Problems on the Finite Hollow Cylinders under the Axially Symmetrical Deformations*

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In this paper, the authors consider three-dimensional elastic problems on a circular hollow cylinder under the axially symmetrical deformations.

The problems on the cylinder are classified into twelve fundamental cases according to boundary conditions. The general solution to various problems was obtained by expanding each elementary formula to the condition in a form of a Fourier and Bessel series and by superposing the stress systems.

As numerical examples, the stress distributions and deformations of a finite hollow cylinder subjected to uniform band pressure were calculated by the digital computer of Kanazawa University (NEAC-2230). This pressure is distributed circularly over the end-surface and over the middle zone of the inner or outer-surface.

1. Introduction

As a problem in the three dimensional theory of elasticity, a study on a finite hollow cylinder under the axially symmetrical deformations has been very important for the design of machine elements such as bushes of the roller chain and wheel hubs. Another important study is the analysis of fitting effects of bolts and the strength of dies under the cold working.

According to the Miyamoto's report⁽¹⁾, however, this problem is very difficult because the boundaries of cylinder are presented by the three kinds of variables. But, Imai⁽²⁾, Saga and Nojima⁽³⁾, applying the Filon's method, gave approximate solutions, Kudo and Matsubara⁽⁴⁾ also showed the stress distribution of cylindrical dies by using the relaxation methods.

On the other hand, recently, the experimental results have been reported on the thick cylinders of steel by Back⁽⁵⁾, on ones of aluminium by Saga⁽⁶⁾. The same analyses have been done by the authors⁽⁷⁾ using the photoelastic technique.

In this paper, the authors gave the general solutions on the finite hollow cylinders under the axially symmetrical load systems by using the stress function reported in the previous paper⁽⁸⁾ and by applying

the Saito's theory⁽⁹⁾ on the short circular cylinder and the Nakahara's theory⁽¹⁰⁾ on the infinite plate with a hole.

As an example, the stress distributions and deformations of the finite hollow cylinders subjected to uniform band pressure were calculated by the digital computer of Kanazawa University (NEAC-2230). This pressure is distributed circularly over each end-surface and over the inner or outer-surface.

2. Basic equations

Denoting the co-ordinate and stress notations as shown in Fig. 1, the equations of equilibrium in case of axisymmetry without torsion become

Fig. 1 Coordinates and stress notations in a solid of revolution

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$$\left. \begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} &= 0 \end{aligned} \right\} \dots\dots\dots(1)$$

and the compatibility equations are

$$\left. \begin{aligned} \nabla^2 \sigma_r - \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{\partial^2 \theta}{\partial r^2} &= 0 \\ \nabla^2 \sigma_\theta + \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{1}{r} \frac{\partial \theta}{\partial r} &= 0 \\ \nabla^2 \sigma_z + \frac{1}{1+\nu} \frac{\partial^2 \theta}{\partial z^2} &= 0 \\ \nabla^2 \tau_{rz} - \frac{1}{r^2} \tau_{rz} + \frac{1}{1+\nu} \frac{\partial^2 \theta}{\partial r \partial z} &= 0 \end{aligned} \right\} \dots\dots\dots(2)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad \theta = \sigma_r + \sigma_\theta + \sigma_z$$

ν : Poisson's ratio

By introducing a stress function ϕ , we have the following stress components. ⁽¹⁾

$$\left. \begin{aligned} \sigma_r &= \frac{\partial}{\partial z} \left(\nu \nabla^2 \phi - \frac{\partial^2 \phi}{\partial r^2} \right) \\ \sigma_\theta &= \frac{\partial}{\partial z} \left(\nu \nabla^2 \phi - \frac{1}{r} \frac{\partial \phi}{\partial r} \right) \\ \sigma_z &= \frac{\partial}{\partial z} \left\{ (2-\nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} \\ \tau_{rz} &= \frac{\partial}{\partial r} \left\{ (1-\nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right\} \end{aligned} \right\} \dots\dots\dots(3)$$

and the radial and axial displacements; u , w are

$$\left. \begin{aligned} u &= -\frac{1+\nu}{E} \frac{\partial^2 \phi}{\partial r \partial z} \\ w &= \frac{1+\nu}{E} \left\{ (1-2\nu) \nabla^2 \phi + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right\} \end{aligned} \right\} \dots\dots\dots(4)$$

Substituting Eqs. (3) into Eqs. (1) and (2), we obtain

$$\nabla^2 \nabla^2 \phi = 0 \dots\dots\dots(5)$$

Consequently, the solution of three-dimensional problems under the axially symmetrical deformations, when body forces are absent, is reduced to the

Fig. 2 Coordinates and stress notations in a finite hollow cylinder

integration of the differential equation (5).

