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Deformation and Stress in a Cylindrical Shell under Concentrated Loading*

(1st Report, Radial Loading)

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As an example of singularity in shells, a thin cylindrical shell subjected to a radial concentrated load with both ends supported is discussed. For a cylindrical shell under a concentrated load, not only the singularity itself but also the whole deformation is important.

Based on K. Mizoguchi's equation, the solutions are obtained by applying double Fourier transforms. The series, however, are not suitable for explaining the singularity at the point of application of the load. Therefore we transform them to obtain a set of series which converges uniformly and a closed form representing the singularity.

The study shows such interesting results that in the vicinity of the load in the circumferential direction the cylinder deforms outwards and gives rise to tensile stress on its surface. These results may be applied to such a machine as a rotary kiln which will possibly suffer fatigue failure by alternated stress due to such deformations.

1. Introduction

As an example of the singularity in shells, a thin cylindrical shell subjected to a radial concentrated load with both ends simply supported is discussed, using the fundamental differential equation on the cylindrical shell introduced by K. Mizoguchi. For a cylindrical shell under a concentrated load, not only the singularity itself but also the whole deformation is important and the shallow shell theory usually used is not sufficient for this purpose. Based on Mizoguchi's equation, the solutions are obtained by applying double Fourier transforms. The series, however, are not suitable for explaining

the singularity at the point of application of the load and the convergency of the series representing the stress resultants and couples is not good in the vicinity of the load. Therefore we transform them into a set of series which converges uniformly and closed terms representing the singularity. The study shows such interesting results that in the vicinity of the point of application of loading in the circumferential direction the cylinder deforms outwards and gives rise to the tensile stress on its surface, as confirmed by experiments. These results may be applied to such a machine as a rotary kiln which will possibly suffer fatigue failure by alternated

Fig. 1 Cylindrical shell subjected to a concentrated radial load

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Fig. 2 Displacements, stress resultants and couples on shell element

stress due to such deformations.

Our procedure will also be applied to problems on singularity of another kind of concentrated loading and moment.

2. Analysis

A circular cylindrical shell subjected to a radial concentrated load with both ends simply supported is shown in Fig. 1 and the behaviour of deforma-

$$A \frac{\partial^8 w}{\partial x^8} + \frac{4B}{a^2} \frac{\partial^6 w}{\partial x^6 \partial \theta^2} + \frac{6C}{a^4} \frac{\partial^6 w}{\partial x^4 \partial \theta^4} + \frac{8-2\nu^2}{a^4} \frac{\partial^6 w}{\partial x^4 \partial \theta^2} + \frac{6L}{a^4} \frac{\partial^4 w}{\partial x^4} + \frac{4}{a^6} \frac{\partial^2}{\partial x^2} \left[\frac{\partial^6 w}{\partial \theta^6} + 2 \frac{\partial^4 w}{\partial \theta^4} + \frac{\partial^2 w}{\partial \theta^2} \right] + \frac{1}{a^8} \frac{\partial^2}{\partial \theta^2} \left[\frac{\partial^6 w}{\partial \theta^6} + 2 \frac{\partial^4 w}{\partial \theta^4} + \frac{\partial^2 w}{\partial \theta^2} \right] = \frac{1}{D} \left[A \frac{\partial^4 p}{\partial x^4} + \frac{2U}{a^2} \frac{\partial^4 p}{\partial x^2 \partial \theta^2} + \frac{B}{a^4} \frac{\partial^4 p}{\partial \theta^4} \right] \quad (1)$$

where w : radial deflection, a : radius of cylinder, h : thickness of cylinder, p : surface load in the radial direction, E : Young's modulus of elasticity, ν : Poisson's ratio, $D = Eh^3/12(1-\nu^2)$: flexural rigidity of cylinder

$$\left. \begin{aligned} A &= 1 + \frac{1}{3} \left(\frac{h}{a} \right)^2, & B &= 1 + \frac{1}{12} \left(\frac{h}{a} \right)^2, & C &= 1 + \frac{1-\nu^2}{72} \left(\frac{h}{a} \right)^2 \\ L &= 2(1-\nu^2) \left(\frac{a}{h} \right)^2 A, & U &= 1 + \frac{1+(1-\nu)^2}{12(1-\nu)} \left(\frac{h}{a} \right)^2 \end{aligned} \right\} \quad (2)$$

As already stated in the previous paper⁽²⁾, it is found that the end plates shown in Fig. 1 behave as if they were thin, even though they are considerably thick compared with the thickness of cylinder. Applying also this assumption of being thin to the present problem, the end conditions can be written as follows:

At $x = \pm l/2$

$$w=0, \quad v=0, \quad T_x=0, \quad M_x=0 \quad (3)$$

These conditions should be satisfied along the whole circumference.

Now, if one imagines an infinitely long cylinder extending in the positive and the negative x -direction and loaded with a series of equidistant loads acting opposite alternately along the generator ($\theta=0^\circ$), it is found that the conditions in Eq. (3) are all satisfied at $x = \pm l/2$. Based on the above idea, we apply double Fourier transform to both sides in Eq. (1).

$$\bar{w}(m, n) = \int_{-l}^l \int_{-\pi}^{\pi} w \cos \frac{m\pi x}{l} \cos n\theta dx d\theta \quad (4)$$

Considering p as δ -function, we have

$$\int_{-l}^l \int_{-\pi}^{\pi} p \cos \frac{m\pi x}{l} \cos n\theta dx d\theta = \frac{P}{2a} (-1)^m - \frac{P}{a} + \frac{P}{2a} (-1)^m = \begin{cases} 0 & (m=0, 2, 4, \dots) \\ -\frac{2P}{a} & (m=1, 3, 5, \dots) \end{cases} \quad (5)$$

