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Application of Optimization Theory for Bounded State Variable Problems to the Operation of Train

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Finding a train operation which minimizes energy consumption for leading the train from one station to the next at the specified instant and stopping it there is dealt with in this paper. Since speed limit is observed strictly in the train operation, the problem should be considered as a bounded state variable problem. It is true that solving a bounded state variable problem is universally considered extremely troublesome, but the author has recently developed a very simple and clear method. The method is applied to this problem and the optimal control law has really been obtained straightforward and definitely. The author's is again confirmed to be very convenient to solve a bounded state variable problem. The research is believed to be the first scientific research on the economization of train operation.

1. Introduction

Let us find such an operation in which the energy consumption becomes the least of all operations in which the train is operated from one station to the next station at the specified instant and stopped there. Since the speed limit is observed strictly in the train operation, the problem should be dealt with as an optimization problem with bounded state variables. Pontryagin and others have presented the jump condition of costate vector as the key to solving the bounded state variable problem⁽¹⁾, but the condition is not so easily applicable to the synthesis of an optimal control system. The author has recently contrived a simple and clear method of solving the bounded state variable problem⁽²⁾ and checked up the convenience of his method through synthesizing the optimal control in several examples. An outline of the method will be given here.

The motion of the controlled system is assumed to be expressed by

$$\dot{x} = f(x, u), \quad (1)$$

where x , an n -vector, is a state vector, u , an r -vector, is a control vector and the constraint for control is assumed to be expressed by $u \in \Omega$. There is naturally an admissible region B within the state space in the bounded state variable problem. For simplicity, the terminal condition is assumed such that t_f is unknown and $x(t_f) = 0$. Although the

state point can physically enter into the prohibited region $\bar{B}$, the restriction is assumed such that the state point must not enter into the prohibited region $\bar{B}$ during the control interval. Both the initial point $x(t)$ and the terminal point, the origin, are, therefore, implicitly assumed to be located within B or on ∂B . Let us denote the equation of ∂B by $h(x) = 0$. Since h necessarily contains at least one component of x , we can always name one component which is contained in h . Let the named component be x_i , the equation of ∂B may then be expressed by

$$x_i = g(\underline{x}), \quad (2)$$

where $\underline{x}$ is $(n-1)$ -vector with x_i deleted from x , or namely,

$$\underline{x} = \{x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}. \quad (3)$$

Criterion function to be minimized is assumed as

$$J = \int_t^{t_f} f_0(x, u) d\tau. \quad (4)$$

To track all possible optimal backward time trajectories emanating from the terminal point is always necessary to find an optimal control law regardless of whether the state vector is bounded or not. The term optimal trajectory, or simply, trajectory may be used instead of optimal backward time trajectory when the simplification does not cause any confusion in the following. Although the trajectories are surely located within the region B during the early stage of backward time, some of them may reach the boundary ∂B in due course of time. The optimal trajectory in the admissible region can connect to the optimal trajectory on the bound-

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ary at the arrival point if the below-mentioned continuity condition is satisfied, whereas the trajectory must end at the arrival point if the condition is not satisfied.

Continuity condition. In order that the trajectory in the admissible region can connect to the trajectory on the boundary at the arrival point A, there must exist such a control in Ω that maximizes the Hamiltonian $H_a(A)$ written in terms of $x(A)$ and $p(A)$ as well as satisfies the equation

$$f_i[x(A), u] = \sum_{k=1, k \neq i}^n \partial g(x)/\partial x_k(A) \cdot f_k[x(A), u] \dots (5)$$

Since a certain condition must be added to the control vector in order that the state point can move on the boundary ∂B , the dimension of control vector for the motion on the boundary becomes $r-1$. Moreover, x_i must not be a state vector component for the motion on the boundary because x_i is statically decided from other components as seen in Eq. (2), and the state vector is necessarily x expressed by Eq. (3). Costate vector is, therefore, also $(n-1)$ -vector for the motion on the boundary. We shall denote the costate vector on the boundary as

$$\phi = \{\phi_1, \phi_2, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_n\} \dots (6)$$

At the connection point A, there exists the important relation

$$\phi_k(A) = p_k(A) + \partial g(x)/\partial x_k(A) \cdot p_i(A) \Big\}_{k=1, 2, \dots, n, k \neq i} \dots (7)$$

which determines the vector ϕ at the point A and thus makes it possible to continue the tracking of the trajectory on the boundary beyond point A.

If the branch-off condition mentioned below is satisfied at a point of the trajectory on the boundary, the state point can either move onward along the trajectory on the boundary or branch off into the admissible region at the point. If the condition, however, is not satisfied, the state point can only move onward along the trajectory on the boundary.