Now, let us take the stress function in the form

$$\phi = F(r, z) + f(\beta r) \cos(\beta z + \alpha) + g(\gamma z) \{A J_0(\gamma r) + B Y_0(\gamma r)\} \dots\dots\dots(6)$$

where

$$\begin{aligned} f(\beta r) &= A' I_0(\beta r) + B' \beta r I_1(\beta r) \\ &\quad + A'' K_0(\beta r) + B'' \beta r K_1(\beta r) \\ g(\gamma z) &= C' \sinh \gamma z + D' \gamma z \cosh \gamma z \\ &\quad + C'' \cosh \gamma z + D'' \gamma z \sinh \gamma z \end{aligned}$$

and $F(r, z)$ is a polynomial expression of r and z , $J_0(\gamma r)$ and $Y_0(\gamma r)$ are the Bessel functions of first and second kind, $I_0(\beta r)$ and $K_0(\beta r)$ are the modified Bessel functions of first and second kind.

3. Classification of the boundary conditions

Considering now that the boundary conditions on the cylinders are given only by the stresses, we can classify the conditions into the normal and shear stresses. Accordingly the problems of load system in practical cases are analyzed by using the principle of superposition. That is, corresponding the stress distributions on the boundary surfaces, we have twelve cases for the finite hollow cylinders as follows. The coordinates and stress notations of the cylinder are presented in Fig. 2.

(1) On the cylinder subjected to the normal stress.

- i. In the cases of even functional distribution of the stress.
Boundary conditions 1st, 2nd and 3rd.
- ii. In the cases of odd functional distribution of the stress.
Boundary conditions 4th, 5th and 6th.

(2) On the cylinder subjected to the shearing stress.

- i. In the cases of even functional distribution of the stress.
Boundary conditions 7th, 8th and 9th.
- ii. In the cases of odd functional distribution of the stress.

Table 1 Classifications of the boundary conditions

Stresses	Stress distribution on the z-axis	σ_r	σ_z	τ_{rz}
Boundaries				
Inner surface ($r=a$)	Even function	1		7
	Odd function	4		10
Outer surface ($r=b$)	Even function	2		8
	Odd function	5		11
End surfaces ($z=\pm h$)	Even function		3	9
	Odd function		6	12

of the stress.

Boundary conditions 10th, 11th and 12th.

These conditions are shown practically in Table 1 according to the boundary surfaces. The tractions on these surfaces are expanded in a series of Fourier and Bessel functions and the stress system for each case is analysed and combining these stress systems, the general solutions are obtained.

Now, in this paper, an elaborate study will be made on the problems of the boundary conditions 1st, 2nd and 3rd as follows:

(1) Boundary condition 1st

$$\left. \begin{aligned} \sigma_r &= a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi z}{h}, \quad \tau_{rz} = 0 \text{ for } r=a \\ \sigma_r &= \tau_{rz} = 0 \text{ for } r=b \\ \sigma_z &= \tau_{rz} = 0 \text{ for } z=\pm h \end{aligned} \right\} \dots (7)$$

(2) Boundary condition 2nd

$$\left. \begin{aligned} \sigma_r &= \tau_{rz} = 0 \text{ for } r=a \\ \sigma_r &= b_0 + \sum_{n=1}^{\infty} b_n \cos \frac{n\pi z}{h}, \quad \tau_{rz} = 0 \text{ for } r=b \\ \sigma_z &= \tau_{rz} = 0 \text{ for } z=\pm h \end{aligned} \right\} \dots (8)$$

(3) Boundary condition 3rd

$$\left. \begin{aligned} \sigma_r &= \tau_{rz} = 0 \text{ for } r=a \\ \sigma_r &= \tau_{rz} = 0 \text{ for } r=b \\ \sigma_z &= h_0 + \sum_{s=1}^{\infty} h_s C_0(\gamma_s r), \quad \tau_{rz} = 0 \text{ for } z=\pm h \end{aligned} \right\} \dots (9)$$

where $C_0(\gamma_s r)$ is the cylindrical function that contains $J_0(\gamma_s r)$ and $Y_0(\gamma_s r)$ of Bessel functions and it is denoted later in this paper.

