Relation between Ultimate Pressure and Thickness of Wall of Cylindrical Pressure Vessels*

By Yasuo SATO** and Fumio NAGAI***

Various formulas and methods for predicting the ultimate pressure of the cylindrical pressure vessels have been investigated. However, concerning the ultimate pressure it seems that there exist some discrepancies between the experimental data and the predicted values for the thick walled cylinder though they coincide for the thin walled cylinder.

In this paper, at first the experimental data on the ultimate pressure for various diameter ratios of mild steel and aluminium cylinders are presented, then comparisons made between the experimental data and some predicted values, and it is pointed out that the predicted value obtained from the stress-strain relation under torsion coincides well with the experiments when a new formula proposed by Nakanishi is used in our calculations as the relation between pure shear and simple shear.

1. Introduction

Various investigations for predicting the ultimate pressure of cylindrical pressure vessels have been carried out until now. Most of them give the ultimate pressure by the empirical formula and others calculate it from the stress-strain relation as the investigation of Crossland. In the latter case, however, the relation between pure shear and simple shear in plastic range is an important problem, since the plastic deformation of a solid bar under torsion is a simple shear and that of a hollow cylinder under internal pressure is considered to be practically a pure shear; the circumferential elongation is caused at the cost of the decrease of the wall and the axial length remains constant while the cylinder is deformed plastically.

There are various formulas concerning the relation between pure shear and simple shear; for instance, the formula based on the incremental strain theory of plasticity is well known, and it was used by Crossland and others in their calculation of the ultimate pressure. Recently, however, Nakanishi proposed a new formula for the relation between pure shear and simple shear.

In this paper the following research works on the ultimate pressure are presented;

1. The experimental data for various diameter ratios of mild steel and aluminium cylinders.

2. The comparisons between these data and predicted values.

One of the predicted values is obtained from an empirical formula based on the mean stress theory, and the others are obtained respectively from the stress-strain relation under torsion by using two formulas concerning the relation between pure shear and simple shear, one being the formula based on the incremental strain theory of plasticity, and the other being Nakanishi’s formula.

2. Experiment (1)

Fig. 1 shows the dimensions of the experimental cylinders, it is made of a solid bar of mild steel (outer diameter 32 mm, carbon content 0.02 %). In the experimental cylinders the outer diameters are $d_o=25$ mm, and the inner diameters are changed variously as shown in Table 1. In this table $K_d$ is the ratio of the outer diameter to the inner

Fig. 1 Mild steel cylinder
diameter, two cylinders are prepared for every ratio, and the solid bar for torsion is also prepared. These test pieces were annealed by heating at 900°C for an hour in an electric furnace, and they were naturally cooled in it.

Fig. 2 shows the device in pressure test of the cylinder. In this figure T is the experimental cylinder, and the oil leakage from the end of the cylinder is avoided by the lens packing C. Internal pressure is applied to the cylinder with an oil pump, a plunger type worked by hand, and it is measured with Bourdon tube pressure gage.

The pressure-expansion curve of the cylinder is obtained by the following way; at first the cylinder is a little deformed by increasing internal pressure, and we wait for a few minutes until the relation between the pressure and the deformation of the cylinder attains equilibrium. And the pressure and the expansion of the outer diameter of the cylinder are measured in that state. In this case the outer diameter is measured by a micrometer. After this measurement the cylinder is again deformed by increasing the internal pressure a little, and the above procedure is repeated until a complete pressure-expansion curve is obtained.

Fig. 3 shows the pressure-expansion curves on the cylinders for various diameter ratios $K_0$, and the maximum pressures of them are the ultimate pressure $p_a$. The deformation of the cylinder is stable until the pressure reaches $p_a$, but its deformation becomes unstable when the pressure reaches $p_a$ and the expansion of the cylinder does not take place uniformly in the range beyond $p_a$, and a contraction occurs locally in the wall of the cylinder. Fig. 4 shows the cross section of the burst cylinder, and a local contraction can be seen clearly.

Table 1 shows the ultimate pressure $p_a$ obtained by pressure test. It can be considered that the stress distribution of the thin hollow cylinder under internal pressure is uniform. Consequently, express-

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$2\tau s = 30.00$ kg/mm²

Fig. 2 Pressure test assembly

Fig. 3 Experimental pressure-expansion curve of mild steel cylinder for various diameter ratios $K_0$

(a) $K_0=1.14$  (b) $K_0=3.13$

Fig. 4 Cross section of burst cylinder of mild steel
ing the nominal shearing stress by \( \tau_b \) at the ultimate pressure, the value of \( \tau_b \) is considered as the constant for a given material, and the relation between \( p_b \) and \( \tau_b \) is

\[
p_b = 2 \tau_b \log K_0 \tag{1}
\]

where \( K_0 \) is the ratio of the outer diameter to the inner diameter of the cylinder at the initial state.