Applying the transforms of Eqs. (4) and (5) to Eq. (1), we obtain

$$\bar{w}(m, n) = -\frac{2a^3 P}{D} \frac{A\mu^4 + 2U\mu^2 n^2 + Bn^4}{H} \quad (6)$$

where $\mu = m\pi a/l$ (7)

$$H = A\mu^8 + 4B\mu^6 n^2 + 6C\mu^4 n^4 - (8-2\nu^2)\mu^4 n^2 + 6L\mu^4 + 4\mu^2 n^2 (n^2 - 1)^2 + n^4 (n^2 - 1)^2 \quad (8)$$

The inversion formula gives the expression of w as follows:

$$w = -\frac{a^3 P}{\pi l D} \sum_{m=1}^{\infty} \frac{\cos \frac{m\pi x}{l}}{\mu^4 + 12(1-\nu^2) \left(\frac{a}{h} \right)^2} - \frac{2a^3 P}{\pi l D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A\mu^4 + 2U\mu^2 n^2 + Bn^4}{H} \cos \frac{m\pi x}{l} \cos n\theta \quad (9)$$

$m=1, 3, 5, 7, \dots, \quad n=1, 2, 3, 4, \dots$

The axial and circumferential displacement u and v are obtained from $w \sim u$ and $w \sim v$ relations⁽³⁾ which do not contain p .

$$u = \frac{a^3 P}{\pi l D} \sum_{m=1}^{\infty} \frac{\nu \sin \frac{m\pi x}{l}}{\mu \left[\mu^4 + 12(1-\nu^2) \left(\frac{a}{h} \right)^2 \right]} + \frac{2a^3 P}{\pi l D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu}{H} \left[\nu A \mu^2 - \frac{(1+\nu)(2-\nu)}{12(1-\nu)} \left(\frac{h}{a} \right)^2 \mu^2 n^2 \right]$$

tions and stresses caused in the cylinder is discussed. The stress resultants, the stress couples, the surface loads per unit area and the displacements in the small element cut out from a cylindrical shell are shown in Fig. 2.

The fundamental differential equation on the cylindrical shell has already been introduced by K. Mizoguchi⁽¹⁾ and is given in the following expression.

Fig. 3 Imagined infinitely long cylinder

$$-\frac{(1+\nu)}{12(1-\nu)}\left(\frac{h}{a}\right)^2 n^4 - V n^2 \left] \sin \frac{m\pi x}{l} \cos n\theta \quad V = 1 - \frac{\nu}{6(1-\nu)}\left(\frac{h}{a}\right)^2 \dots\dots\dots(10)$$

$$v = \frac{2a^3 P}{\pi l D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n}{H} \left[\frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a}\right)^2 \mu^4 + \frac{4-3\nu+\nu^2}{12(1-\nu)} \left(\frac{h}{a}\right)^2 \mu^2 n^2 \right. \\ \left. + (2+\nu) \mu^2 + \frac{1}{12} \left(\frac{h}{a}\right)^2 n^4 + n^2 \right] \cos \frac{m\pi x}{l} \sin n\theta \quad m=1,3,5,7,\dots, \quad n=1,2,3,4, \dots\dots\dots(11)$$

Substituting Eqs. (9), (10) and (11) into the equations of stress resultants and couples expressed by the displacements u , v and $w^{(4)}$, we obtain

$$T_x = -\frac{24P}{\pi l} \left(\frac{a}{h}\right)^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{H} \left[\frac{(2-\nu)}{12} \left(\frac{h}{a}\right)^2 \mu^4 n^2 + (1-\nu^2) I \mu^2 n^2 \right. \\ \left. + \frac{(1-\nu)^2}{12} \left(\frac{h}{a}\right)^2 \mu^2 n^4 - \frac{\nu}{12} \left(\frac{h}{a}\right)^2 n^4 (n^2-1) \right] \cos \frac{m\pi x}{l} \cos n\theta \dots\dots\dots(12)$$

$$T_\theta = -\frac{12P}{\pi l} \left(\frac{a}{h}\right)^2 \sum_{m=1}^{\infty} \frac{(1-\nu^2) \cos \frac{m\pi x}{l}}{\mu^4 + 12(1-\nu^2) \left(\frac{a}{h}\right)^2} - \frac{24P}{\pi l} \left(\frac{a}{h}\right)^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{H} \left[A(1-\nu^2) \mu^4 - \frac{4-\nu^2}{12} \left(\frac{h}{a}\right)^2 \mu^4 n^2 \right. \\ \left. - \frac{1}{3} \left(\frac{h}{a}\right)^2 \mu^2 n^2 (n^2-1) - \frac{1}{12} \left(\frac{h}{a}\right)^2 n^4 (n^2-1) \right] \cos \frac{m\pi x}{l} \cos n\theta \dots\dots\dots(13)$$

$$S = -\frac{12(1-\nu)P}{\pi l} \left(\frac{a}{h}\right)^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu n}{H} \left[2(1+\nu) I \mu^2 + \frac{(1-\nu)}{6} \left(\frac{h}{a}\right)^2 \mu^2 n^2 \right. \\ \left. - \frac{\nu}{6(1-\nu)} \left(\frac{h}{a}\right)^2 n^2 (n^2-1) + \frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a}\right)^2 \mu^4 \right] \sin \frac{m\pi x}{l} \sin n\theta \quad I = 1 + \frac{\nu}{6(1+\nu)} \left(\frac{h}{a}\right)^2 \dots\dots\dots(14)$$