4. Problems on the boundary conditions 1st, 2nd and 3rd

According to Eq. (6) as the stress function, we find

$$\begin{aligned} \phi &= A_0 \frac{z^3}{6} + B_0 z \ln r + C_0 \frac{zr^2}{2} \\ &+ \sum_{n=1}^{\infty} \frac{1}{\beta_n^3} \left\{ A_n' I_0(\beta_n r) + B_n' \beta_n r I_1(\beta_n r) + A_n'' K_0(\beta_n r) + B_n'' \beta_n r K_1(\beta_n r) \right\} \sin \beta_n z \\ &+ \sum_{s=1}^{\infty} \frac{1}{\gamma_s^3} (C_s' \sinh \gamma_s z + D_s' \gamma_s z \cosh \gamma_s z) \left\{ J_0(\gamma_s r) - J_1(\gamma_s b) Y_0(\gamma_s r) / Y_1(\gamma_s b) \right\} \dots (10) \end{aligned}$$

where β_n , γ_s are $n\pi/h$ and λ_s/a , then λ_s is the positive root of the following equation.

$$J_1(\lambda_s) Y_1\left(\frac{b}{a} \lambda_s\right) - J_1\left(\frac{b}{a} \lambda_s\right) Y_1(\lambda_s) = 0 \dots (11)$$

This solution λ_s , if b/a is small, can be obtained from the M'Mahon's formula⁽¹²⁾, and if b/a is large, then by using the trial and error method. For reference, the calculated values of λ_s are shown in Table 2 on $b/a=2.0$, 4.0 and 8.0.

Now, from the Bessel functions $Y_\mu(\gamma_s r)$ and $J_\mu(\gamma_s r)$ being the solutions of the same kind of differential equation, we define

$$C_\mu(\gamma_s r) = J_\mu(\gamma_s r) - J_1(\gamma_s b) Y_\mu(\gamma_s r) / Y_1(\gamma_s b) \dots (12)$$

and so, from Eq. (11) we find

$$C_1(\gamma_s a) = C_1(\gamma_s b) = 0 \dots (13)$$

By using the cylindrical function $C_\mu(\gamma_s r)$, the stress function (10) can be rewritten as follows:

$$\begin{aligned} \phi &= A_0 \frac{z^3}{6} + B_0 z \ln r + C_0 \frac{zr^2}{2} \\ &+ \sum_{n=1}^{\infty} \frac{1}{\beta_n^3} \left\{ A_n' I_0(\beta_n r) + B_n' \beta_n r I_1(\beta_n r) + A_n'' K_0(\beta_n r) + B_n'' \beta_n r K_1(\beta_n r) \right\} \sin \beta_n z \\ &+ \sum_{s=1}^{\infty} \frac{1}{\gamma_s^3} (C_s' \sinh \gamma_s z + D_s' \gamma_s z \cosh \gamma_s z) C_0(\gamma_s r) \end{aligned}$$

Table 2 Values of λ_s

s	λ_s of $b/a=2.0$	λ_s of $b/a=4.0$	λ_s of $b/a=8.0$	s	λ_s of $b/a=2.0$	λ_s of $b/a=4.0$	λ_s of $b/a=8.0$
1	3.196 578	1.111 877	0.499 822 1	11	34.562 941	11.527 271	4.946 090
2	6.312 355	2.134 230	0.935 129 1	12	37.704 082	12.573 800	5.394 140
3	9.444 465	3.169 740	1.375 190	13	40.845 293	13.620 430	5.842 300
4	12.581 202	4.210 407	1.818 292	14	43.986 558	14.667 141	6.290 548
5	15.719 854	5.253 493	2.263 174	15	47.127 867	15.713 916	6.738 868
6	18.859 477	6.297 868	2.709 143	16	50.269 211	16.760 743	7.187 245
7	21.999 658	7.343 020	3.155 816	17	53.410 585	17.807 613	7.635 671
8	25.140 190	8.388 668	3.602 965	18	56.551 982	18.854 520	8.084 137
9	28.280 957	9.434 652	4.050 461	19	59.693 401	19.901 457	8.532 637
10	31.421 889	10.480 875	4.498 183	20	62.834 836	20.948 420	8.981 167

Substituting this in Eqs. (3) and (4), we obtain the following formulas of stresses and displacement components, where $I_0(\beta_n r)$, $K_0(\beta_n r)$ and the like are shown as I_{0r} , K_{0r} and so on.

$$\sigma_r = A_0 \nu + \frac{B_0}{r^2} + C_0(2\nu - 1) + \sum_{n=1}^{\infty} A_n \cos \beta_n z + \sum_{s=1}^{\infty} \left\{ B_s C_0(\gamma_s r) + C_s C_1(\gamma_s r) / \gamma_s r \right\} \quad (14)$$

where

$$\left. \begin{aligned} A_n &= A_n' (-I_{0r} + I_{1r} / \beta_n r) + B_n' \{ -(1-2\nu) I_{0r} - \beta_n r I_{1r} \} \\ &\quad + A_n'' (-K_{0r} - K_{1r} / \beta_n r) + B_n'' \{ (1-2\nu) K_{0r} - \beta_n r K_{1r} \} \\ B_s &= C_s' \cosh \gamma_s z + D_s' \{ (1+2\nu) \cosh \gamma_s z + \gamma_s z \sinh \gamma_s z \} \\ C_s &= -C_s' \cosh \gamma_s z - D_s' (\cosh \gamma_s z + \gamma_s z \sinh \gamma_s z) \end{aligned} \right\} \quad (14)'$$