This is one of the empirical formulas to predict the ultimate pressure and it is based on the mean stress theory. Calculating the value of \( \tau_b \) from the experimental data shown in Table 1, it is \( \tau_b = 15.0 \text{ kg/mm}^2 \). In the last column of Table 1 the ultimate pressure \( p_b \) is shown in terms of dimensionless values, \( p_b/2 \tau_b \).

Fig. 5 shows the relation between the twisting moment \( M \) and the shearing strain \( \gamma \) of a solid bar of 13 mm outer diameter of mild steel. The \( M-\gamma \) relation is obtained in the same way as the pressure test mentioned above; at first the test piece is little deformed by increasing the twisting moment and we wait for a few minutes until the \( M-\gamma \) relation attains equilibrium. And the twisting moment and the angle of torsion are measured at that state. After this measurement the test piece is again deformed by increasing the twisting moment a little, and the above procedure is repeated successively. In this figure the curve denoted by \( \tau \gamma \) shows the relation between the shearing stress \( \tau \) and the shearing strain \( \gamma \), and this relation is obtained by applying the following relation to the \( M-\gamma \) curve.

\[
\tau \gamma = \frac{12}{\pi d_0^8} \frac{(M + \frac{\gamma}{3} \frac{dM}{d\gamma})}{d_0^8} \tag{2}
\]

Fig. 6 is the experimental result on the ultimate pressure \( p_b \) and the diameter ratio \( K_0 \) shown in Table 1, with \( p_b/2 \tau_b \) taken in ordinate, and \( K_0 \) in abscissa. In this figure the curve denoted by \( \log K_0 \) shows the \( p_b-K_0 \) relation expressed by Eq. (1). The curve \( C \) in this figure is the \( p_b-K_0 \) relation calculated by the method of Crossland and others. As shown in Fig. 6 these curves and the experimental data all coincide well in the small range of \( K_0 \), but the experimental data deviate from these curves as the diameter ratio increases, and they ride on a curve shown by a thick solid line. It is natural that the experimental data lie closer to the thin solid line. The table below shows the ultimate pressure of aluminium cylinder in experiment.

Table 2 Ultimate pressure of aluminium cylinder in experiment

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<th>Number of experimental cylinder</th>
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under the curve log $K_0$ as the stress distribution is not uniform in the thick walled cylinders. Concerning the discrepancy between the experimental data and the calculated result of Crossland and others, it seems that there are some reasons though their difference is not so large. It is an important purpose of this paper to make clear the reason why the calculated results of Crossland and others do not coincide with the experiments. For this purpose it seems necessary to accumulate the accurate experimental data on the $p_n-K_0$ relation for the wide range of the diameter ratio.

3. Experiment (2)

Every experimental cylinder is made of a solid bar of 30 mm outer diameter called commercially pure aluminium, and a solid bar for torsion is also prepared. These test pieces were annealed by heating at 350°C for an hour in an electric furnace, and they were naturally cooled in it. The dimension of the cylinder and the ultimate pressure obtained by pressure test are shown in Table 2, and the experiment is widely performed with sixteen kinds of cylinders having a wide range of diameter ratios from $K_0=1.24$ up to $K_0=12.07$. In this material $\tau_s=3.5$ kg/mm$^2$ which is calculated from the ultimate pressure $p_n$ of the thin walled cylinder in Table 2 as well as experiment (1). In the last column the ultimate pressure is shown by $p_n/2\tau_s$. And a torsion test is also carried out and the relation between shearing stress and strain is obtained as in experiment (1).

The marks of $*$ in Fig. 7 denote the experimental data of pressure test shown in Table 2, $p_n/2\tau_s$ being taken in ordinate, and $K_0$ in abscissa. The curve C in this figure represents the calculating result by the method of Crossland and others. These experimental data agree with the curve log $K_i$ and the curve C in the small range of diameter ratios, but they deviate from these curves as the diameter ratio increases, and ride on a thick solid line as in experiment (1).

4. Relation between pure shear and simple shear

In the comparisons on the ultimate pressure between the experiment and the calculation of Crossland and others, they do not coincide in the large range of $K_0$, though they coincide in the small range of $K_0$ as shown in Figs. 6 and 7. From this fact, it may be considered that the relation between pure shear and simple shear used in their calculation is not correct for the large plastic strain.