$$M_x = \frac{aP}{\pi l} \sum_{m=1}^{\infty} \frac{\mu^2 \cos \frac{m\pi x}{l}}{\mu^4 + 12(1-\nu^2) \left(\frac{a}{h}\right)^2} + \frac{2aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{H} \left[A \mu^6 + (2U + \nu V) \mu^4 n^2 - \nu(2+\nu) \mu^2 n^2 \right. \\ \left. + \left\{ 1 + 2\nu + \frac{1-\nu^2}{12} \left(\frac{h}{a}\right)^2 \right\} \mu^2 n^4 + \nu n^4 (n^2-1) \right] \cos \frac{m\pi x}{l} \cos n\theta \dots\dots\dots(15)$$

$$M_\theta = \frac{aP}{\pi l} \sum_{m=1}^{\infty} \frac{\nu \mu^2 \cos \frac{m\pi x}{l}}{\mu^4 + 12(1-\nu^2) \left(\frac{a}{h}\right)^2} + \frac{2aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{H} \left[\nu A \mu^6 + (2\nu U + V) \mu^4 n^2 \right. \\ \left. + (2+\nu) \mu^2 n^2 (n^2-1) + n^4 (n^2-1) \right] \cos \frac{m\pi x}{l} \cos n\theta \dots\dots\dots(16)$$

$$\mathfrak{M} = -\frac{2aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1-\nu) \mu n}{H} \left[V \mu^4 + 2J' \mu^2 n^2 - (2+\nu) \mu^2 + n^2 (n^2-1) \right] \sin \frac{m\pi x}{l} \sin n\theta \dots\dots\dots(17)$$

$$J' = 1 - \frac{\nu}{24} \left(\frac{h}{a}\right)^2$$

$$N_x = -\frac{P}{\pi l} \sum_{m=1}^{\infty} \frac{\mu^3 \sin \frac{m\pi x}{l}}{\mu^4 + 12(1-\nu^2) \left(\frac{a}{h}\right)^2} - \frac{2P}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu}{H} \left[A \mu^6 + (2U + V) \mu^4 n^2 \right. \\ \left. + \left\{ 3 + \frac{1-\nu}{12} \left(\frac{h}{a}\right)^2 \right\} \mu^2 n^4 - (2+\nu) \mu^2 n^2 + n^4 (n^2-1) \right] \sin \frac{m\pi x}{l} \cos n\theta \dots\dots\dots(18)$$

$$N_\theta = -\frac{2P}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n}{H} \left[\left\{ 1 + \frac{\nu}{6} \left(\frac{h}{a}\right)^2 \right\} \mu^6 + \left\{ 3 + \frac{\nu(1-\nu)}{12} \left(\frac{h}{a}\right)^2 \right\} \mu^4 n^2 \right. \\ \left. - (1-\nu)(2+\nu) \mu^4 + 3 \mu^2 n^2 (n^2-1) + n^4 (n^2-1) \right] \cos \frac{m\pi x}{l} \sin n\theta \dots\dots\dots(19)$$

$$S' = -\frac{2P}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu n}{H} \left[(1-\nu)(2+3\nu) \left\{ 1 + \frac{12(1+\nu)}{(2+3\nu)} \left(\frac{a}{h}\right)^2 \right\} \mu^2 \right. \\ \left. - (1-\nu^2) \left\{ 1 - \frac{\nu}{12(1+\nu)} \left(\frac{h}{a}\right)^2 \right\} \mu^2 n^2 + \left\{ 1 + \frac{\nu}{6} \left(\frac{h}{a}\right)^2 \right\} \mu^4 - n^2 (n^2-1) \right] \sin \frac{m\pi x}{l} \sin n\theta \dots\dots\dots(20)$$

As the convergency of these double series representing the stress resultants and the stress couples is, in general, unsatisfactory, especially in the vicinity of the point of application of the load, direct computation

would be very laborious. Thus in order to investigate in more detail the property of these series which will have singularities, the following procedure is taken.

In the course of analysis, in general, a small quantity $(h/a)^2$ should not be neglected compared with unity.⁽⁵⁾ But if we proceed to no further analysis, the small quantity $(h/a)^2$ may be ignored compared with unity. Consequently we have

$$H - (\mu^2 + n^2)^4 = [6L - (8 - 2\nu^2)n^2]\mu^4 - n^2(2n^2 - 1)(4\mu^2 + n^2) \equiv \varphi(\mu, n) \quad (21)$$

The next relation will also hold.

$$\sum \sum \frac{f(\mu, n)}{H} \cos \frac{m\pi x}{l} \cos n\theta = \sum \sum \frac{f(\mu, n)}{(\mu^2 + n^2)^4} \cos \frac{m\pi x}{l} \cos n\theta - \sum \sum \frac{f(\mu, n)\varphi(\mu, n)}{(\mu^2 + n^2)^4 H} \cos \frac{m\pi x}{l} \cos n\theta \quad (22)$$

where $f(\mu, n)$ is of 6th or 7th order with respect to μ or n for the stress resultants, the stress couples and the transverse shearing forces, respectively.