$$\sigma_\theta = A_0 \nu - \frac{B_0}{r^2} + C_0(2\nu - 1) + \sum_{n=1}^{\infty} D_n \cos \beta_n z + \sum_{s=1}^{\infty} \left\{ E_s C_0(\gamma_s r) + F_s C_1(\gamma_s r) / \gamma_s r \right\} \quad (15)$$

where

$$\left. \begin{aligned} D_n &= -A_n' I_{1r} / \beta_n r - (1-2\nu) B_n' I_{0r} + A_n'' K_{1r} / \beta_n r + (1-2\nu) B_n'' K_{0r} \\ E_s &= 2\nu D_s' \cosh \gamma_s z \\ F_s &= C_s' \cosh \gamma_s z + D_s' (\cosh \gamma_s z + \gamma_s z \sinh \gamma_s z) \end{aligned} \right\} \quad (15)'$$

$$\sigma_z = A_0(1-\nu) + 2C_0(2-\nu) + \sum_{n=1}^{\infty} G_n \cos \beta_n z + \sum_{s=1}^{\infty} H_s C_0(\gamma_s r) \quad (16)$$

where

$$\left. \begin{aligned} G_n &= A_n' I_{0r} + B_n' \{ 2(2-\nu) I_{0r} + \beta_n r I_{1r} \} + A_n'' K_{0r} + B_n'' \{ -2(2-\nu) K_{0r} + \beta_n r K_{1r} \} \\ H_s &= -C_s' \cosh \gamma_s z + D_s' \{ (1-2\nu) \cosh \gamma_s z - \gamma_s z \sinh \gamma_s z \} \end{aligned} \right\} \quad (16)'$$

$$\tau_{rz} = \sum_{n=1}^{\infty} L_n \sin \beta_n z + \sum_{s=1}^{\infty} M_s C_1(\gamma_s r) \quad (17)$$

where

$$\left. \begin{aligned} L_n &= A_n' I_{1r} + B_n' \{ 2(1-\nu) I_{1r} + \beta_n r I_{0r} \} - A_n'' K_{1r} + B_n'' \{ 2(1-\nu) K_{1r} - \beta_n r K_{0r} \} \\ M_s &= C_s' \sinh \gamma_s z + D_s' (2\nu \sinh \gamma_s z + \gamma_s z \cosh \gamma_s z) \end{aligned} \right\} \quad (17)'$$

$$u = -\frac{1+\nu}{E} \left(C_0 r + \frac{B_0}{r} \right) + \frac{1+\nu}{E} \sum_{n=1}^{\infty} N_n \cos \beta_n z + \frac{1+\nu}{E} \sum_{s=1}^{\infty} O_s C_1(\gamma_s r) \quad (18)$$

where

$$\left. \begin{aligned} N_n &= \frac{1}{\beta_n} (-A_n' I_{1r} - B_n' \beta_n r I_{0r} + A_n'' K_{1r} + B_n'' \beta_n r K_{0r}) \\ O_s &= \frac{1}{\gamma_s} \{ C_s' \cosh \gamma_s z + D_s' (\cosh \gamma_s z + \gamma_s z \sinh \gamma_s z) \} \end{aligned} \right\} \quad (18)'$$

$$w = \frac{1+\nu}{E} \left\{ (1-2\nu) A_0 + 4(1-\nu) C_0 \right\} z + \frac{1+\nu}{E} \sum_{n=1}^{\infty} P_n \sin \beta_n z + \frac{1+\nu}{E} \sum_{s=1}^{\infty} Q_s C_0(\gamma_s r) \quad (19)$$

where

$$\left. \begin{aligned} P_n &= \frac{1}{\beta_n} \left[A_n' I_{0r} + B_n' \{ 4(1-\nu) I_{0r} + \beta_n r I_{1r} \} + A_n'' K_{0r} - B_n'' \{ 4(1-\nu) K_{0r} - \beta_n r K_{1r} \} \right] \\ Q_s &= -\frac{1}{\gamma_s} \left[C_s' \sinh \gamma_s z + D_s' \{ \gamma_s z \cosh \gamma_s z - 2(1-2\nu) \sinh \gamma_s z \} \right] \end{aligned} \right\} \quad (19)'$$