Branch-off condition. In order that the trajectory on the boundary can branch off into the admissible region at point B, it is necessary that control u , which maximizes the Hamiltonian $H_a(\rho, u)$

$$= \sum_{k=1, k \neq i}^n \{\phi_k(B) - \partial g(x)/\partial x_k(B) \cdot \rho\} \times f_k[x(B), u] + \rho f_i[x(B), u] - f_0[x(B), u] \dots (8)$$

and makes it zero, surely move the state point into the admissible region from point B.

When the branch-off condition is satisfied, the value of ρ is decided simultaneously with control u .

* The condition means that the train is required to arrive at the next station at t_f .

Then, the important relation

$$\left. \begin{aligned} p_k(B) &= \phi_k(B) - \partial g(x)/\partial x_k(B) \cdot \rho \\ k &= 1, 2, \dots, n, \quad k \neq i \end{aligned} \right\} \dots (9)$$

$$p_i(B) = \rho$$

determines the vector p at point A and thus makes it possible to continue the tracking of the trajectory again through the admissible region.

By making use of continuity and branch-off conditions, we can track any trajectory over the period until $t \rightarrow \infty$. It is clear that the optimal control law is obtained after completion of tracking of all possible trajectories. The above-outlined method for solving the bounded state variable problem is applied to the optimization of train operation in this paper.

2. Statement of the problem

The motion of a train will be approximately described by

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u \end{aligned} \right\} \dots (10)$$

where x_1 denotes normalized train position, the origin of which is taken at the next station and the positive axis is taken in the direction of train motion, and x_2 denotes normalized velocity of train. u is a normalized control variable, where $u > 0$ means the exertion of traction force due to the action of prime mover and $u < 0$ means the application of brake, and u is assumed to take the value in the range

$$-1 \leq u \leq u_T/u_B \dots (11)$$

End point condition is such that $x_1(t_f) = 0$ and $x_2(t_f) = 0$, with t_f being specified*. Since the energy consumption should be minimized, the criterion function must be

$$J = \int_0^{t_f} \frac{1}{2} (u + |u|) x_2 d\tau \dots (12)$$

Moreover, the restriction $x_2 \leq v_m$ is added on account of regulation of speed limit. Besides the restriction, the constraints $x_2 \geq 0$ and $x_1 \geq 0$ should be considered in order not to bring about such solutions that the train comes to stop or moves backward on its way to the next station, but since they can easily be dealt with without resorting to any particular method of solving the bounded state variable problem, the substance which makes the problem interesting and complicates it is certainly the restriction $x_2 \leq v_m$.

Since the final instant is specified, an artificial state variable described by $x_3 = t_f - t$ must be introduced and thus the dimensions of state vector must be considered three. Costate vector, therefore, is naturally three dimensional, but since the component p_3 corresponding to the artificially introduced component x_3 is always a constant $-\mu_3$, the differential equations with respect to p_1 and p_2 only are suf-

ficient for the definition of costate vector. They are defined by the following differential equations and boundary conditions.

$$\dot{p}_1=0, \quad \dot{p}_2=-p_1+p_2+\frac{1}{2}(u+|u|) \dots\dots\dots (13)$$

$$p_1(t_f)=-\mu_1, \quad p_2(t_f)=-\mu_2 \dots\dots\dots (14)$$

In the above, $-\mu_i$, $i=1, 2, 3$, are undetermined multipliers. Now, Hamiltonian H is written as

$$H=p_1x_2+p_2(-x_2+u)-(-\mu_3)-\frac{1}{2}(u+|u|x_2) \dots\dots\dots (15)$$

3. Tracking of optimal backward time trajectories in admissible region

First of all, let us track all possible backward time trajectories emanating from the origin as far as x_2 reaches v_m or 0, and find an optimal control law for the region occupied by such trajectories.

It is true that among the operations which satisfy the optimal control condition derived from maximum principle, there is such an operation which causes the situation where the train arrives at the next station too early and waits there for the specified final time t_f , wastefully*. The value of J in such an operation, however, is certainly greater than that in an operation which leads the train to the next station just at t_f and stops it there, satisfying of course the optimal control condition. Therefore, the operation which causes too early arrival must not be taken up. It can, therefore, be said that x_2 is positive during the control interval. Then, from Eq. (15), the optimal control condition is obtained as

$$u=\left\{\begin{array}{l} -1, \quad p_2<0 \\ [-1, 0], \quad p_2=0 \\ 0, \quad 0<p_2<x_2 \\ [0, u_T/u_B], \quad p_2=x_2 \\ u_T/u_B, \quad x_2<p_2 \end{array}\right\} \dots\dots\dots (16)$$

As seen in Eq. (16), optimal control will be indeterminate when p_2 is equal to 0 or x_2 , but when the situation passes in an instant, it does not relate to a singular solution at all. If, however, there exists a solution under the condition that the situation $p_2=0$ or $p_2=x_2$ persists and $H=0$ is kept during a finite interval**, such a solution is a singular solution which satisfies the optimal condition and must be regarded as the candidate of optimal solution as well as the regular solution which satisfies the optimal condition. In fact, it is confirmed that both kinds

of singular solution can arise in this problem. They are called type 2 and 3 singular solutions respectively. We shall track backward time trajectories in the following order of regular solution, type 2 singular solution and type 3 singular solution.