Recently Nakanishi proposed a new formula on the relation between pure shear and simple shear in plastic range. Here the outline of his formula will be described and then compared with that used in the calculation of Crossland and others.

The deformation of pure shear takes place when the state of principal stresses, $\sigma_1$, $\sigma_2$, and $\sigma_3$ is as follows;

$$\sigma_1>\sigma_2>\sigma_3, \sigma_3=0, \text{ and } \sigma_2=\frac{1}{2}\sigma_1$$

This state of stresses is similar to that of thin walled cylinder subjected to internal pressure and it can be divided into a hydrostatic tension and shearing stress shown in Fig. 8 (a). On the other hand the deformation due to simple shear takes place in a bar subjected to twisting moment and the state of shearing stress in this case is shown in Fig. 8 (b). Expressing the deformations under these shearing stresses by taking unit circles in the material, these circles deform respectively to ellipses as shown in the figures. Let the tensile
strain in pure shear be \( \varepsilon \) or \( \mu = 1 + \varepsilon \), and the shearing strain in simple shear \( \gamma \), and the deformations will be the same when the following relation holds.

\[
\gamma = \mu - 1 \quad \mu \quad \mu
\]

(2)

Let the shearing stress in simple shear be \( \tau \), at \( \gamma \) and that in pure shear \( 1/2 \sigma \), at \( \varepsilon \) or \( \mu \), and assume that, starting from the equal deformations and adding small increments \( d\mu \) and \( d\gamma \), where

\[
d\gamma = (\mu + 1/\mu) d\mu \quad d\gamma
\]

Then the works of the incremental deformations are equal, and we get

\[
\sigma \mu \frac{d\mu}{\mu} = \tau \mu \frac{d\gamma}{\mu}
\]

From above equations

\[
\tau = \frac{\sigma}{\mu + 1/\mu}
\]

(3)

where \( \gamma = \mu - 1/\mu \).

Equation (3) is presented by Nakanishi as the formula on the relation between pure shear and simple shear. When the deformation is not so large, he also proposes that this formula can be put approximately

\[
\tau = \frac{1/2}{\sigma}
\]

(4)

where \( \gamma = 2 \log \mu \).

The formula shown by Eq. (4) is equal to that induced from the incremental strain theory of plasticity and used by Crossland and others in their calculation.

Moreover, there is the following formula shown by Manning on the relation(3) between pure shear and simple shear.

\[
\tau = \frac{1}{2} \sigma
\]

where \( \gamma = \frac{1}{2} (\mu - 1/\mu) \).

This formula is also an approximation and discussion on it will be omitted in this paper.

Now supposing that Nakanishi’s formula is a correct one as the relation between pure shear and simple shear in plastic range, it seems that the discrepancy on the ultimate pressure between the experiments and calculations of Crossland and others is caused by Eq. (4), and also it is expected that this discrepancy can be eliminated if we use Eq. (3) instead of Eq. (4) in our calculation. So we will calculate the ultimate pressure using Eq. (3) and the results will be compared with the experimental data.

5. Calculation of ultimate pressure

If the stress distribution of the cylinder which is subjected to internal pressure is known, the pressure-expansion curve of the cylinder will be calculated is also known. So at first the stress distribution will be discussed.

5-1 Stress distribution

The stress distribution of the thin walled cylinder under internal pressure is considered to be uniform, and its deformation is plastically pure shear extended in the circumferential direction as much as the decrease of the thickness of wall. In this case the actual stress is influenced by double effects of the decrease of the thickness of wall and the increase of the diameter, and the relation between the actual stress \( \sigma \) and the nominal stress \( \sigma_0 \) is as follows:

\[
\sigma = \sigma_0 (1 + \varepsilon)^2
\]

(5)

where \( \varepsilon \) is tensile strain.

This relation is similar as to the shearing stress, or

\[
\tau = \tau_0 (1 + \varepsilon)^2
\]

(5)'

and internal pressure is

\[
p = 2 \pi \rho \log \rho
\]

The relation between the ultimate pressure \( p_u \) and the nominal shearing stress \( \tau_u \) at that state is given similarly as shown by Eq. (1).

In the thick walled cylinder the stress distribution is not uniform, and internal pressure can not be obtained so easily as in the thin walled cylinder. It is ascertained experimentally that the axial length of the cylinder keeps constant plastically when the thick walled cylinder is subjected to internal pressure. From this fact it is hitherto assumed that the deformation of the thick walled cylinder is pure shear like that of the thin walled cylinder. Here under this assumption the stress distribution of the cylinder and internal pressure will be calculated.