For any boundary condition 1st, 2nd or 3rd, the inner, outer cylindrical surfaces $r=a$, b and both end planes $z=\pm h$ of the finite cylinder are free from shearing force, then we ought to have

$$L_n|_{r=a}=0, \quad L_n|_{r=b}=0, \quad M_s|_{z=\pm h}=0 \quad (20)$$

or using Eq. (17)

$$\left. \begin{aligned} A_n' I_{1a} + B_n' \{ 2(1-\nu) I_{1a} + \beta_n a I_{0a} \} - A_n'' K_{1a} + B_n'' \{ 2(1-\nu) K_{1a} - \beta_n a K_{0a} \} &= 0 \\ A_n' I_{1b} + B_n' \{ 2(1-\nu) I_{1b} + \beta_n b I_{0b} \} - A_n'' K_{1b} + B_n'' \{ 2(1-\nu) K_{1b} - \beta_n b K_{0b} \} &= 0 \end{aligned} \right\} \quad (21)$$

$$C_s' \sinh \gamma_s h + D_s' (2\nu \sinh \gamma_s h + \gamma_s h \cosh \gamma_s h) = 0 \quad (22)$$

Now, considering that the first term of series in Eq. (14), when r equals to a or b , is shown as

$\sum_{n=1}^{\infty} A_n \cos \beta_n z$ or $\sum_{n=1}^{\infty} B_n \cos \beta_n z$, we find

$$\left. \begin{aligned} A_n' (-I_{0a} + I_{1a} / \beta_n a) + B_n' \{ -(1-2\nu) I_{0a} - \beta_n a I_{1a} \} + A_n'' (-K_{0a} - K_{1a} / \beta_n a) \\ + B_n'' \{ (1-2\nu) K_{0a} - \beta_n a K_{1a} \} = A_n \\ A_n' (-I_{0b} + I_{1b} / \beta_n b) + B_n' \{ -(1-2\nu) I_{0b} - \beta_n b I_{1b} \} + A_n'' (-K_{0b} - K_{1b} / \beta_n b) \\ + B_n'' \{ (1-2\nu) K_{0b} - \beta_n b K_{1b} \} = B_n \end{aligned} \right\} \quad (23)$$

Combining Eqs. (21) with (23), we have the following relations:

$$\left. \begin{aligned} A_n' &= \frac{\Delta_A^1}{\Delta_n} \bar{A}_n + \frac{\Delta_B^1}{\Delta_n} \bar{B}_n, & B_n' &= \frac{\Delta_A^2}{\Delta_n} \bar{A}_n + \frac{\Delta_B^2}{\Delta_n} \bar{B}_n \\ A_n'' &= \frac{\Delta_A^3}{\Delta_n} \bar{A}_n + \frac{\Delta_B^3}{\Delta_n} \bar{B}_n, & B_n'' &= \frac{\Delta_A^4}{\Delta_n} \bar{A}_n + \frac{\Delta_B^4}{\Delta_n} \bar{B}_n \end{aligned} \right\} \dots\dots\dots (24)$$

where $\Delta_A^1, \Delta_A^2, \Delta_A^3, \Delta_A^4, \Delta_B^1, \Delta_B^2, \Delta_B^3$ and Δ_B^4 are the cofactors of $a_3, b_3, c_3, d_3, a_4, b_4, c_4$ and d_4 in the following determinant:

$$\Delta_n = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

and the factors $a_1, a_2, \dots$ are taken as follows:

$$\begin{aligned} a_1 &= I_{1a}, & b_1 &= 2(1-\nu)I_{1a} + \beta_n a I_{0a}, & c_1 &= -K_{1a}, & d_1 &= 2(1-\nu)K_{1a} - \beta_n a K_{0a} \\ a_2 &= I_{1b}, & b_2 &= 2(1-\nu)I_{1b} + \beta_n b I_{0b}, & c_2 &= -K_{1b}, & d_2 &= 2(1-\nu)K_{1b} - \beta_n b K_{0b} \\ a_3 &= -I_{0a} + I_{1a}/\beta_n a, & b_3 &= -(1-2\nu)I_{0a} - \beta_n a I_{1a}, & c_3 &= -K_{0a} - K_{1a}/\beta_n a, & d_3 &= (1-2\nu)K_{0a} - \beta_n a K_{1a} \\ a_4 &= -I_{0b} + I_{1b}/\beta_n b, & b_4 &= -(1-2\nu)I_{0b} - \beta_n b I_{1b}, & c_4 &= -K_{0b} - K_{1b}/\beta_n b, & d_4 &= (1-2\nu)K_{0b} - \beta_n b K_{1b} \end{aligned}$$

Besides, when z equals to $\pm h$, we consider that the second term of series in Eq. (16) is shown as $\sum_{s=1}^{\infty} C_s C_0(\gamma_s r)$.

