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Numerical Solutions of Two-Dimensional Elastic, Plastic Problems by Conformal Mapping and Finite-Difference Method*

(1st Report, Elastic Solution of a Strip with Circular
Notches under Tension)

By Minoru HAMADA** and Hiroshi KITAGAWA***

In this research, a numerical method for treating two-dimensional elastic, plastic problems is presented, in which the region of any given body is mapped into a lattice region by means of conformal mapping, and then the finite-difference method is applied.

In this first report, the elastic problem of a circularly notched strip under tension is treated. To this problem, H. Politsky has once applied a method similar to the method in this study. Therefore, the authors try to improve his method in accuracy and extent of its applicability.

The calculated results coincide well with Kikukawa's experimental data in maximum stress values as well as in the stress distribution, and also with Wolf's approximate solutions with regard to the stress gradient at the notch bottom.

1. Introduction

The finite-difference method has been widely used in solving many two-dimensional elasticity problems. In most works using this method, which have been carried on until now, the finite-difference nets were made up of straight lines based on such co-ordinate systems as rectangular or oblique Cartesian, etc., then it was difficult for the real region consisting of curved boundary to be replaced with the lattice region precisely. This defect of the finite-difference method is serious, because the

maximum stress or the stress coming into question appears at the place where the accuracy of the finite-difference approximation is worst for the reason above mentioned.

In this paper, in order to remove this defect, the curvilinear co-ordinate, which is fit for the real boundary, is obtained first of all, and then the finite-difference net is formed of this co-ordinate. In other words, any given region of problem is mapped conformally into a simple shaped region which is easily represented by the nets and the finite-difference approximation is applied to the differential equation in the mapped plane.

This idea of solution was first suggested by H. Poritsky et al.⁽³⁾, but thereafter no one has developed their method. In this paper, considering the

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** Professor, Faculty of Engineering, Osaka University, Miyakojima-ku Osaka.

*** Graduate Student, Division for Research of Engineering, Graduate School, Osaka University.

high speed numerical calculation by digital computer, the authors improve his method in accuracy and extent of its applicability, and they establish a method to solve the elastic problem of a strip with arbitrarily shaped notches under tension.

Concerning the stress concentration factor α of a symmetrically notched strip, Neuber's approximate formula⁽⁴⁾ has been widely used to estimate its value for arbitrary notch depth. But as the result of an exact solution for semicircular notch⁽¹⁾ and some experimental investigations for U-shaped notch^{(2) (6) (9)} which have been carried out subsequently, a part of assumptions of Neuber's theory have become doubtful⁽⁶⁾, and moreover there are considerable differences between the individual experimental results. Therefore, the authors attempt in this paper to present some evidence to settle the above problem.

2. Basic equations

The plane elasticity problem is mathematically equivalent to determining the Airy's stress function χ which satisfies in the region of the given body the partial differential equation,

$$\nabla^4 \chi = 0, \quad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \quad (1)$$

and also satisfies the prescribed values of χ , and $\partial\chi/\partial n$ i.e. the normal derivatives of χ along the boundary of the region. In terms of χ , the stress components are given by,

$$\left. \begin{aligned} \sigma_x &= \partial^2 \chi / \partial y^2, & \sigma_y &= \partial^2 \chi / \partial x^2, \\ \tau_{xy} &= -\partial^2 \chi / \partial x \partial y \end{aligned} \right\} \quad (2)$$

Assuming that the region is simply connected, the boundary values of χ and $\partial\chi/\partial n$ are determined by

$$\left. \begin{aligned} \partial\chi/\partial x &= -\int Y_n ds, & \partial\chi/\partial y &= \int X_n ds, \\ \chi &= x \partial\chi/\partial x + y \partial\chi/\partial y + \int (x Y_n - y X_n) ds \end{aligned} \right\} \quad (3)$$

where X_n , Y_n are the components of the boundary traction in x , y directions, respectively (Fig. 1).