3.1 Regular solution trajectories and cross sections of them

It is clear that brake must be applied in the final stage of control interval in order to stop the train at the next station just at the specified instant t_f . Since brake is applied when $p_2<0$ in regular solution and the value must be -1 (maximum brake), p_2 must be negative in the final stage of control interval, though it may be just zero at t_f if it surely assumes negative values before t_f . In other words, the train may be stopped at the next station in such a way either that maximum brake is applied continually even after the complete stop or that maximum brake is applied so long as the train moves but is switched to arbitrary value just when the train is stopped. Thus it results that $-\mu_2\leq 0$.

We shall call t_f-t backward time and denote it by t again in the following for convenience. Applying the condition $H=0$ at $t=0$ we obtain $-\mu_3=-(-\mu_2)$ or 0 according to $-\mu_2<0$ or $=0$. Then, it is always right to say $-\mu_3=-(-\mu_2)$. Optimal backward time trajectories are determined by the following relations.

$$\left.\begin{array}{l} \dot{x}_1=-x_2, \quad \dot{x}_2=x_2-u, \quad x_3=t \\ \dot{p}_2=(-\mu_1)-p_2-\frac{1}{2}(u+|u|) \\ x_1(0)=0, \quad x_2(0)=0, \quad p_2(0)=-\mu_2\leq 0 \\ u \text{ obeys to Eq. (16)} \end{array}\right\} \dots\dots\dots (17)$$

In the case $-\mu_2=0$, $-\mu_1$ must be negative in order that $u=-1$ may be surely applied at $t=0+$, but $-\mu_1$ may assume arbitrary values if $-\mu_2$ is negative, however small its magnitude is. By the way, when $-\mu_1=0$ and $-\mu_2=0$, the situation $p_2=0$ persists perpetually and type 2 singular solution will result.

$u=-1$ is applied during at least the early stage of backward time and the following solution is obtained.

$$\left.\begin{array}{l} x_2=e^t-1 \\ p_2=\{(-\mu_2)-(-\mu_1)\}e^{-t}+(-\mu_1) \end{array}\right\} \dots\dots\dots (18)$$

It can be said that if $-\mu_2<-v_m(-\mu_1)$, x_2 will reach v_m (it means that the state point reaches the boundary $x_2=v_m$), with p_2 being negative. Domain a of Fig. 1 gives the condition to yield such a trajectory. Under the condition other than that of domain a in the fourth quadrant of the figure, p_2 turns positive before x_2 reaches v_m and the situation $0<p_2<x_2$ arises. Then the optimal control turns out $u=0$ (coasting). The switching time t_1 from maximum brake to coasting is given by

$$t_1=\ln\{-(-\mu_2)/(-\mu_1)+1\} \dots\dots\dots (19)$$

* Minimum time solution is obtained in the extremal case, where the wasted time at the next station is the greatest. It can be shown that the value of J increases as the wasted time.

** Non-infinitesimal small interval is meant by finite interval.

and the form of solution changes to

$$\begin{aligned} x_2 &= -(-\mu_2)/\{-(-\mu_2)+(-\mu_1)\} \cdot e^t \} \dots\dots\dots(20) \\ p_2 &= \{(-\mu_2)-(-\mu_1)\}e^{-t}+(-\mu_1) \end{aligned}$$

after t_1 . As seen from Fig. 2, which shows the time evolutions of x_2 and p_2 , both x_2 and p_2 increase after t_1 , and x_2 may reach v_m with the situation $0 < p_2 < x_2$ being kept for some conditions of $\{-\mu_1, -\mu_2\}$ or p_2 exceeds x_2 before x_2 reaches v_m for the other conditions. The condition that x_2 reaches v_m with the situation $0 < p_2 < x_2$ being kept is given by

$$-\mu_2 \begin{cases} < \frac{1}{4}(-\mu_1)^2, & -\mu_1 < 2v_m \\ = v_m, & -\mu_1 = 2v_m \\ \leq -v_m(-\mu_1) + v_m^2, & -\mu_1 > 2v_m \end{cases} \dots\dots\dots(21)$$

and is represented by domain b in Fig. 1. The remaining domain c represents the condition that p_2 exceeds x_2 before x_2 reaches v_m and thus the application of $u=u_T/u_B$ (full pull) is made. The switching time t_2 from coasting to full pull is given by

$$t_2 = \ln \frac{\{-(-\mu_2)+(-\mu_1)\} \{(-\mu_1)-\sqrt{(-\mu_1)^2+4(-\mu_1)}\}}{-2(-\mu_2)} \dots\dots\dots(22)$$

Clearly, x_2 lessens itself after t_2 and it can be further shown that the situation $x_2 < p_2$ continues until x_2 reaches 0. Of course, tracking of the trajectory will be stopped when x_2 reaches 0. It may be interesting to note that $x_2=0$ means the starting of the train in the forward time; in other words, it means one station on this side.

