_i and λ_i which satisfies the condition (59) at upstream infinity. Then, constants Γ_i , Θ_i , G_i and E_i are determined from Eqs. (31) and (32).

Thus, using η_i obtained from Eq. (60) with the initial, approximate values of λ_i and ν_i , we can perform the numerical integration of the flow equations. This procedure is iterated using a closer approximate value of λ_i until a solution is obtained which satisfies the condition (59) at the exit of the machine.

6. Flow in a multi-stage axial turbomachine

Since the solution for a multi-stage axial turbomachine is basically a series of the solution for a single blade row which has been given in the previous chapter, the flow of the former can be also readily obtained by stacking the flow of the latter in the axial direction. The boundary conditions at the inlet and the exit of the machine are to be satisfied as in chapter 5 and, using the continuity relations at the leading and trailing edges of all the blade rows, the flows in each

blade row and in each duct are joined.

In practice, the axial length of the gaps between the blade rows is usually very small compared with the axial chord length of the blades, so that we may neglect them. Then, the flow conditions at the inlet of the blade row can be obtained directly from those at the exit of the preceding blade row, that is, considering the continuity of the angular momentum and the total enthalpy at the boundary of rotor and stator,

$$\left. \begin{aligned} \xi_{*r} &= \xi_{*s} \\ \left(\frac{\partial \xi}{\partial \sigma} \right)_{*r} &= \left(\frac{\partial \xi}{\partial \sigma} \right)_{*s} \\ J_r &= J_s - \omega \left(\frac{\partial \xi}{\partial \sigma} \right)_{*s} \\ K_r &= K_s - \frac{1}{2} \omega \left(\frac{\partial \xi}{\partial \sigma} \right)_{*s} \lambda_s - \omega \xi_{*s} \end{aligned} \right\} \dots \dots \dots (61)$$

or, using β_* ,

$$\left. \begin{aligned} \beta_{*r} &= \beta_{*s} - \omega \sqrt{\frac{R}{2\nu_s}} \\ \left(\frac{\partial \beta}{\partial \sigma} \right)_{*r} &= \left(\frac{\partial \beta}{\partial \sigma} \right)_{*s} - \omega \left(\frac{B}{4} \sqrt{\frac{2}{R\nu_s}} - \mu_s \sqrt{\frac{R}{2\nu_s}} \right) \\ J_r &= J_s - \omega \sqrt{\frac{R}{2\nu_s}} \beta_{*s} \left\{ \frac{1}{\beta_{*s}} \left(\frac{\partial \beta}{\partial \sigma} \right)_{*s} + \mu_{*s} + \frac{B}{2R} \right\} \\ K_r &= K_s - \frac{1}{2} J_s \lambda_s - \omega \sqrt{\frac{R}{2\nu_s}} \beta_{*s} + \frac{1}{2} J_r \lambda_s \end{aligned} \right\} \dots \dots \dots (62)$$

Where subscripts r and s denote rotor and stator respectively. Continuity of η , λ , μ and ν must also exist at the boundary.

7. Comparison with exact calculations

7.1 Compressible flow in a single-stage turbine

In this chapter comparison with two accurate calculations has been made, in order to investigate the accuracy of the present theory.

Wu⁽⁸⁾ obtained the solution of the compressible flow through a single-stage turbine by solving the full partial differential equations of the flow numerically with a finite-difference method. The boss ratio of the turbine is 0.6 and the inner-and outer-casing radii are constant. The aspect ratio of rotor and stator blade (based on axial chord length) is 2.67. The ratio of inlet velocity of the uniform flow to rotor tip speed is 0.65, and inlet Mach

Fig. 4 Flow in a multi-stage axial turbomachine

number is 0.3205. From these and $n=1.333$,

$$\rho_{-\infty}/\rho_{0,-\infty}=0.95033, \quad H_{-\infty}/V_t^2=12.546$$

and we get

$$Z=-\infty: \Gamma=0, \quad \Theta=0, \quad E=0, \quad G=12.546V_t^2$$

$$R/t^2=\text{const.}=1.36, \quad B/t^2=\text{const.}=0.64,$$

$$r_*/t=0.8246 \quad m/(\rho_{0,-\infty}V_t t^2)=1.242,$$

Since the inner and outer casings of this turbine are cylindrical, R' , B' and R'' are equal to zero. The gradient of entropy $(\partial\phi/\partial\sigma)_*$ is also neglected.