Now, if (x, y) plane is mapped into (u, v) plane by the analytic function,

Fig. 1 Boundary traction and normal, tangential directions

$$W = u + iv = F(z), \quad z = x + iy \quad (4)$$

the previous equations are rewritten in the following forms with the independent variables (u, v) ;

$$\left. \begin{aligned} \nabla_{uv}^2 (h^2 \nabla_{uv}^2 \chi) &= 0 \\ \nabla_{uv}^2 &= \partial^2/\partial u^2 + \partial^2/\partial v^2, \quad h^2 = (\partial u/\partial x)^2 + (\partial u/\partial y)^2 \end{aligned} \right\} \quad (5)$$

or

$$\nabla_{uv}^2 \chi = \zeta/h^2, \quad \nabla_{uv}^2 \zeta = 0 \quad (6)$$

and

$$\left. \begin{aligned} \sigma_u &= h^2 \partial^2 \chi / \partial v^2 + h \{ (\partial \chi / \partial v) (\partial h / \partial v) \\ &\quad - (\partial \chi / \partial u) (\partial h / \partial u) \} \\ \sigma_v &= h^2 \partial^2 \chi / \partial u^2 + h \{ (\partial \chi / \partial u) (\partial h / \partial u) \\ &\quad - (\partial \chi / \partial v) (\partial h / \partial v) \} \\ \tau_{uv} &= -h^2 \partial^2 \chi / \partial u \partial v - h \{ (\partial \chi / \partial u) (\partial h / \partial v) \\ &\quad - (\partial \chi / \partial v) (\partial h / \partial u) \} \end{aligned} \right\} \quad (7)$$

The stress components σ_x , σ_y and τ_{xy} are given by

$$\left. \begin{aligned} \sigma_x &= \sigma_u \cos^2 \theta + \sigma_v \sin^2 \theta - 2\tau_{uv} \cos \theta \sin \theta \\ \sigma_y &= \sigma_u \sin^2 \theta + \sigma_v \cos^2 \theta + 2\tau_{uv} \cos \theta \sin \theta \\ \tau_{xy} &= (\sigma_v - \sigma_u) \cos \theta \sin \theta + \tau_{uv} (\cos^2 \theta - \sin^2 \theta) \end{aligned} \right\} \quad (8)$$

where, $\sin \theta = h \partial y / \partial u$, $\cos \theta = h \partial y / \partial v$ (9)

3. Method of solution

3.1 Numerical calculation of mapping function

In this problem, the shape of the body is a strip with variable width as shown in Fig. 2 (a), then it is convenient to map the given region into a rectangle [Fig. 2 (b)]. This conformal mapping is achieved by determining the function satisfying the following equations;

$$\left. \begin{aligned} \nabla^2 u(x, y) &= 0, & \nabla^2 v(x, y) &= 0, \\ \partial u / \partial x &= \partial v / \partial y, & \partial v / \partial x &= -\partial u / \partial y \\ &\text{in the region,} \\ v &= v_a, \text{ on AB, } & v &= v_b, \text{ on CD,} \\ u &= u_c, \text{ on BC, } & u &= u_d, \text{ on AD} \end{aligned} \right\} \quad (10)$$

However, in solving the above equations numerically, the disadvantage mentioned previously is involved. To avoid this, Eqs.(10) are replaced by,

Fig. 2

$$\left. \begin{aligned} \nabla_{uv}^2 x(u, v) &= 0, \quad \nabla_{uv}^2 y(u, v) = 0, \\ \partial x / \partial u &= \partial y / \partial v, \quad \partial y / \partial u = -\partial x / \partial v, \\ &\text{in the region,} \\ f_1(x, y) &= 0, \text{ on } A^*B^*, \quad f_2(x, y) = 0, \text{ on } C^*D^*, \\ \partial y / \partial u &= -\partial x / \partial v = 0, \text{ on } B^*C^* \text{ and } D^*A^* \end{aligned} \right\} \quad \dots\dots\dots(11)$$

where $f_1(x, y) = 0$, $f_2(x, y) = 0$ are the functions which represent the geometric shapes of boundaries AB, CD, respectively. Eq.(11) can be solved easily in mapped plane (u, v) by the finite-difference method. The procedure of numerical calculation is summarized as follows:

(1) Assume the boundary values of y on A^*B^* and C^*D^* properly.