In summary, it turns out that all possible regular trajectories are divided into three kinds according to the value of $\{-\mu_1, -\mu_2\}$. Clearly, they

Fig. 1 Partitioning of $-\mu_1, -\mu_2$ plane

Fig. 2 Time evolutions of x_2 and p_2

spread in three dimensional state space and hence graphical representation requires cross section with a certain plane. It may be said that cross section with $x_3=T$ plane is the most convenient, where T is the difference between the final time t_f and the present time t or the remaining interval.

Since the shape of cross section changes discontinuously for the gradual change of the values of T , the cross section should be presented for every section of T as shown in Fig. 3, where optimal control is also entered with the intention to utilize the cross section as it is for the optimal control law diagram. Equations for various curves Γ_i 's in the figure are listed up in the Appendix. (likewise in the later figures)

3.2 Type 2 singular solution

Type 2 singular solution arises when $-\mu_1=0$ and $-\mu_2=0$, in other words, under the condition of the origin of Fig. 1. It is clear that regular solution cannot transfer to type 2 singular solution and vice versa in a particular control interval. Therefore, if a singular solution is assumed to arise, the solution must be singular all over the control interval. Although the trajectory itself is not definite because u assumes indefinite values between -1 and 0 , the extent within which the trajectories are located in the three dimensional state space is definite and thus the contour of the cross section with $x_3=T$ plane is definite as shown in Fig. 4.

Arbitrary control from the range between -1

Fig. 3 Cross sections of regular solution trajectories within admissible region and optimal control

and 0 is approved when the plotted state point is located within the area of the cross section. The freedom, however, will rather make the actual operation inconvenient in the case of manual control or make the synthesis of control device inconvenient if it is decided to perform the control automatically. It is convenient to make it a rule to apply $u=-1$ and $u=0$ when the plotted state point is located within the area and on the boundary curve Γ_1 respectively. Such a rule will necessarily lead the train to the next station just at t_f and stop it there.

3.3 Type 3 singular solution

In type 3 singular solution which arises when the situation $p_2=x_2$ persists, u assumes indefinite values between 0 and u_T/u_B , but it can be said that $\frac{1}{2}(u+|u|)$ is necessarily equal to u because u is not negative. From the second row of Eq. (10) and Eq. (13), we obtain $x_2=p_2=u=\frac{1}{2}(-\mu_1)$. The singular solution means such a solution where the train keeps a constant speed of certain value below v_m and u also must be constant so as to maintain the constant speed. Since p_2 is necessarily positive in this singular solution, the singular solution can not arise in the final stage (in the sense of forward time) of control interval. The singular solution, therefore, necessarily comes from the regular solution. The transfer from the regular solution can occur at the instant when p_2 overtakes x_2 , or namely at t_2 . From Eqs. (20) and (22), we obtain

$$x_2(t_2) = \frac{1}{2} \{ (-\mu_1) - \sqrt{(-\mu_1)^2 + 4(-\mu_2)} \} \dots (23)$$

It is clear from the statement mentioned already that if $x_2(t_2)$ given above is equal to $\frac{1}{2}(-\mu_1)$, the transfer from the regular solution can occur. Clearly, the condition of transfer is

$$-\mu_2 = -\frac{1}{4}(-\mu_1)^2 \dots (24)$$

which means that the transfer may arise when $\{-\mu_1, -\mu_2\}$ assumes the value on the parabolic part of the borderline between b and c of Fig. 1. Under this condition t_1 expressed by Eq. (19) be-

comes $t_1 = \ln \left\{ \left(\frac{1}{4} \right) (-\mu_1) + 1 \right\}$, while t_2 expressed by Eq. (22) becomes the instant when the state point enters into this singular solution. The instant which shall be denoted by t_s becomes

$$t_s = \ln \left\{ \frac{1}{2} (-\mu_1) + 2 \right\} \dots (25)$$

Then, it can be concluded that $t_s = t_1 + \ln 2$ and $x_2(t_s) = 2x_2(t_1)$. From the above fact, the forward time optimal trajectory including type 3 singular solution turns out as follows. Assume the state point to be in this singular solution at first, then the train to be running at constant speed of some value below v_m . Operation will be switched to the one of $u=0$ (coasting) at certain instant, after which the train speed will decrease gradually due to the rolling resistance. The operation will next be switched to the one of $u=-1$ (maximum brake) at the time when the train speed becomes one half of the value of constant speed in the singular solution. After running for a while under maximum brake, the train will arrive at the next station just at t_f and stop there. It may be added that the coasting period is always $\ln 2$.