Hence

$$-C_s' \cosh \gamma_s h + D_s' \{ (1-2\nu) \cosh \gamma_s h - \gamma_s h \sinh \gamma_s h \} = C_s \dots\dots\dots (25)$$

From Eqs. (22) and (25),

$$C_s' = -(2\nu \sinh \gamma_s h + \gamma_s h \cosh \gamma_s h) \frac{\bar{C}_s}{\Omega_s}, \quad D_s' = \sinh \gamma_s h \frac{C_s}{\Omega_s} \dots\dots\dots (26)$$

where

$$\Omega_s = \sinh \gamma_s h \cosh \gamma_s h + \gamma_s h$$

Consequently, as the expressions for the stresses and displacement components, we find the following coefficients by Eqs. (24) and (26).

For σ_r

$$\left. \begin{aligned} A_n &= \frac{\bar{A}_n}{\Delta_n} \left[\Delta_A^1 (-I_{0r} + I_{1r}/\beta_n r) + \Delta_A^2 \{ -(1-2\nu)I_{0r} - \beta_n r I_{1r} \} \right. \\ &\quad \left. + \Delta_A^3 (-K_{0r} - K_{1r}/\beta_n r) + \Delta_A^4 \{ (1-2\nu)K_{0r} - \beta_n r K_{1r} \} \right] \\ &\quad + \frac{\bar{B}_n}{\Delta_n} \left[\Delta_B^1 (-I_{0r} + I_{1r}/\beta_n r) + \Delta_B^2 \{ -(1-2\nu)I_{0r} - \beta_n r I_{1r} \} \right. \\ &\quad \left. + \Delta_B^3 (-K_{0r} - K_{1r}/\beta_n r) + \Delta_B^4 \{ (1-2\nu)K_{0r} - \beta_n r K_{1r} \} \right] \\ B_s &= \frac{\bar{C}_s}{\Omega_s} \{ (\sinh \gamma_s h - \gamma_s h \cosh \gamma_s h) \cosh \gamma_s z + \gamma_s z \sinh \gamma_s h \sinh \gamma_s z \} \\ C_s &= \frac{\bar{C}_s}{\Omega_s} \left[-\gamma_s z \sinh \gamma_s h \sinh \gamma_s z + \{ \gamma_s h \cosh \gamma_s h - (1-2\nu) \sinh \gamma_s h \} \cosh \gamma_s z \right] \end{aligned} \right\} \dots\dots\dots (27)$$

For σ_θ

$$\left. \begin{aligned} D_n &= \frac{\bar{A}_n}{\Delta_n} \left\{ -\Delta_A^1 I_{1r}/\beta_n r - \Delta_A^2 (1-2\nu)I_{0r} + \Delta_A^3 K_{1r}/\beta_n r + (1-2\nu)\Delta_A^4 K_{0r} \right\} \\ &\quad + \frac{\bar{B}_n}{\Delta_n} \left\{ -\Delta_B^1 I_{1r}/\beta_n r - \Delta_B^2 (1-2\nu)I_{0r} + \Delta_B^3 K_{1r}/\beta_n r + (1-2\nu)\Delta_B^4 K_{0r} \right\} \\ E_s &= \frac{\bar{C}_s}{\Omega_s} 2\nu \sinh \gamma_s h \cosh \gamma_s z \\ F_s &= \frac{\bar{C}_s}{\Omega_s} \left[\{ (1-2\nu) \sinh \gamma_s h - \gamma_s h \cosh \gamma_s h \} \cosh \gamma_s z + \gamma_s z \sinh \gamma_s h \sinh \gamma_s z \right] \end{aligned} \right\} \dots\dots\dots (28)$$

For σ_z

$$\left. \begin{aligned} G_n &= \frac{\bar{A}_n}{\Delta_n} \left[\Delta_A^1 I_{0r} + \Delta_A^2 \{ 2(2-\nu)I_{0r} + \beta_n r I_{1r} \} + \Delta_A^3 K_{0r} + \Delta_A^4 \{ -2(2-\nu)K_{0r} + \beta_n r K_{1r} \} \right] \\ &\quad + \frac{\bar{B}_n}{\Delta_n} \left[\Delta_B^1 I_{0r} + \Delta_B^2 \{ 2(2-\nu)I_{0r} + \beta_n r I_{1r} \} + \Delta_B^3 K_{0r} + \Delta_B^4 \{ -2(2-\nu)K_{0r} + \beta_n r K_{1r} \} \right] \\ H_s &= \frac{\bar{C}_s}{\Omega_s} \{ (\gamma_s h \cosh \gamma_s h + \sinh \gamma_s h) \cosh \gamma_s z - \gamma_s z \sinh \gamma_s h \sinh \gamma_s z \} \end{aligned} \right\} \dots\dots\dots (29)$$