(2) Calculate the harmonic function y satisfying $\nabla_{uv}^2 y = 0$ in the region and the assumed values on the boundary by the finite-difference method with the nets formed in (u, v) plane.

(3) Calculate the boundary values of x by means of numerical integration of the following equation,

$$x = - \int (\partial y / \partial u) dv + x_0, \text{ or } x = \int (\partial y / \partial v) du + x_0 \quad \dots\dots\dots(12)$$

These are the integral forms of Chauchy-Riemann equations $\partial x / \partial u = \partial y / \partial v$ or $\partial x / \partial v = -\partial y / \partial u$, respectively.

(4) Correct the boundary values of y from the newly determined set of x -values.

(5) Repeat steps (1) to (4) until boundary values of y converge.

To be concrete, steps (2) and (3) are carried on as follows;

Step (2) As the approximate finite-difference equation for Laplace equation, K-type operator is used. Namely, concerning the lattice region shown in Fig. 3, $\nabla_{uv}^2 y = 0$ is replaced by

$$\left. \begin{aligned} 4S_1 + S_2 - 20y_0 &= 0 \\ S_1 &= y_1 + y_2 + y_3 + y_4, \quad S_2 = y_5 + y_6 + y_7 + y_8 \end{aligned} \right\} \quad \dots\dots\dots(13)$$

Numerical computations of Eqs.(13) can be easily carried out by such procedure as SLOR method⁽⁸⁾.

Step (3) As Eqs.(12) must be integrated

numerically for each interval of lattice, the following equation which is obtained from the modified Bessel interpolation formula is used. That is, the integral between the points 0 and 1 in Fig. 4,

$$I = \int_0^1 F(t) dt \text{ is approximately replaced by}$$

$$I = \frac{\delta}{1440} \{ 11(F_{-2} + F_3) - 93(F_{-1} + F_2) + 802(F_0 + F_1) \} \quad \dots\dots\dots(14)$$

The convergence of the above procedure is dependent on the tangent of boundary dy/dx , and from experience, it is certain to converge for about $|dy/dx| < 2$.

This method to calculate the mapping function is conceived by the authors in the light of the reference (7).

3.2 Calculation of the stress function

Concerning the square lattice shown in Fig. 3, Eqs.(6) are replaced approximately by

$$\left. \begin{aligned} \chi_1 + \chi_2 + \chi_3 + \chi_4 - 4\chi_0 &= \delta^2 \zeta_0 / h_0^2 \\ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 - 4\zeta_0 &= 0 \end{aligned} \right\} \quad \dots\dots\dots(15)$$

To solve Eqs.(15), Wolf's method is applied. This method is as follows;

(1) Assume the values of χ in the region as reasonably as possible considering the boundary conditions.

(2) By means of Eqs.(15), estimate the boundary values of ζ from the newly obtained χ -values.

(3) Solve Eq.(15)₂ numerically for the region.

(4) Solve Eq.(15)₁ in the same manner as the above.

(5) Repeat steps (2) to (4) until both the values of χ and ζ converge.

Among the above five steps, (2) must be carried out most carefully for good convergence. The equation to estimate the boundary values of ζ is obtained in the following manner. Taylor expansion about the boundary point 0 in Fig. 5 gives,

$$\begin{aligned} \chi_2 &= \chi_0 + \delta \frac{\partial \chi}{\partial v} \Big|_0 + \frac{\delta^2}{2!} \frac{\partial^2 \chi}{\partial v^2} \Big|_0 \\ &\quad + \frac{\delta^3}{3!} \frac{\partial^3 \chi}{\partial v^3} \Big|_0 + \frac{\delta^4}{4!} \frac{\partial^4 \chi}{\partial v^4} \Big|_0 + O(\delta^5) \end{aligned}$$