The first series in the right-hand sides of Eq. (22) does not always converge uniformly, the second series, however, will have finite values for all values of x and θ , even in case where $f(\mu, n)$ is of 7th order with respect to μ and n , because H is of 8th and φ is of 6th order with respect to μ and n . Thus it is only the first series that needs transformation. For this purpose the following relation of the series is used.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\cos n\theta}{(n^2 + \mu^2)^4} = & -\frac{1}{2\mu^5} + \frac{\pi\theta^2 \cosh \mu(\theta - \pi)}{16\mu^5 \sinh \mu\pi} - \frac{\pi\theta^3 \sinh \mu(\theta - \pi)}{96\mu^4 \sinh \mu\pi} + \frac{\pi^2\theta^2 \cosh \mu\theta}{32\mu^4 \sinh^2 \mu\pi} \\ & + \frac{\pi}{32\mu^4} \left[\frac{5\pi}{\mu^2 \sinh^2 \mu\pi} + \frac{4\pi^2 \cosh \mu\pi}{\mu \sinh^3 \mu\pi} + \frac{2\pi^3 \cosh^2 \mu\pi}{\sinh^4 \mu\pi} - \frac{2\pi^3}{3 \sinh^2 \mu\pi} \right] \cosh \mu\theta \\ & + \frac{5\pi \cosh \mu(\theta - \pi)}{32\mu^7 \sinh \mu\pi} - \frac{\pi}{32\mu^4} \left[\frac{4\pi}{\mu \sinh^2 \mu\pi} + \frac{2\pi^2 \cosh \mu\pi}{\sinh^3 \mu\pi} \right] \theta \sinh \mu\theta - \frac{5\pi\theta \sinh \mu(\theta - \pi)}{32\mu^6 \sinh \mu\pi} \quad (23) \\ & (0 \leq \theta \leq 2\pi) \end{aligned}$$

Using Eq. (23), the double series of the first term in the right-hand sides of Eq. (22) can be represented in single ones. The single series containing $\sinh^2 \mu\pi$, $\sinh^3 \mu\pi$ and $\sinh^4 \mu\pi$ in the denominator converge rapidly. For the terms containing $\sinh \mu\pi$ in their denominators the following relations can be employed.

$$\frac{\cosh \mu(\theta - \pi)}{\sinh \mu\pi} = e^{-\mu\pi} \frac{\cosh \mu\theta}{\sinh \mu\pi} + e^{-\mu\theta}, \quad \frac{\sinh \mu(\theta - \pi)}{\sinh \mu\pi} = e^{-\mu\pi} \frac{\sinh \mu\theta}{\sinh \mu\pi} - e^{-\mu\theta} \quad (24)$$

In the case where $f(\mu, n)$ is of the 5th, 6th or 7th order with respect to μ , the coefficients for $\sinh \mu(\theta - \pi)/\sinh \mu\pi$, $\cosh \mu(\theta - \pi)/\sinh \mu\pi$ in Eq. (23) are given in the following table. But they are for θ and μ except the constant coefficients.

Table 1

Order of $f(\mu, n)$	5	6	7
$\frac{\cosh \mu(\theta - \pi)}{\sinh \mu\pi}$	$\theta^2, 1/\mu^2$	$\theta^2\mu, 1/\mu$	$\theta^2\mu^2, \mu^0$
$\frac{\sinh \mu(\theta - \pi)}{\sinh \mu\pi}$	$\theta^3\mu, \theta/\mu$	$\theta^3\mu^2, \theta$	$\theta^3\mu^3, \theta\mu$

If $f(\mu, n)$ is of 5th order, all terms containing θ become 0 for $x \rightarrow 0$, $\theta \rightarrow 0$ and the term $1/\mu^2$ has a finite value. If $f(\mu, n)$ is of 6th order, only the term $1/\mu$ tends to ∞ in the form of $\ln 2[\cosh \pi a\theta/l - \cos \pi x/l]$ for $x \rightarrow 0$, $\theta \rightarrow 0$ and the other terms have finite values. In the case of 7th order of $f(\mu, n)$ all the terms tend to ∞ for $x \rightarrow 0$, $\theta \rightarrow 0$. Thus

transformation should be performed for all expressions for the stress resultants and couples and the main parts of them should be given in closed forms. For this purpose, we use the following relations.

$$\left. \begin{aligned} \sum_{m=1}^{\infty} e^{-\mu\theta} \sin \frac{\mu x}{a} &= \frac{\sin \frac{\pi x}{l}}{4} \left[\frac{1}{\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l}} + \frac{1}{\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l}} \right] \\ \sum_{m=1}^{\infty} e^{-\mu\theta} \cos \frac{\mu x}{a} &= \frac{\sinh \frac{\pi a\theta}{l}}{4} \left[\frac{1}{\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l}} - \frac{1}{\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l}} \right] \\ \sum_{m=1}^{\infty} \frac{e^{-\mu\theta}}{\mu} \cos \frac{\mu x}{a} &= \frac{l}{4\pi a} \left[\ln 2 \left(\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l} \right) - \ln 2 \left(\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l} \right) \right] \end{aligned} \right\} \quad (25)$$

$m=1, 3, 5, 7, \dots, \quad (0 \leq \theta \leq \pi)$

Employing the above expressions, transformation may be performed. As no analytical computation is made

in further discussion, it is permitted to ignore the small quantity $(h/a)^2$ compared with unity. If we denote by $\tilde{M}_x, \tilde{M}_\theta, \dots$ the values of $M_x, M_\theta, \dots$ with the terms transformed by the above abbreviation, we obtain the following expressions.