Fig. 9 Deformation of thick walled cylinder
Now, the deformation of the thick walled cylinder (initial inner radius is $r_1$, and outer radius $r_0$) subjected to internal pressure will be considered by taking a concentric thin walled cylinder (initial radius is $r$, and its thickness of wall $dr$) in it as shown in Fig. 9. Supposing that the thin walled cylinder is deformed as follows by the deformation of the thick walled cylinder; radius $r$ expands to $r'$, the thickness $dr$ contracts to $dr'$, and the strain in this state is $\varepsilon$.

Then

$$r' = r(1+\varepsilon) \quad (6')$$

Under the assumption the axial length of the cylinder does not change plastically, so the sectional area keeps constant always. Consequently the relation between $dr$ and $dr'$ is

$$dr' = \frac{dr}{1+\varepsilon} \quad (6')$$

The strain $\varepsilon$ at $r$ can be also obtained as the function of $\varepsilon_0$, where $\varepsilon_0$ is the strain at the outer surface of the thick walled cylinder, or

$$\varepsilon_0^2 - \varepsilon^2 = r_0^2 - r^2,$$

$$\varepsilon_0^2 - \varepsilon^2 = r_0^2 (1+\varepsilon_0)^2 - r^2 (1+\varepsilon)^2,$$

$$\varepsilon = \sqrt{1 + \frac{K_0^2}{K^2} (2\varepsilon_0 + \varepsilon_0^3) - 1} \quad (7)$$

where $K_0 = \frac{r_0}{r_1}$, and $K = \frac{r}{r_1}$.

This equation gives the strain distribution on the cross section of the cylinder when the strain at the outer surface of the cylinder is known.

Applying Eq. (3) to the $\tau$-$T$ relation obtained in torsion test as mentioned above, the stress-strain relation under pure shear can be known, and the strain distribution of the cylinder is given by Eq. (7). Consequently the distribution of shearing stress $\tau$ on the cross section of the cylinder can be obtained, if the strain $\varepsilon_0$ at the outer diameter is assumed. The distribution of radial stress $\sigma_r$ of the cylinder will be calculated from that of shearing stress, as the following relation between $\tau$ and $\sigma_r$ holds in the cylinder.

$$-\sigma_r = 2\int_{r_0'}^r \frac{r'}{r'} \frac{dr'}{r} \quad (8)$$

From Eqs. (5'), (6) and (6'), Eq. (8) becomes

$$-\sigma_r = 2\int_{r_0}^r \frac{\tau_0}{r} \frac{dr}{r} \quad (8')$$

The magnitude of internal pressure of the cylinder is equal to that of the radial stress at the inner surface of the cylinder, so the internal pressure can be easily obtained from the distribution of radial stress. If the calculation of internal pressure is performed for various strains at the outer surface of the cylinder, not only the pressure-expansion curve can be known, but also the ultimate pressure of the cylinder.

5.2 Pressure-expansion curve

The calculation will be a bother if we want to obtain the pressure-expansion curve by the above method. However there is a convenient method as used by Crossland and others, and it is as follows; at first we perform the calculations of the distribution curves of the strain and the radial stress on the thicker walled cylinder than that of the cylinder to be calculated, and then their distribution curves will be used for predicting the ultimate pressure of the cylinder as mentioned in the latter.