The axial variations of the angular momentum $\xi/(tV_t)$ in the stator and rotor row are given as in Fig. 5 which are those used in Wu's calculation⁽⁸⁾. At any constant z -plane, the angular momentum is constant with respect to r , so that the flow in the turbine corresponds to the free-vortex type. The HIPAC-103 electronic digital computer was used for obtaining the solutions by a Runge-Gill method.

The results of η , λ and μ are shown in Fig. 6. Since the flow is free-vortex type and we assume blades exert no radial forces, η , λ and μ would be equal to zero for the incompressible flow. Therefore, the results of Fig. 6 show the effects of the

Fig. 5 Variations of angular momentum

Fig. 6 η , λ , μ

fluid compressibility. ν is shown in Fig. 7.

Non-dimensional axial velocities, radial velocities and densities derived from η , λ , μ and ν are shown in Figs. 8 to 10. In these figures the results obtained by Wu are also plotted. At the inner and outer casings the results obtained by the two methods agree closely as at mean radius, and these results show that the present theory is sufficiently accurate for practical use. In Fig. 11, Mach

Fig. 7 ν

Fig. 8 Variation of axial velocity

Fig. 9 Variation of radial velocity

numbers are shown. In the rotor row, the relative Mach numbers to the blade are given.

Figs. 12 and 13 show the effect of varying the boss-ratio of this turbine. The results were obtained for a boss-ratio 0.5 with the same M_∞ , w_∞/V_t , $\rho_\infty/\rho_{0,-\infty}$, H_∞/V_t^2 as used in the previous calculation. As the boss-ratio is decreased, the radial displacement of the streamline is increased.

Fig. 10 Variation of density

Fig. 11 Variation of Mach number

Figs. 14 and 15 show the effect of varying the inlet Mach number from 0.3205 to 0.37 (with $w_\infty/V_t = 0.65$, we obtain $H_\infty/V_t^2 = 9.4819$, $\rho_\infty/\rho_{0,-\infty} = 0.93449$). The boss-ratio is 0.6 in both cases. As the Mach number is increased the effect of the curvature of the meridional streamline becomes large. The shapes of the meridional streamlines are shown in Fig. 16. λ is usually small and from Eq. (6)

$$\sigma = \Psi - 1 - \frac{\lambda}{2}(\Psi^2 - 2\Psi) \dots \dots \dots (63)$$

$$\text{where } \Psi \equiv \frac{4\pi}{m}\varphi$$

Fig. 16 shows the streamlines $\sigma = \frac{\lambda}{2}$ for $\Psi = 1$ which lie on r_* at upstream infinity. For the incompressible flow ($M_\infty = 0$), this streamline is apparently parallel to the z axis throughout the turbine (in Fig. 16 the radial scale of the streamlines is magnified five times). As the Mach number is increased the radial displacement of the streamline is increased, and the effect of the axial curvature of the streamline becomes large.

7.2 Incompressible flow through an actuator disc placed in a diverging annular passage

In the previous report⁽⁹⁾, the author has given

Fig. 12 Effect of varying the boss ratio: λ , μ

Fig. 13 Effect of varying the boss ratio: ν

an exact solution of the nonviscous, incompressible flow through an actuator disc placed in a diverging or converging annular passage. This is an extension of Bragg and Hawthorne's solution⁽¹⁰⁾ The present theory is compared with this exact solution is made.

The shape of the diverging flow passage used in the calculation is shown in Fig. 17. An actuator disc is placed at $z=0$. The uniform flow without swirl flows into the passage and at $z=0$ the flow gets discontinuously a tangential velocity. The total enthalpy is constant with respect to r . The tangential and axial velocity distributions at downstream infinity are respectively given as in Figs. 18 and 19. In Fig. 19 the axial velocities obtained

Fig. 14 Effect of varying the inlet Mach number: λ , μ

Fig. 15 Effect of varying the inlet Mach number: ν

Fig. 16 Deflection of meridional streamlines (the radial shift of streamlines magnified five times)

by both methods are shown and agree fairly well. From these, the constants G , Γ and Θ are determined as follows,

$$z < 0: G=0, \quad \Gamma=0, \quad \Theta=0$$

$$z \geq 0: G=0, \quad \Gamma=0.0480, \quad \Theta=1.036$$

The axial velocities are shown in Fig. 20 compared with the results of exact solution. The results of the two methods agree very closely at

Fig. 17 Flow through an actuator disc in a diverging channel

Fig. 18 Tangential velocity distribution at downstream infinity

Fig. 19 Axial velocity distribution at downstream infinity

Fig. 20 Variation of axial velocity

Fig. 21 Radial displacement of the middle streamline at all the positions. The shape of the meridional streamline which lies on $r_* = 1.164$ at the upstream infinity is shown in Fig. 17. Fig. 21 shows the radial displacement of this streamline from $r_*(\delta \equiv r - r_*)$, and there is also good agreement of the streamline shape between the two solutions.