$$\begin{aligned}\tilde{M}_x &= \frac{aP}{\pi l} \sum_{m=1}^{\infty} \frac{\mu^2 \cos \frac{\mu x}{a}}{\mu^4 + 12(1-\nu^2) \left(\frac{a}{h}\right)^2} \\ &+ \frac{2aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{H} \left[\mu^6 + (2+\nu)\mu^4 n^2 + (1+2\nu)\mu^2 n^4 - \nu(2+\nu)\mu^2 n^2 + \nu n^4(n^2-1) \right] \cos \frac{\mu x}{a} \cos n\theta \\ &= -\frac{12(1-\nu^2)aP}{\pi l} \sum_{m=1}^{\infty} \frac{\cos \frac{\mu x}{a}}{\mu^2 \left[\left(\frac{h}{a}\right)^2 \mu^4 + 12(1-\nu^2) \right]} - \frac{2\nu aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[(2+\nu)\mu^2 + n^2]n^2}{H} \cos \frac{\mu x}{a} \cos n\theta \\ &- \frac{2aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\mu^2 + \nu n^2)\varphi(\mu, n)}{(\mu^2 + n^2)^2 H} \cos \frac{\mu x}{a} \cos n\theta - \frac{(1-\nu)aP}{2l} \theta \sum_{m=1}^{\infty} e^{-\mu\pi} \frac{\sinh \mu\theta}{\sinh \mu\pi} \cos \frac{\mu x}{a} \\ &+ \frac{(1+\nu)aP}{2l} \sum_{m=1}^{\infty} e^{-\mu\pi} \frac{\cosh \mu\theta}{\mu \sinh \mu\pi} \cos \frac{\mu x}{a} + \frac{(1-\nu)\pi aP}{2l} \sum_{m=1}^{\infty} \frac{\cosh \mu\theta}{\sinh^2 \mu\pi} \cos \frac{\mu x}{a} \\ &+ \frac{(1-\nu)aP}{8l} \theta \sinh \frac{\pi a\theta}{l} \left[\frac{1}{\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l}} - \frac{1}{\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l}} \right] \\ &+ \frac{(1+\nu)P}{8\pi} \left[\ln 2 \left(\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l} \right) - \ln 2 \left(\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l} \right) \right] \quad (0 \leq \theta \leq \pi) \dots\dots\dots(26)\end{aligned}$$

$$\begin{aligned}\tilde{M}_\theta &= \frac{\nu aP}{\pi l} \sum_{m=1}^{\infty} \frac{\mu^2 \cos \frac{\mu x}{a}}{\mu^4 + 12(1-\nu^2) \left(\frac{a}{h}\right)^2} \\ &+ \frac{2aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{H} \left[\nu\mu^6 + (2\nu+1)\mu^4 n^2 + (2+\nu)\mu^2 n^2(n^2-1) + n^4(n^2-1) \right] \cos \frac{\mu x}{a} \cos n\theta \\ &= -\frac{12\nu(1-\nu^2)aP}{\pi l} \sum_{m=1}^{\infty} \frac{\cos \frac{\mu x}{a}}{\mu^2 \left[\left(\frac{h}{a}\right)^2 \mu^4 + 12(1-\nu^2) \right]} - \frac{2aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[(2+\nu)\mu^2 + n^2]n^2}{H} \cos \frac{\mu x}{a} \cos n\theta \\ &- \frac{2aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\nu\mu^2 + n^2)\varphi(\mu, n)}{(\mu^2 + n^2)^2 H} \cos \frac{\mu x}{a} \cos n\theta \\ &+ \frac{(1-\nu)aP}{2l} \theta \sum_{m=1}^{\infty} e^{-\mu\pi} \frac{\sinh \mu\theta}{\sinh \mu\pi} \cos \frac{\mu x}{a} - \frac{(1-\nu)\pi aP}{2l} \sum_{m=1}^{\infty} \frac{\cosh \mu\theta}{\sinh^2 \mu\pi} \cos \frac{\mu x}{a} \\ &+ \frac{(1+\nu)aP}{2l} \sum_{m=1}^{\infty} e^{-\mu\pi} \frac{\cosh \mu\theta}{\mu \sinh \mu\pi} \cos \frac{\mu x}{a} \\ &- \frac{(1-\nu)aP}{8l} \theta \sinh \frac{\pi a\theta}{l} \left[\frac{1}{\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l}} - \frac{1}{\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l}} \right] \\ &+ \frac{(1+\nu)P}{8\pi} \left[\ln 2 \left(\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l} \right) - \ln 2 \left(\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l} \right) \right] \quad (0 \leq \theta \leq \pi) \dots\dots\dots(27)\end{aligned}$$

$$\begin{aligned}\tilde{\mathfrak{M}} &= -\frac{2aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1-\nu)\mu n [\mu^4 + 2\mu^2 n^2 + n^2(n^2-1) - (2+\nu)\mu^2]}{H} \sin \frac{\mu x}{a} \sin n\theta \\ &= \frac{2(1-\nu)aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu n \varphi(\mu, n)}{(\mu^2 + n^2)^2 H} \sin \frac{\mu x}{a} \sin n\theta \\ &+ \frac{2(1-\nu)aP}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu n [(2+\nu)\mu^2 + n^2]}{H} \sin \frac{\mu x}{a} \sin n\theta \\ &- \frac{(1-\nu)aP}{2l} \theta \sum_{m=1}^{\infty} e^{-\mu\pi} \frac{\cosh \mu\theta}{\sinh \mu\pi} \sin \frac{\mu x}{a} + \frac{(1-\nu)\pi aP}{2l} \sum_{m=1}^{\infty} \frac{\sinh \mu\theta}{\sinh^2 \mu\pi} \sin \frac{\mu x}{a} \\ &- \frac{(1-\nu)aP}{8l} \theta \sin \frac{\pi x}{l} \left[\frac{1}{\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l}} + \frac{1}{\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l}} \right] \quad (0 \leq \theta \leq \pi) \dots\dots\dots(28)\end{aligned}$$