Now, let us find the equation of optimal backward time trajectory in this case. As in the case of regular solution, $u=-1$ is applied in the early stage of backward time, and after t_1 , $u=0$ is applied. The solution can enter into this singular solution at $t=t_s$ by changing u to $\frac{1}{2}(-\mu_1)$, namely to the value of train speed $x_2(t_s)$. This singular solution may be kept during any period T_s , and at the later time T_s the state point can escape from this singular solution into regular solution by applying $u=u_T/u_B$ or $u=0$. In due time x_2 will reach 0 or v_m in accordance to the application of $u=u_T/u_B$ or $u=0$.

There are, therefore, two families of trajectories which include type 3 singular solution: one is the family of trajectories with $u=u_T/u_B$ after singular solution, and the other is the one with $u=0$ after singular solution. It can be seen that each family has two parameters t_1 and t_s , and thus has an area in the cross section with $x_3=T$ plane. The shape of the cross section changes discontinuously for the gradual change of the value of T as shown in Fig. 5.

3.4 Summary

Now we can say that all possible backward time trajectories emanating from the origin have been tracked as far as x_2 becomes v_m or 0, and the optimal control has been obtained simultaneously for every point on the trajectories. Moreover, we obtained cross sections of Figs. 3 ~ 5. Since these cross sections do not overlap each other, they can be

Fig. 4 Cross sections of type 2 singular solution trajectories

superposed on one sheet of cross section. The sheet of cross section thus obtained is nothing but the optimal control law diagram for the initial points such that the state point will be able to reach the origin directly, or in other words, the train speed never amounts to the limit speed during the control interval. Insertion of the superposed cross section, however, is omitted here in order to avoid redundancy.

4. Continuity of optimal backward time trajectory

Equation (2) $x_t = g(x)$ reduces to $x_2 = v_m$ in this problem and hence $\partial g(x)/\partial x_k$, $k=1, 2$, are zero. Equation (5), therefore, reduces to $-v_m + u = 0$. As mentioned already, there are three kinds of trajectory which reaches the boundary $x_2 = v_m$ from the admissible region. Continuity condition for each kind of trajectory will be examined in the following.

4.1 Regular solution

If $\{-\mu_1, -\mu_2\}$ assumes the value in the domain a of Fig. 1, control is definite as $u = -1$ at the point where the trajectory reaches the boundary $x_2 = v_m$ and it is easily seen that the continuity condition is not satisfied. Apparently, if $\{-\mu_1, -\mu_2\}$ assumes the value in the domain c of Fig. 1, the trajectory never reaches the boundary $x_2 = v_m$ but $x_2 = 0$. Even if $\{-\mu_1, -\mu_2\}$ assumes the value in the domain b of Fig. 1, the continuity condition is not generally satisfied because the control is definite as $u = 0$ at the arrival point. If $\{-\mu_1, -\mu_2\}$, however, assumes the value on the borderline $-\mu_2 = -v_m(-\mu_1)$ of domain b, the arrival point at $x_2 = v_m$ just coincides with the first switching point and the control becomes indefinite within the range between -1 and 0 . However, since this range does not include the control $u = v_m$ which can move the state point on the boundary, the continuity condition is not satisfied in this case either. Finally, consider the case where $\{-\mu_1, -\mu_2\}$ assumes the value on the other borderline $-\mu_2 = -v_m(-\mu_1) + v_m^2$ of domain b. The arrival point at $x_2 = v_m$ will now coincide with the second switching point and the control becomes indefinite within the range between 0 and u_T/u_B . Control $u = v_m$ is now included in this range, hence

the continuity condition is satisfied in this case. The conclusion is that the state point can go on the boundary $x_2 = v_m$ by switching u to v_m from 0 , which has been the value of u so far, at the arrival point. Positions $x(A)$ of arrival point and costate vector $p(A)$ are expressed by

$$\begin{aligned} x(A) &= \begin{bmatrix} -v_m + 1 \\ v_m \\ t_1 + \ln\{v_m/(e^{t_1} - 1)\} \end{bmatrix}, \\ p(A) &= \begin{bmatrix} v_m^2/(v_m + 1 - e^{t_1}) \\ v_m \\ v_m^2(e^{t_1} - 1)/(v_m + 1 - e^{t_1}) \end{bmatrix}, \dots\dots\dots (26) \end{aligned}$$

with t_1 being a parameter, where, the range of the variation of t_1 is $\ln\left(\frac{1}{2}v_m + 1\right) \sim \ln(v_m + 1)$.