For τ_{zr}
$$\left. \begin{aligned} L_n &= \frac{\bar{A}_n}{\Delta_n} \left[\Delta_A^1 I_{1r} + \Delta_A^2 \{2(1-\nu)I_{1r} + \beta_n r I_{0r}\} - \Delta_A^3 K_{1r} + \Delta_A^4 \{2(1-\nu)K_{1r} - \beta_n r K_{0r}\} \right] \\ &+ \frac{\bar{B}_n}{\Delta_n} \left[\Delta_B^1 I_{1r} + \Delta_B^2 \{2(1-\nu)I_{1r} + \beta_n r I_{0r}\} - \Delta_B^3 K_{1r} + \Delta_B^4 \{2(1-\nu)K_{1r} - \beta_n r K_{0r}\} \right] \\ M_s &= \frac{C_s}{\Omega_s} (-\gamma_s h \cosh \gamma_s h \sinh \gamma_s z + \gamma_s z \sinh \gamma_s h \cosh \gamma_s z) \end{aligned} \right\} \dots\dots\dots (30)$$

For displacement u
$$\left. \begin{aligned} N_n &= \frac{\bar{A}_n}{\beta_n \Delta_n} (-\Delta_A^1 I_{1r} - \Delta_A^2 \beta_n r I_{0r} + \Delta_A^3 K_{1r} + \Delta_A^4 \beta_n r K_{0r}) \\ &+ \frac{\bar{B}_n}{\beta_n \Delta_n} (-\Delta_B^1 I_{1r} - \Delta_B^2 \beta_n r I_{0r} + \Delta_B^3 K_{1r} + \Delta_B^4 \beta_n r K_{0r}) \\ O_s &= \frac{\bar{C}_s}{\gamma_s \Omega_s} \left[\{(1-2\nu) \sinh \gamma_s h - \gamma_s h \cosh \gamma_s h\} \cosh \gamma_s z + \gamma_s z \sinh \gamma_s h \sinh \gamma_s z \right] \end{aligned} \right\} \dots\dots\dots (31)$$

For displacement w
$$\left. \begin{aligned} P_n &= \frac{\bar{A}_n}{\beta_n \Delta_n} \left[\Delta_A^1 I_{0r} + \Delta_A^2 \{4(1-\nu)I_{0r} + \beta_n r I_{1r}\} + \Delta_A^3 K_{0r} - \Delta_A^4 \{4(1-\nu)K_{0r} - \beta_n r K_{1r}\} \right] \\ &+ \frac{\bar{B}_n}{\beta_n \Delta_n} \left[\Delta_B^1 I_{0r} + \Delta_B^2 \{4(1-\nu)I_{0r} + \beta_n r I_{1r}\} + \Delta_B^3 K_{0r} - \Delta_B^4 \{4(1-\nu)K_{0r} - \beta_n r K_{1r}\} \right] \\ Q_s &= \frac{\bar{C}_s}{\gamma_s \Omega_s} \left[\{2(1-\nu) \sinh \gamma_s h + \gamma_s h \cosh \gamma_s h\} \sinh \gamma_s z - \gamma_s z \cosh \gamma_s h \sinh \gamma_s z \right] \end{aligned} \right\} \dots\dots\dots (32)$$

From these expressions, the normal stresses at the boundary surfaces of the objective finite hollow cylinder are

$$\sigma_r \Big|_{r=a} = A_0 \nu + \frac{B_0}{a^2} + C_0(2\nu-1) + \sum_{n=1}^{\infty} \bar{A}_n \cos \beta_n z + \sum_{s=1}^{\infty} \frac{\bar{C}_s C_0(\gamma_s a)}{\Omega_s} \left[(\sinh \gamma_s h - \gamma_s h \cosh \gamma_s h) \cosh \gamma_s z + \gamma_s z \sinh \gamma_s h \sinh \gamma_s z \right] \dots\dots\dots (33)$$

$$\sigma_r \Big|_{r=b} = A_0 \nu + \frac{B_0}{b^2} + C_0(2\nu-1) + \sum_{n=1}^{\infty} \bar{B}_n \cos \beta_n z + \sum_{s=1}^{\infty} \frac{\bar{C}_s C_0(\gamma_s b)}{\Omega_s} \left[(\sinh \gamma_s h - \gamma_s h \cosh \gamma_s h) \cosh \gamma_s z + \gamma_s z \sinh \gamma_s h \sinh \gamma_s z \right] \dots\dots\dots (34)$$

$$\sigma_z \Big|_{z=\pm h} = A_0(1-\nu) + C_0 \cdot 2(2-\nu) + \sum_{s=1}^{\infty} \bar{C}_s C_0(\gamma_s r) + \sum_{n=1}^{\infty} (-1)^n \frac{\bar{A}_n}{\Delta_n} \left[\Delta_A^1 I_{0r} + \Delta_A^2 \{2(2-\nu)I_{0r} + \beta_n r I_{1r}\} + \Delta_A^3 K_{0r} - \Delta_A^4 \{2(2-\nu)K_{0r} - \beta_n r K_{1r}\} \right] + \sum_{n=1}^{\infty} (-1)^n \frac{\bar{B}_n}{\Delta_n} \left[\Delta_B^1 I_{0r} + \Delta_B^2 \{2(2-\nu)I_{0r} + \beta_n r I_{1r}\} + \Delta_B^3 K_{0r} - \Delta_B^4 \{2(2-\nu)K_{0r} - \beta_n r K_{1r}\} \right] \dots\dots\dots (35)$$