8. Conclusion

A theory of three-dimensional compressible flow in axial turbomachines with varying boundary shapes was presented. Assuming the axial velocity profiles and the specific mass flow profiles to be expressible by a single parameter respectively, the ordinary differential equations of the flow were derived, and the numerical method of solution was shown. By using the present theory the flow through the blade rows can be obtained taking

account of the effects of slope and curvature of the meridional streamline, the fluid compressibility and the boundary shapes of the passage.

Comparing the present theory with two accurate calculations, it was shown that the approximate expressions of the axial velocity profiles and the specific mass flow profiles [Eqs. (1) and (2)] were sufficiently accurate for practical purposes and that the present theory might be generally applicable in practice.

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Discussion

M. MATSUKI and S. FUJII: (1) The discussers remember that the idea of using a single parameter for the flow in axial turbomachines has been presented for the first time by Whitehead and Beavers. The author is to be congratulated on developing a theory taking into account the fluid compressibility. As a question concerning such a parameter theory, we think as follows: Firstly, a streamline displacement would not be necessarily small if the flow pattern deviated largely from a free-vortex type. As a numerical example, the author has shown only the flow of a free-vortex type, however, the discussers wish to know the applicability of the present theory for other flow patterns (for example, for forced-vortex type). Secondly, the accuracy of this theory seems to deteriorate in the case where the variation of boundary shapes is large. Particularly, it appears that the theory is not applicable for machines with constant outer casing and varying inner casing radii.

We have the opinion that the method of repeating the streamline approximation is to be applied for the detailed design of, for example, jet engine's compressors, because an error in the flow calculation in the vicinity of outer and inner casing affects largely the performance estimation of the machine. The present theory would be useful for gaining an insight into the design or off-design performance of the machine.

Y. KAGEYAMA: (2) The discussor considers it is essential that the expressions of Eqs. (1) and (2) have a sufficient accuracy, so that the author had better confirm the universality of these expressions by comparing with other exact calculations or with the calculation on the more accurate expression of Eqs. (1) and (2).

T. IKUI: (3) The present theory would be useful for axial turbomachines, however, in actual machines the streamline displacement due to boundary layers and tip clearances appear larger than

those due to the effect of fluid compressibility. It would be better that the effects of the distortion of velocity profiles due to the fluid viscosity are taken into account in the theory.

Compare the curve in section 7.1 with the result by Whitehead's theory.

Author's closure

(1) In regard to the first question, considering the fluid compressibility, the radial displacement of streamlines is not necessarily small for a flow of a free-vortex type. Similarly, it would not be necessarily possible that the streamline displacement becomes large in other types of flow. By taking the axial distance δz small in Fig. 2, we will be able to make the radial displacement $\delta\sigma$ as small as we desire, so that the theory may also be applicable for other flow patterns with sufficient accuracy. Concerning the second question, the effects of the slope and the curvature of the inner casing are taken into account through the terms of R' and R'' like those of the outer casing, so that the theory may also be applicable for such a case. The author does not consider the usual method of radial equilibrium to be more accurate, because the shape of streamlines has to be assumed. Of course, the two-parameters theory will be suitable also for such a problem suggested by the discussers.

(2) The author agrees to that opinion. Con-

sidering the very complicated three-dimensional flow in the vicinity of walls in actual machines, the expressions of Eqs. (1) and (2) are sufficiently accurate in practice. We would be able to improve the accuracy of Eqs. (1) and (2), if we took more terms of Taylor's expansion; however, the rapid increase of the computational labour may decrease the practicality of the theory, and the effect of it appears not so large to the author.

(3) In regard to the effects pointed out by Professor Ikui, the author has not yet considered them in the present theory; however, the effects of the growth of boundary layers at walls will be treated approximately in the theory, by giving the outer and inner casing radii considering the displacement thickness. Hereafter, the author wishes to investigate the problem.

For the example in the section 7.1, Whitehead's theory gives the results $\eta, \lambda, \mu \equiv 0$, so that the results of Fig. 6 show the effects of compressibility apparently.

P.S.: Recently, Beavers proposed another parameter theory in ARC C.P. 755 (1965). Also he used parameter expressions of flow which were given to ρ and ρw and he neglected the term $u \frac{\partial u}{\partial r}$ in the equation of motion, Eq. (14), in order to make flow equations linear.