4.2 Type 2 singular solution

In type 2 singular solution, control u is indefinite within the range between -1 and 0 . Of course, this is also true when the trajectory meets the boundary $x_2 = v_m$, and hence the continuity condition is not satisfied.

4.3 Type 3 singular solution

As mentioned already, type 3 singular solution appears when $\{-\mu_1, -\mu_2\}$ assumes the value on the parabolic part of the borderline between domain b and c of Fig. 1. It is easily seen that when $-\mu_1$ is less than $2v_m$ the value of constant speed in singular solution is less than v_m , and hence the trajectory which meets the boundary $x_2 = v_m$ is necessarily the part of the trajectory which has escaped from this singular solution under the control $u = 0$. The continuity condition is not fully satisfied in this case. If $-\mu_1 = 2v_m$, however, the point at which the solution is just entering into this singular solution is located on $x_2 = v_m$. Since the singular solution in this case is nothing but the movement on the boundary $x_2 = v_m$, it may be right to say that the continuity condition is satisfied in this case. However, since the value of $\{-\mu_1, -\mu_2\}$ in this case, namely $\{2v_m, -v_m^2\}$ is just covered by the range of $\{-\mu_1, \mu_2\}$ considered in section 4.1, the connecting point need not be regarded as newly appearing.

Summarizing the above statement, the continuity condition is satisfied only when $\{-\mu_1, -\mu_2\}$ can be assumed on the straight part of the borderline

Fig. 5 Cross sections of trajectories involving type 3 singular solution

between domain b and c of Fig. 1. It should be added that the state vector and the costate vector at the arrival point are given by Eq. (26).

5. Tracking of optimal backward time trajectories on boundary

State vector on the boundary is a 2-vector denoted by $\{x_1, x_3\}$ and the state vector differential equation reduces to

$$\begin{cases} \dot{x}_1 = v_m \\ \dot{x}_3 = -1 \end{cases} \quad (27)$$

It should be noticed that Eq. (27) does not contain u and the optimization problem on the boundary has degenerated to a mere analysis of free motion. The reason why u disappears from the equations of motion on the boundary lies in the fact that the control in this problem is not a vector but a scalar, or speaking in more detail, the control is determined as $u=v_m$ only from the condition that the state point should move on the boundary

The backward time equations are written as

$$\begin{cases} \dot{x}_1 = -v_m \\ \dot{x}_3 = 1 \end{cases} \quad (28)$$

Of course, costate vector too is a 2-vector and since, as is seen easily, the differential equations for costate variables become $\dot{\phi}_1=0$ and $\dot{\phi}_3=0$, ϕ_1 and ϕ_3 are both constants. Let B be an arbitrary point on the trajectory on the boundary, then $\underline{x}(B)$ and $\underline{\phi}(B)$ will be calculated as

$$\begin{cases} \underline{x}(B) = \begin{bmatrix} (-v_m + t_1) - v_m(t - t_1) \\ + v_m \ln\{v_m/(e^{t_1} - 1)\} \end{bmatrix} \\ \underline{p}(B) = \begin{bmatrix} v_m^2/(v_m + 1 - e^{t_1}) \\ v_m^2(e^{t_1} - 1)/(v_m + 1 - e^{t_1}) \end{bmatrix} \end{cases} \quad (29)$$

By the way, since the time when the state point arrived at the boundary from the admissible region is expressed by $t_1 + \ln\{v_m/(e^{t_1} - 1)\}$, it turns out that the state point has moved on the boundary during $T_B = t - t_1 - \ln\{v_m/(e^{t_1} - 1)\}$ till the present time t .

Although the trajectories on the boundary have a parameter t_1 , the cross section with the plane $x_3=T$ is apparently a straight line $x_2=v_m$.

6. Branch-off of optimal backward time trajectory

Assuming that the trajectory on the boundary can branch off into the admissible region at point B, we obtain

$$\begin{cases} p_1(B) = \phi_1(B) = v_m^2/(v_m + 1 - e^{t_1}) \\ p_2(B) = \rho \\ p_3(B) = \phi_3(B) = v_m^2(e^{t_1} - 1)/(v_m + 1 - e^{t_1}) \end{cases} \quad (30)$$

from Eqs. (9) and (29). Now, Hamiltonian at this point is

$$H = v_m^2/(v_m + 1 - e^{t_1}) v_m + \rho(-v_m + u) - v_m^2(e^{t_1} - 1)/(v_m + 1 - e^{t_1}) - \frac{1}{2}(u + |u|)v_m \quad (31)$$