$$\begin{aligned}
\tilde{N}_x = & -\frac{P}{\pi l} \sum_{m=1}^{\infty} \frac{\mu^3 \sin \frac{\mu x}{a}}{\mu^4 + 12(1-\nu^2) \left(\frac{a}{h}\right)^2} \\
& - \frac{2P}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu [\mu^6 + 3\mu^4 n^2 + 3\mu^2 n^4 - (2+\nu)\mu^2 n^2 + n^4(n^2-1)]}{H} \sin \frac{\mu x}{a} \cos n\theta \\
= & -\frac{12(1-\nu^2)P}{\pi l} \sum_{m=1}^{\infty} \frac{\sin \frac{\mu x}{a}}{\mu \left[\left(\frac{h}{a}\right)^2 \mu^4 + 12(1-\nu^2) \right]} + \frac{2P}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu n^2 [(2+\nu)\mu^2 + n^2]}{H} \sin \frac{\mu x}{a} \cos n\theta \\
& + \frac{2P}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu \varphi(\mu, n)}{(\mu^2 + n^2)H} \sin \frac{\mu x}{a} \cos n\theta - \frac{P}{l} \sum_{m=1}^{\infty} e^{-\mu\pi} \frac{\cosh \mu\theta}{\sinh \mu\pi} \sin \frac{\mu x}{a} \\
& - \frac{P}{4l} \sin \frac{\pi x}{l} \left[\frac{1}{\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l}} + \frac{1}{\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l}} \right] \quad (0 \leq \theta \leq \pi) \dots\dots\dots (29)
\end{aligned}$$

$$\begin{aligned}
\tilde{N}_\theta = & -\frac{2P}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n}{H} \left[\mu^6 + 3\mu^2 n^2 (\mu^2 + n^2) + n^4 (n^2 - 1) - (1-\nu)(2+\nu)\mu^4 - 3\mu^2 n^2 \right] \cos \frac{\mu x}{a} \sin n\theta \\
= & \frac{2P}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n [(2+\nu)\mu^2 + n^2] [(1-\nu)\mu^2 + n^2]}{H} \cos \frac{\mu x}{a} \sin n\theta \\
& + \frac{2P}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n \varphi(\mu, n)}{(\mu^2 + n^2)H} \cos \frac{\mu x}{a} \sin n\theta + \frac{P}{l} \sum_{m=1}^{\infty} e^{-\mu\pi} \frac{\sinh \mu\theta}{\sinh \mu\pi} \cos \frac{\mu x}{a} \\
& + \frac{P}{4l} \sinh \frac{\pi a\theta}{l} \left[\frac{1}{\cosh \frac{\pi a\theta}{l} + \cos \frac{\pi x}{l}} - \frac{1}{\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l}} \right] \quad (0 \leq \theta \leq \pi) \dots\dots\dots (30)
\end{aligned}$$

T_x and T_θ have also singularities at the point of application of the load. But the stresses due to them are small in the order of h/a compared with those due to M_x and M_θ . Therefore the computation for the former is worked out without transformation. We see that the forms of singularities of the above equations are such as $\ln 2 \left(\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l} \right)$ or $\sin \frac{\pi x}{l} / \left[\cosh \frac{\pi a\theta}{l} - \cos \frac{\pi x}{l} \right]$ etc.

3. Experiment

Some experiments are carried out to confirm the results of analysis. Two kinds of cylinders are prepared for experiments. Their dimensions are:—

- (i) $l/a=4$, $a/h=30$, $h=8$ mm
- (ii) $l/a=4$, $a/h=50$, $h=4.5$ mm

- ① Cylindrical shell
- ② Supporting plate of cylindrical shell
- ③ Load cell, ④ Frame work,
- ⑤ Hydraulic jack

Fig. 4 Experimental apparatus

A sketch of the apparatus for experiment is shown in Fig. 4. A radial concentrated force is applied through a small steel ball put on one side of a load cell ③ inserted between the frame work ④ and the cylinder ①. The load is measured by the load cell. In the experiment the radial deflection w is measured by dial gauges and the stresses σ_x and σ_y produced on the surface of the cylinder are measured by wire strain gauge.

4. Results

4.1 Displacement

As for the radial displacement w which is thought to be important in practical use, numerical calculation is worked out by Eq. (9). The first term in right-hand sides of Eq. (9) is an axisymmetrical expression independent of θ and is so small compared to the second term in the range $x/l \geq 0.2$ that it may be neglected without appreciable error. The number of terms taken in numerical computation, in the case of double series, is about 20 or 30 for n or m respectively in the vicinity of the point of application of the load and as the value of x or θ increases it may be reduced, because the convergency of series becomes higher.

The numerical and experimental results are shown in Fig. 5 which illustrates diagrams for w/w_0 , where w_0 is the maximum deflection calculated from the elementary beam theory and has a value of $Pl^3/48\pi Ea^3h$.

As regards the variation of radial deflection

along the circumference through a point of loading, we see that the cylinder deflects inwards at a point of loading and outwards as the reaction of load at $\theta=15^\circ\sim30^\circ$ and after attaining a maximum value at $\theta=40^\circ\sim50^\circ$, it comes gradually near to the base circle and has zero value at about $\theta=80^\circ\sim90^\circ$. For a larger angle than $\theta=90^\circ$ the deflection is no longer significant, having a downward value at the bottom. At any other circumferential cross-section the deflection has the same tendency, and the larger is

w_0 : the maximum deflection calculated from beam theory

(a) The variation of w/w_0 along circumference

(b) The variation of w/w_0 along generator
Fig. 5 Diagram of radial deflection w/w_0

the distance in x -direction from the point of loading, the smaller is the deflection and reduces to zero at the ends of the cylinder. It is found that the influence of a concentrated load on deflections is much smaller for the cross-section of $\theta \geq 90^\circ$ than that for upper half, but not so small as that calculated by beam theory; even at the bottom of the cylinder ($\theta=180^\circ$, $x/l=0$) the deflection is about 5 times greater than that of beam for $a/h=30$ and about 11 times for $a/h=50$. The deflection directly under the point of loading is extraordinarily large and is more than 100 times compared with that calculated by beam theory for $a/h=30$ and more than 200 times for $a/h=50$, while the maximum deflection under line load is about 20 ($a/h=50$)⁽⁶⁾. These facts show how large the effect of a concentrated load will be; practical importance, however, is rather attached to the phenomenon of protrusion. The maximum value and the position of protrusion are as follows:

$$\begin{aligned} a/h=30 \quad w/w_0 &\approx 50 \quad (\theta \approx 48^\circ) \\ a/h=50 \quad w/w_0 &\approx 80 \quad (\theta \approx 42^\circ) \end{aligned}$$

These results show that as the ratio a/h increases the position of protrusion approaches the position of loading and the value w/w_0 becomes larger.