Fig. 3 Finite-difference net

Fig. 4 Numerical integration between points 0 and 1

Fig. 5 Net near boundary

$$\chi_4 = \chi_0 - \delta \left. \frac{\partial \chi}{\partial v} \right|_0 + \frac{\delta^2}{2!} \left. \frac{\partial^2 \chi}{\partial v^2} \right|_0 - \frac{\delta^3}{3!} \left. \frac{\partial^3 \chi}{\partial v^3} \right|_0 + \frac{\delta^4}{4!} \left. \frac{\partial^4 \chi}{\partial v^4} \right|_0 - O(\delta^5)$$

Subtract each side of the above equations, then

$$\chi_2 = \chi_4 + 2\delta \left. \frac{\partial \chi}{\partial v} \right|_0 + \frac{\delta^3}{3} \left. \frac{\partial^3 \chi}{\partial v^3} \right|_0 + O(\delta^5)$$

Now, from the first of Eqs. (6),

$$\partial^2 \chi / \partial v^2 = \zeta / h^2 - \partial^2 \chi / \partial u^2$$

Differentiation gives

$$\frac{\partial^3 \chi}{\partial v^3} = \frac{\partial}{\partial v} \left(\frac{\zeta}{h^2} \right) - \frac{\partial^2}{\partial u^2} \left(\frac{\partial \chi}{\partial v} \right)$$

From the above expressions, the χ -value on the fictitious nodal point 2 in Fig. 5 is evaluated as,

$$\chi_2^{(n)} = \chi_4^{(n)} + 2\delta \left. \frac{\partial \chi}{\partial v} \right|_0 - \frac{\delta^3}{3} \left. \frac{\partial^2}{\partial u^2} \left(\frac{\partial \chi}{\partial v} \right) \right|_0 + \frac{\delta^3}{3} \left. \frac{\partial}{\partial v} \left(\frac{\zeta^{(n)}}{h^2} \right) \right|_0 \quad \dots\dots\dots(16)$$

Substitute Eq.(16) into the first of Eq.(15), then ζ on the boundary is given. Though $\partial \chi / \partial v$ is prescribed along the boundary, for χ_4 and ζ in Eq.(16) the values obtained at that step ought to be used.

In order to improve the convergence of the above procedures (1) to (5), the following two processes are applied.

(a) When the n 'th iteration has been finished, the improved boundary values of ζ , $\zeta_b^{(n+1)}$, are calculated by the next equation,

$$\zeta_b^{(n+1)} = \zeta_b^{(n)} + \omega \{ \hat{\zeta}_b^{(n+1)} - \zeta_b^{(n)} \} \quad \dots\dots\dots(17)$$

where $\hat{\zeta}_b^{(n+1)}$ is the estimate made from the first of Eqs.(15) and (16); ω is the parameter which controls the divergence of ζ_b and generally $0 < \omega < 1$; ω is taken smaller as the fluctuation of ζ_b is severe at each time of iteration.

(b) On the other hand, $\zeta_b^{(n+1)}$ can be also estimated by the following equations.

$$\left. \begin{aligned} \zeta_b^{(n+1)} &= \zeta_b^{(n-2)} - (\Delta_{n-2} \zeta)^2 / \Delta_{n-1}^2 \zeta \\ \Delta_{n-2} \zeta &= \zeta_b^{(n-1)} - \zeta_b^{(n-2)} \\ \Delta_{n-1}^2 \zeta &= \zeta_b^{(n)} - 2\zeta_b^{(n-1)} + \zeta_b^{(n-2)} \end{aligned} \right\} \quad \dots\dots\dots(18)$$

Eq.(17) is applied each time and Eqs.(18) is every four times of iteration.