From the continuity condition, it is necessary that u which maximizes H expressed by Eq. (31) and makes it zero do surely move the state point at B into the admissible region in order that the trajectory can branch off into the admissible region. First of all, it is seen that u which maximizes H is described by

$$u = \begin{cases} u_T/u_B, & \rho > v_m \\ [0, u_T/u_B], & \rho = v_m \\ 0, & 0 < \rho < v_m \\ [-1, 0], & \rho = 0 \\ -1, & \rho < 0 \end{cases} \quad (32)$$

in terms of unknown parameter ρ . It may be easily seen that ρ and u which satisfy Eq. (32) and simultaneously make the value of H zero are confined to $\rho=v_m$ and $u=[0, u_T/u_B]$. On the other hand, it is apparent that the control which moves the state point on the boundary into the admissible region must be in the range between v_m and u_T/u_B . Since $u=[0, u_T/u_B]$ does surely include such a control, the branch-off condition is satisfied at point B. The conclusion is that the state point can enter into the admissible region from the arbitrary point B by applying any control from the range between v_m and u_T/u_B . As mentioned above, ρ , namely $p_2(B)$, is v_m and the position $x(B)$ of branch-off point and the costate vector $p(B)$ are given by

$$\begin{cases} x(B) = \begin{bmatrix} -v_m + t_1 - v_m T_B \\ v_m \\ t_1 + \ln\{v_m/(e^{t_1} - 1)\} + T_B \end{bmatrix} \\ p(B) = \begin{bmatrix} v_m^2/(v_m + 1 - e^{t_1}) \\ v_m \\ v_m^2(e^{t_1} - 1)/(v_m + 1 - e^{t_1}) \end{bmatrix} \end{cases} \quad (33)$$

7. The second tracking of optimal backward time trajectory

Although the state point at point B branches off into the admissible region by applying any control from the range between v_m and u_T/u_B , optimal control will be definite as $u=u_T/u_B$ immediately after the state point branches off. The reason is stated below. The values of x_2 and p_2 at the time Δt after the state point branches off are expressed by

$$\begin{cases} x_2(t + \Delta t) = v_m + v_m \Delta t - u \Delta t \\ p_2(t + \Delta t) = v_m + \{v_m^2/(v_m + 1 - e^{t_1})\} \Delta t - v_m \Delta t - u \Delta t \end{cases} \quad (34)$$

Since the range of variation of t_1 is $\ln\left(\frac{1}{2}v_m + 1\right) \sim \ln(v_m + 1)$ as mentioned already, the situation of $p_2(t + \Delta t) > x_2(t + \Delta t)$ is assured and hence u , the optimal control, must assume definite value of u_T/u_B . After branch-off, x_2 decreases, but it can be shown that the situation $p_2 > x_2$ surely continues until x_2 reaches 0, or in other words $u=u_T/u_B$ is applied all the time after branch-off.

The trajectory of the second motion in the admissible region after branch-off is obtained as below.

$$\left. \begin{aligned} &\text{for } t \text{ of } t_1 + \ln\{v_m/(e^{t_1}-1)\} + T_B \leq t \leq t_1 + \ln\{v_m/(e^{t_1}-1)\} + T_B + \ln\{(u_T/u_B)/(u_T/u_B - v_m)\} \\ &x_1 = t_1 - v_m T_B + (u_T/u_B)[t_1 + \ln\{v_m/(e^{t_1}-1)\} + T_B - 1] - (u_T/u_B)t + (u_T/u_B - v_m)(e^{t_1}-1)/v_m \cdot e^{-t_1} - T_B + t, \\ &x_2 = u_T/u_B - (u_T/u_B - v_m)/v_m \cdot e^{-t_1} - T_B + t, \quad x_3 = t, \quad u = u_T/u_B \end{aligned} \right\} \quad \dots\dots\dots (35)$$

Eq. (35) expresses a family of trajectories with two parameters t_1 and T_B and thus has an area in the cross section with $x_3 = T$ plane. The shape of the cross section changes discontinuously for the gradual change of the value of T as shown in Fig. 6.

8. Optimal control law

We have completed the tracking of all possible optimal backward time trajectories through the admissible region, on the boundary and through the admissible region again and have obtained cross sections with $x_3 = T$ plane. As mentioned already, three kinds of cross section corresponding to the three families of trajectories within the admissible region, Figs. 3~5, do not overlap each other, and it is moreover seen that either Fig. 6 does not overlap any one of them. This fact means that all possible backward time trajectories emanating from the origin except type 2 singular solution do not intersect each other in the three dimensional state space. Hence the diagram which indicates all possible trajectories just as it is results in an optimal control law diagram. Thus, by superposing Figs. 3~6 into one sheet, the optimal control law for this problem is obtained as shown in Fig. 7.