As regards the deflections along the generator, the absolute values of them decrease as the distance from the mid-plane increases. As easily be seen from the figure only the deflection curve at $\theta=30^\circ$ for $a/h=30$ has different tendency from others. Being different from the behaviour of the circumferential deflection the signs of deflection along generator remain unchanged. Comparing the analytical results with experimental ones, we see that they are in good agreement with each other, especially in the case of $a/h=30$ than in the case of $a/h=50$, showing some influence due to the rigidity of the end plates; i.e., the end plates used for both cylinders are of the same thickness (1 inch), the influence of which, consequently, is more effective on the latter than on the former.

4.2 Stress resultant

4.2.1 T_x As already stated T_x is calculated by Eq. (12) before transformation. The number of terms taken in numerical calculation is almost the same as in the case of w . Numerical results are shown in Fig. 6, which illustrates the dimensionless value $t_x = T_x/h/Pl/4\pi a^2 h$, where $Pl/4\pi a^2$ is the maximum stress resultant calculated from the beam theory.

The variations of t_x along axial and circumferential direction resemble, as a whole, those of deflection w , showing that negative parts of deflection w correspond to compression of the cylinder and the positive ones correspond to tension. The values of

t_x except the point of loading are not so large as in the case of w . The maximum tensile stress t_x is as follows:

$$\begin{aligned} t_x &\doteq 4.7 & (x=0, \theta \doteq 50^\circ) & \text{ for } a/h=30 \\ t_x &\doteq 6.9 & (x=0, \theta \doteq 45^\circ) & \text{ for } a/h=50 \end{aligned}$$

It will be seen that the variations of t_x are like those of w and for the cross-section $\theta \geq 90^\circ$ the effect of a concentrated load on t_x is small.

4.2.2 T_θ The number of terms taken in numerical calculation is almost the same as T_x or w . The first term in the right-hand sides in Eq. (13) (an axi-symmetrical term) can be neglected compared with the second one without appreciable error for $x/l \geq 0.1$. As in the case of t_x dimensionless values $t_\theta = T_\theta/h/Pl/4\pi a^2h$ are shown in Fig. 7.

The variations of t_θ are quite different from those of t_x or w ; it varies from compression to tension along the generator ($\theta=0^\circ$) having a maximum tensile value at a very near point to the load ($x/l = 0.03 \sim 0.07$) and reduces to zero at the ends keeping its sign. This is an interesting phenomenon occurring only for t_θ . On the other hand, along the circumference, it varies from compression to tension for $\theta=30^\circ \sim 50^\circ$, the values, however, are always very small showing practically no effect of the load on $\theta \geq 90^\circ$.

4.3 Stress couple

4.3.1 M_x The double series in Eq. (15) (M_x before transformation) does not converge uniformly and the numerical computation is very

(a) The variation of t_x along circumference

(a) The variation of t_θ along circumference

(b) The variation of t_x along generator

Fig. 6 Diagram of stress resultant $t_x = T_x/Pl/4\pi a^2$

(b) The variation of t_θ along generator

Fig. 7 Diagram of stress resultant $t_\theta = T_\theta/Pl/4\pi a^2$

laborious even using an electronic computer, especially at the near point to loading. The double series in Eq. (26) ($\tilde{M}_x$ after transformation), however, are uniformly convergent, so the numerical calculation is carried out by this equation and 10 terms or so are sufficient for this calculation. As the single series in the same equation are rapidly convergent, it is sufficient to take only the first few terms. The property of singularity is represented by the last term $\ln 2 \left(\cosh \frac{\pi a \theta}{l} - \cos \frac{\pi x}{l} \right)$.

Numerical results are shown in Fig. 8 for dimensionless value $m_x = 6M_x/h^2/Pl/4\pi a^2h$. To make clear the relations between the stress couples and deflection or stress resultants, the sign of co-ordinate is taken inversely. The variation of m_x along the circumference resembles that of w or t_x . The

(a) The variation of m_x along circumference

(b) The variation of m_x along generator

Fig. 8 Diagram of stress couple $m_x = 6M_x/h^2/Pl/4\pi a^2h$

maximum values except the near point to loading are about 1.7~1.8 times larger than those of t_x for both cylinders and the position of maximum value is $\theta \approx 30^\circ$ for $a/h=30$ and $\theta \approx 20^\circ$ for $a/h=50$. The variation of m_x along the generator ($\theta=0^\circ$) is very interesting and each curve for both cylinders has an extreme value at $x/l=0.05 \sim 0.07$ or $x/l=0.1 \sim 0.15$. We can obtain the axial stress σ_x by combining t_x and m_x .

4.3.2 M_θ

M_θ is calculated from Eq. (27) after transformation. Comparing $\tilde{M}_\theta$ with $\tilde{M}_x$, it can easily be found that only the coefficient of double series in the third term of the right-hand sides in Eq. (27) is different from $\tilde{M}_x$ and the other terms may be used if only the signs and constant coefficients of $\tilde{M}_x$ are changed. The numerical results are shown in Fig. 9 for dimensionless value

(a) The variation of m_θ along circumference

(b) The variation of m_θ along generator

Fig. 9 Diagram of stress couple $m_\theta/6M_\theta/h^2/Pl/4\pi a^2$

$m_\theta = 6M_\theta/h^2/Pl/4\pi a^2h$, taking inverse sign for the co-ordinates as in the case of m_x .