From χ -values, stress components are given by

$$\left. \begin{aligned} \sigma_{u,0} &= \{ h_0^2 (\chi_2 + \chi_4 - 2\chi_0) + h_0 (\chi_2 - \chi_4) (h_2 - h_4) / 4 \} \\ &\quad - h_0 (\chi_1 - \chi_3) (h_1 - h_3) / 4 \} / \delta^2 \\ \sigma_{v,0} &= \{ h_0^2 (\chi_1 + \chi_3 - 2\chi_0) + h_0 (\chi_1 - \chi_3) (h_1 - h_3) / 4 \} \\ &\quad - h_0 (\chi_2 - \chi_4) (h_2 - h_4) / 4 \} / \delta^2 \\ \tau_{uv,0} &= - \{ h_0^2 (\chi_5 - \chi_6 + \chi_7 - \chi_8) + h_0 (\chi_1 - \chi_3) \\ &\quad \times (h_2 - h_4) + h_0 (\chi_2 - \chi_4) (h_1 - h_3) \} / 4\delta^2 \end{aligned} \right\} \quad \dots\dots\dots(19)$$

4. Numerical results

This numerical method is applied to the problems of tension of strips with symmetric circular notches. Because of symmetry, only a quarter of

the whole region may be considered. Assume that the half width of the strip is unit and the strip is subjected to the uniform stress $\sigma_x = 1$ at place far away from the notch. The dimensions and the shape of notch are given by the radius of the notch, r and the distance between its center and the edge of the strip, e [Fig. 6 (a)].

4.1 On the calculation of mapping function

Under the condition that,

$$\left. \begin{aligned} v=1 \text{ on BCD, } v=0 \text{ on OA,} \\ u=0 \text{ on DO, } u=3 \text{ on AB,} \end{aligned} \right\} \quad \dots\dots\dots(20)$$

in Fig. 6 (a)

map the region OABCD in (x, y) plane into the rectangle O*A*B*C*D* in (u, v) plane [Fig. 6(b)]. If the corner C in Fig. 6(a) is sharp, C is the singular point of the mapping function. In order to avoid it, this corner is made round with a small circular arc (radius ρr).

The boundary conditions to calculate the mapping function are,

$$\left. \begin{aligned} \text{on B*C*D*}, \quad & y = 1 + e - \sqrt{r^2 - x^2}, \quad 0 \leq x < x_1 \\ & y = 1 - \rho r + \sqrt{(\rho r)^2 - (x - x_2)^2}, \quad x_1 \leq x < x_2 \\ & y = 1, \quad x_2 \leq x \\ & x_1 = r \sqrt{1 - (e/r + \rho)^2 / (1 + \rho)^2} \\ & \quad x_2 = (1 + \rho)x_1 \end{aligned} \right\} \quad \dots\dots\dots(21)$$

on A*O*, $y=0$
on B*A*, $\partial y / \partial u = 0$
on O*D*, $\partial y / \partial u = 0$

As stated before, the procedure to calculate the mapping function will probably diverge in the case of $|dy/dx| > 2$, then it is impossible to apply this method to the problem with the deeper than semi-circular notch, even if the corner C is made round.

4.2 On the calculation of the stress function

In this case, the boundary condition to determine χ is, in the (x, y) plane,

$$\left. \begin{aligned} \partial\chi/\partial x=0, \partial\chi/\partial y=y, \chi=y^2/2, & \text{ on AB,} \\ \partial\chi/\partial x=0, \partial\chi/\partial y=1, \chi=y-\frac{1}{2}, & \text{ on BCD,} \\ \partial\chi/\partial x=0, & \text{ on DO, and } \partial\chi/\partial y=0, & \text{ on AO} \end{aligned} \right\} \quad \dots\dots(22)$$

$$\left. \begin{aligned} \text{and transforming into the forms in } (u, v) \text{ plane,} \\ \partial\chi/\partial u=0, \partial\chi/\partial v=v, \chi=v^2/2, & \text{ on A*B*,} \\ \partial\chi/\partial v=\partial y/\partial v, \chi=y-\frac{1}{2}, & \text{ on B*C*D*,} \\ \partial\chi/\partial u=0, & \text{ on D*O*, and } \partial\chi/\partial v=0, & \text{ on A*O*} \end{aligned} \right\} \quad \dots\dots(23)$$

Also, the conditions of ζ in (u, v) plane are,

$$\left. \begin{aligned} \zeta=\sigma_u+\sigma_v=1, & \text{ on A*B*,} \\ \text{Eqs. (15)}_1 \text{ and (16),} & \text{ on B*C*D*,} \\ \partial\zeta/\partial u=0, & \text{ on D*O*, and } \partial\zeta/\partial v=0, & \text{ on A*O*} \end{aligned} \right\} \quad \dots\dots(24)$$

The maximum stress appears at the bottom of the notch D in Fig. 6 (a), and its magnitude is expressed by ζ_D . Then the stress concentration factor α is given by

$$\alpha = \zeta_D(1+e-r) \quad \dots\dots(25)$$

4.3 Numerical results

The computations are carried out for the cases of six kinds of notches characterized by the parameters a/r and t/b .