How to make use of Fig. 7 in order to execute optimal operation of train is explained here. From the specified arrival time t_f and the present instant t , $T = t_f - t$ is continually determined. Choose a proper diagram corresponding to T among all diagrams shown in Fig. 7, and plot $\{x_1, x_2\}$, or namely the present train position and the present train

speed into the diagram, then optimal control u is immediately obtained. To apply u thus found to the train is, of course, the optimal operation at that instant. Applying the control instructed by the optimal control law, or namely Fig. 7 continually in this way, the train will surely arrive at the next station just at the specified time and stop there. Important is the condition that moreover the energy consumption on the way to the next station be surely minimum.

A time function which describes the time evolution of optimal control through all the control interval is called optimal control sequence. It indicates, for example, when coasting should be commenced or when maximum brake operation should be started. It is, however, to be noticed here that the driver of the train need not worry about such a control sequence in the future at all. What is necessary for him to do is to apply the control just instructed by optimal control law all the time. It is true that an operation such that $u(t)$ generated according to an optimal control sequence is applied to the train

brings about a correct optimal control if no disturbance influences the system, but since the operation does not consider any counter plan against disturbance, the operation should be called open loop control scheme. On the other hand, an operation such that u just instructed by the optimal control law is applied to the train all the time is surely a closed loop control scheme and is considered the best possible control scheme.

The optimal operation mentioned in this paper may be performed through the driver or may be performed automatically through the inert device if some degree of complication of the actual control device is not minded.

9. Conclusion

The problem to find an optimal train operation which minimizes the energy consumption is solved in this paper. Since the speed limit is strictly observed in the train operation, the problem must be considered a bounded state variable problem. Although to solve the bounded state variable problem has been extremely troublesome, it is not so, now that the author has recently developed a very simple and clear method. The method is applied here and an optimal control law has been found straightforward and clear-cut as expected. Author's method is again confirmed to be very convenient and powerful.

Considerable idealization has been made on the equations of motion for the train in this report, but the basic point seems to have been revealed about the optimal operation of train. The author believes that the report will serve to make the beginning of scientific and reliable research on the economization of train operation for which huge amount of energy is consumed everyday.

Appendix

$$\begin{aligned}
 \Gamma_0: & x_1 = -e^{t_1} + t_1 + 1, \quad x_2 = e^{t_1} - 1 \\
 \Gamma_1: & x_1 = -(1 - e^{-t_1})e^T + t_1, \quad x_2 = (1 - e^{-t_1})e^T \\
 \Gamma_2: & x_1 = -\{1 - (1 + u_T/u_B)e^{-t_1}\}e^T + (1 + T) \cdot u_T/u_B \\
 & \quad + (1 + u_T/u_B)t_1 \\
 & x_2 = \{1 - (1 + u_T/u_B)e^{-t_1}\}e^T + u_T/u_B \\
 \Gamma_3: & x_1 = (u_T/u_B - v_m)/v_m \cdot (1 - e^{-t_1})e^T \\
 & \quad + (1 + u_T/u_B)t_1 + (u_T/u_B) \cdot \ln\{v_m/(e^{t_1} - 1)\} \\
 & \quad - (1 + T) \cdot u_T/u_B \\
 & x_2 = -(u_T/u_B - v_m)/v_m \cdot (1 - e^{-t_1})e^T + u_T/u_B \\
 \Gamma_4: & x_1 = \left(1 + \frac{1}{2}u_T/u_B\right)e^{-t_1}e^T + (1 + u_T/u_B)t_1 - e^T \\
 & \quad - u_T/u_B \cdot T - u_T/u_B + u_T/u_B \cdot \ln 2 \\
 & x_2 = -\left(1 + \frac{1}{2}u_T/u_B\right)e^{-t_1}e^T + e^T + u_T/u_B \\
 \Gamma_5: & x_1 = -2(e^{t_1} - 1)(T - t_1 - \ln 2 + 1) + t_1, \\
 & x_2 = 2(e^{t_1} - 1) \\
 \Gamma_6: & x_1 = \{(u_T/u_B - v_m)/(v_m + 2)\}e^{T - T_s} \\
 & \quad + (u_T/u_B - v_m)e^{T_s} - (1 + T) \cdot u_T/u_B \\
 & \quad + \ln\left(\frac{1}{2}v_m + 1\right) + u_T/u_B \cdot \ln(v_m + 2) \\
 & x_2 = -\{(u_T/u_B - v_m)/(v_m + 2)\}e^{T - T_s} + u_T/u_B \\
 \Gamma_7: & x_1 = \{(u_T/u_B - v_m)/(v_m + 1)\}e^{T - T_s - v_m T_B} \\
 & \quad + (1 + u_T/u_B)\ln(v_m + 1) + (T_B - T - 1) \\
 & \quad \times u_T/u_B \\
 & x_2 = -\{(u_T/u_B - v_m)/(v_m + 1)\}e^{T - T_s - v_m T_B}
 \end{aligned}$$

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