In above calculation the distribution of the radial stress can be obtained more simply when we use Eq. (8)' instead of Eq. (8), because Eq. (8) shows the relation between radial stress and shearing stress on the deformed cylinder. On the other hand Eq. (8)' shows that on the initial cylinder. Fig. 10 shows the distribution of strain $\varepsilon$, shearing stress $\tau_0/\tau_B$, and radial stress $-\sigma_r/2\tau_B$ on a mild steel cylinder in experiment (1), when the diameter ratio of the cylinder is $K_0=30$, and the strain at the outer surface is $\varepsilon_0=0.5\%$. The distribution of strain $\varepsilon$ is obtained by Eq. (7), and this has no relation to the material. The distribution of shearing stress $\tau_0/\tau_B$ is obtained by the following procedure; at first the relation between shearing stress $\tau$ and strain $\varepsilon$ under pure shear is obtained by applying Eq. (3) to the $\tau$-$\varepsilon$ relation shown in Fig. 5, and then the distribution of shearing stress $\tau_0/\tau_B$ is calculated by applying Eqs. (5)' and (7) to the $\tau$-$\varepsilon$ relation. In this case shearing stress $\tau_0$ is expressed as nominal stress. Applying Eq. (8)' to the distribution of shearing stress $\tau_0/\tau_B$ the distribution of radial stress, $-\sigma_r/2\tau_B$, can be calculated as shown in the figure, and this calculation is made graphically. The pressure-expansion curves of various
diameter ratios of the cylinders can be calculated by using the distribution curves of $\varepsilon$ and $-\sigma/2\tau_b$ shown in Fig. 10. The calculating method on the cylinder of $K_o=5$, is as follows; at first the deformation of that cylinder is assumed, for instance, the strain $\varepsilon=1.5$ percent at the outer surface. The value of $K$ where strain is 1.5 percent in this figure is $K=17.0$ as shown by the point A. As the diameter ratio of this cylinder is $K_o=5$, the inner surface of this cylinder is shown by the point B of $K=17/5=3.4$. The stress distribution of the cylinder at this state is given by the curve $A' C'B'$. The distribution of radial stress of the cylinder is also given by the curve $A'C'B''$. However the radial stress at the outer surface is zero, so the radial stress in this case is not for the abscissa BA of the initial cylinder, but for the horizontal line $B''A''$ through the point $A''$. Consequently the internal pressure is given by the corresponding value to the length $B''B'''$ in this figure, and this is the same as the difference of the radial stresses at the points, A and B, in the original thick walled cylinder. This value of radial stress gives the internal pressure of the cylinder of $K_o=5$ when it is deformed $\varepsilon=1.5$ percent at the outer surface. Repeating above procedure for various values of strain at the outer surface, and obtaining internal pressure, the pressure-expansion curve can be obtained, and the ultimate pressure is also known from this curve. Similarly the pressure-expansion curves and the ultimate pressures are calculated on the cylinders of various diameter ratios $K_o$.

6. Comparison between calculation and experiment

The calculation on the pressure-expansion curve and the ultimate pressure will be compared with the experimental data.

In Fig. 11, a comparison is made on the pressure-expansion curve between the calculation and the experimental data shown in Fig. 3, and this comparison indicates they coincide very well. The $p_b/2\tau_b-K_o$ relation denoted by the thick solid line in Fig. 6 is that obtained from the calculated pressure-expansion curves for various $K_o$ of the cylinders. In this case the calculated results also coincide very well with the experimental data as we expected.

The same comparison on the ultimate pressure of aluminium cylinder as in experiment (2) is also made. The $p_b/2\tau_b-K_o$ relation denoted by the thick solid line in Fig. 7 is that obtained from the calculation as in the case of mild steel cylinder. The calculated result and the experimental data also coincide very well as we expected.

For reference, when Eq. (4) is used in the calculation as the relation between pure shear and simple shear, the distributions of $\tau_b/\tau_b$ and $-\sigma_b/2\tau_b$ on mild steel cylinder are shown by two thin solid lines in Fig. 10. These two distribution curves do not coincide with that obtained by our calculation in the large range of strain, though they coincide in the small range of strain. This is the reason why the ultimate pressure calculated by the method of Crossland and others does not coincide with the experimental data in the large range of $K_o$.

Marin and others discussed the ultimate pressure which was predicted by twenty odd kinds of formulas proposed hitherto. In their discussion they indicate that the calculated result of Crossland and others coincides with the experimental data better than those calculated by the other formulas, but there exists a few percent errors in it. And they also obtain experimentally the $p_b-K_o$ relations on the cylinder of a few kinds of material. One of them is that on the cylinder of aluminium alloy (6061-T4). As this is, of course, a ductile material, it seems that their $p_b-K_o$ relation of aluminium alloy coincides with our experimental data shown in Fig. 7. The experimental data of Marin and others are shown by the marks of • in
Fig. 7. Their experimental data coincide very well with our experimental data as we expected.

Previously we performed an experiment on brass cylinders having the diameter ratio $K_0 < 2.2$, and considered that the $p_d - K_0$ relation of these cylinders was given by Eq. (1) at that time. In this case the thickness of wall of the cylinder was rather thin, so its consideration was approximately correct, but generally speaking it is not correct as mentioned above.

7. Conclusions

Concerning the ultimate pressure for various diameter ratios of the cylinder, the calculating results coincide with the experimental data when Nakanishi’s formula is used in the calculation. From this fact it may be concluded that his relation expressed by Eq. (3) is correct as the relation between pure shear and simple shear in plastic range. If we use Eq. (8) instead of Eq. (8) in the calculation of the radial stress, the calculation can be performed simply and it is convenient actually to predict the pressure-expansion curve, or the ultimate pressure.

References