Fig. 7 presents the stress concentration factors and Fig. 8 the distribution of stress components σ_x, σ_y in the section $x=0$. The above results are obtained for the lattice-region with net-interval $\delta=0.1$.

5. Discussion

(1) The results in Figs. 7 and 8 are predicted to involve the errors which are estimated at about 1% in view of the accuracy of the mapping

Fig. 7 Stress concentration factor

function, the inherent errors of the finite-difference approximation and the lattice interval.

(2) In comparison with the exact solution for the semi-circular notch⁽¹⁾, the finite-difference results agree with them with a difference not exceeding 1%. In other cases shown in Fig. 7, the

Fig. 8 Stress distributions on $x=0$ section

accuracy is expected better, because it is supposed that the error of this method increases as the depth of the notch increases.

(3) As shown in Fig. 8, the results obtained by the finite-difference method coincide well with Kikukawa's experimental data⁽⁶⁾ in maximum stresses at the notch bottom as well as the stress distributions in the section $x=0$. Recently Kikukawa's data were supported by Flynn⁽⁹⁾ with photo-elastic experiment.

(4) The gradient of the stress σ_x at the bottom of the notch,

$$\theta = r \frac{1}{\sigma_{\max}} \frac{\partial \sigma_x}{\partial y} \dots \dots \dots (26)$$

is given in Table 1 together with Wolf's approximate solutions⁽⁵⁾. The difference between both

Fig. 9 Notched specimen

Table 1 Gradient of stress σ_x at notch bottom, θ

a/r	t/b	Finite-difference	Wolf ⁽⁵⁾
2.0	0.21	2.25	2.29
	0.28	2.27	2.23
	0.333	2.20	2.20
1.5	0.143	2.34	2.29
	0.25	2.26	2.28
	0.4	2.16	2.19
1.0	0.2	2.34	2.36
	0.35	2.23	2.23
	0.5	2.17	2.17

Table 2 Radii of arcs rounding corner C

a/r	t/b	ρ
2.0	0.21	0.25
	0.28	0.5
	0.333	0.5
1.5	0.143	0.25
	0.25	0.25
	0.4	0.5
1.0	0.2	0.25
	0.35	0.3
	0.5	0.5

Table 3 Relation between α and ρ (experimental data)

ρ	α	ρ	α
0	1.60	0.6	1.60
0.2	1.60	0.8	1.58
0.4	1.59	1.0	1.59

results is within 5%. It seems that especially in the case of $a/r=2.0$, agreement is poor. It may be caused by the defect of our solution, which is interpreted to be, owing to relative coarseness of finite-difference net to notch dimension, wrong approximation of given shape of region near the notch by lattice.

(5) The radii of the arcs with which the corner C in Fig. 6 (a) is made round are tabulated in Table 2. In order to investigate the influence of this treatment on the stress distribution, tensile tests of notched specimens made of steel (Fig. 9) are carried out, and by means of wire-strain gage bonded at the bottom of the notch, the maximum strain is measured when ρ is changed from 0 to 1. According to the experimental results shown in Table 3, it is concluded that as long as ρ is less than 1, the treatment of rounding off the corner C has little influence on the stress distribution.

6. Conclusion

A method for treating two-dimensional elasticity problems is demonstrated and in the present paper stress concentration factors of symmetrically notched strip under tension are numerically estimated by this method.

The calculated results support the Kikukawa's experimental data and prove that Neuber's solutions give somewhat unsafe values, and also fairly well coincide with Wolf's approximate solutions about stress gradient at notch bottom.

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