The variations of m_θ along generator and circumference resemble m_x except the uppermost variation ($\theta=0^\circ$) which is, unlike m_x , monotonous having no extreme value. The maximum value and the position of it except the point of loading are as follows:

$$\begin{aligned} a/h=30 \quad m_\theta &= -17.6 \quad (\theta=27^\circ) \\ a/h=50 \quad m_\theta &= -25.3 \quad (\theta=21^\circ) \end{aligned}$$

They are almost two times greater than m_x . Similar to t_x and m_x , with the increase of a/h these maximum values become larger and the position of them approaches closer to the loading point. The

(a) The variation of axial stress along circumference

(b) The variation of axial stress along generator
Fig. 10 Axial stress on surfaces of the cylindrical shell

circumferential stress σ_θ can be calculated from t_θ and m_θ .

4.4 Stress

The stress on the surface of the cylinder may be written as follows:

$$\left. \begin{aligned} (\sigma_x)_{\text{out}} &= t_x - m_x, & (\sigma_\theta)_{\text{out}} &= t_\theta - m_\theta \\ (\sigma_x)_{\text{in}} &= t_x + m_x, & (\sigma_\theta)_{\text{in}} &= t_\theta + m_\theta \end{aligned} \right\} \dots\dots\dots (31)$$

where $(\)_{\text{out}}$ and $(\)_{\text{in}}$ mean the outer and inner surface of the cylinder, respectively. As already mentioned t_x , t_θ and m_x , m_θ are dimensionless stresses based on the stress resultants and the stress couples.

The axial stress σ_x on the outer and inner sur-

(a) The variation of circumferential stress along circumference

(b) The variation of circumferential stress along generator

Fig. 11 Circumferential stress on surfaces of the cylindrical shell

face of the cylinder is shown in Figs. 10 (a) and (b) and the circumferential stress σ_θ in Figs. 11 (a) and (b). The experimental results measured by wire strain gauge are plotted on (b) in Figs. 10 and 11. The stresses on the outer surface of cylinder have a similar tendency to the deflection w . Near the point of loading along the circumference, tensile stresses occur having the following maximum values:

$$\begin{array}{llll} (\sigma_x)_{\text{out}} : a/h=30 & 9.8 & a/h=50 & 16.0 \\ (\sigma_x)_{\text{out}} : a/h=30 & 13.7 & a/h=50 & 21.0 \end{array}$$

On the contrary, on the inner surface along the circumference compressive stresses occur, the maximum values of which are as follows:

$$\begin{array}{llll} (\sigma_x)_{\text{in}} : a/h=30 & -8.2 & & \\ & a/h=50 & -13.8 & \\ (\sigma_\theta)_{\text{in}} : a/h=30 & -21.2 & & \\ & a/h=50 & -32.3 & \end{array}$$

The variation of σ_x along the uppermost generator ($\theta=0^\circ$) on the inner surface, as shown in Fig.10(b), is quite different from that of outer surface having maximum compressive stress close to the point of loading, the maximum values of which are as follows:

$$\begin{array}{ll} (\sigma_x)_{\text{in}} : a/h=30 & -9.3 \\ & a/h=50 & -14.7 \end{array}$$

The variation of σ_θ along generator on the outer and inner surface resembles each other.

Summing up the results above mentioned σ_θ is, in general, greater than σ_x on both inner and outer surface. As regards σ_x the tensile stress on outer surface is greater than that on inner surface and for σ_θ the compressive stress on inner surface is greater than that of outer surface. With the increase of a/h these values become larger and the position of them approaches to the point of loading. These results may be applicable for the determination of a dangerous cross section in designing machines subjected to repeated stress such as a rotary kiln.

The numerical results are in good agreement with those of experiments. It is found that our assumption that the end plates behave as if they were thin plates rather than rigid ones is appropriate; because the effect of end plates as rigid bodies is limited to the vicinity of the ends and almost negligible at a small distance from the ends. Thus, as is expected, our assumption is not incompatible with practical use.

5. Conclusions

The conclusions are as follows:—

- (1) The stress resultants and the stress cou-

ples given in the form of double Fourier series are transformed into uniformly convergent series and closed terms which have a singularity at the point of application of the load. The stresses due to stress resultants are of h/a order compared with those of stress couples. The radial deflection w , however, as in the case of flat plate has a finite value at the point of loading.

(2) The variation of the radial deflection w along the circumference through a point of loading is so interesting that the deflection at the point of loading is considerably larger, as is expected, than that calculated from beam theory and the cylinder protrudes outwards at a small distance from the point of loading, then deflects again inwards and at the lowest position ($\theta=180^\circ$) it protrudes outwards. Both the axial and circumferential stress have singularities at the point of loading and at a distance from this point along the circumference they become tensile on the outer surface and compressive on the inner surface. The important matter is that with an increase of a/h the maximum stresses corresponding to the parts of protrusion of deflection w become larger and the positions of them approach to the point of loading. These results may be generally applicable for the investigation of such a machine as a rotary kiln which will possibly suffer fatigue failure through repeated stresses.

(3) For the cross-section of $\theta \geq 90^\circ$ the effect of a concentrated load on the deflection and the stress is so small as to be left out of the question, therefore we may suppose the behaviour of the cylinder subjected to a pair of equal and opposite concentrated radial loads as has been often treated.

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