Expanding each term of Eqs. (33), (34) and (35) in a Fourier and Bessel series, we have

$$\sigma_r \Big|_{r=a} = A_0 \nu + \frac{B_0}{a^2} + C_0(2\nu-1) + \sum_{n=1}^{\infty} \left[\bar{A}_n + \sum_{s=1}^{\infty} \bar{C}_s \frac{(-1)^n 4\gamma_s h (n\pi)^2 \sinh^2 \gamma_s h C_0(\gamma_s a)}{\Omega_s \{(\gamma_s h)^2 + (n\pi)^2\}^2} \right] \cos \beta_n z \dots\dots\dots (36)$$

$$\sigma_r \Big|_{r=b} = A_0 \nu + \frac{B_0}{b^2} + C_0(2\nu-1) + \sum_{n=1}^{\infty} \left[\bar{B}_n + \sum_{s=1}^{\infty} \bar{C}_s \frac{(-1)^n 4\gamma_s h (n\pi)^2 \sinh^2 \gamma_s h C_0(\gamma_s b)}{\Omega_s \{(\gamma_s h)^2 + (n\pi)^2\}^2} \right] \cos \beta_n z \dots\dots\dots (37)$$

$$\sigma_z \Big|_{z=\pm h} = A_0(1-\nu) + C_0 \cdot 2(2-\nu) + \sum_{s=1}^{\infty} \left[\bar{C}_s + \sum_{n=1}^{\infty} \left\{ \frac{\bar{A}_n}{\Delta_n} \frac{2(-1)^n}{b^2 C_0(\gamma_s b)^2 - a^2 C_0(\gamma_s a)^2} \frac{S_A}{\beta_n^2 + \gamma_s^2} + \frac{\bar{B}_n}{\Delta_n} \frac{2(-1)^n}{b^2 C_0(\gamma_s b)^2 - a^2 C_0(\gamma_s a)^2} \frac{S_B}{\beta_n^2 + \gamma_s^2} \right\} \right] C_0(\gamma_s r) \dots\dots\dots (38)$$

where
$$S_A = \{\Delta_A^1 + 2(2-\nu)\Delta_A^2\}(\beta_n b I_{1b} C_{0b} - \beta_n a I_{1a} C_{0a}) + \Delta_A^2 \{\beta_n^2 b^2 I_{0b} C_{0b} - \beta_n^2 a^2 I_{0a} C_{0a} - \frac{2\beta_n^2}{\beta_n^2 + \gamma_s^2} (\beta_n b I_{1b} C_{0b} - \beta_n a I_{1a} C_{0a})\} + \{\Delta_A^3 - 2(2-\nu)\Delta_A^4\}(-\beta_n b K_{1b} C_{0b} + \beta_n a K_{1a} C_{0a}) + \Delta_A^4 \{-\beta_n^2 b^2 K_{0b} C_{0b} + \beta_n^2 a^2 K_{0a} C_{0a} + \frac{2\beta_n^2}{\beta_n^2 + \gamma_s^2} (-\beta_n b K_{1b} C_{0b} + \beta_n a K_{1a} C_{0a})\}$$

$$S_B = \{\Delta_B^1 + 2(2-\nu)\Delta_B^2\}(\beta_n b I_{1b} C_{0b} - \beta_n a I_{1a} C_{0a}) + \Delta_B^2 \{\beta_n^2 b^2 I_{0b} C_{0b} - \beta_n^2 a^2 I_{0a} C_{0a} - \frac{2\beta_n^2}{\beta_n^2 + \gamma_s^2} (\beta_n b I_{1b} C_{0b} - \beta_n a I_{1a} C_{0a})\} + \{\Delta_B^3 - 2(2-\nu)\Delta_B^4\}(-\beta_n b K_{1b} C_{0b} + \beta_n a K_{1a} C_{0a}) + \Delta_B^4 \{-\beta_n^2 b^2 K_{0b} C_{0b} + \beta_n^2 a^2 K_{0a} C_{0a} + \frac{2\beta_n^2}{\beta_n^2 + \gamma_s^2} (-\beta_n b K_{1b} C_{0b} + \beta_n a K_{1a} C_{0a})\}$$

Consider now that these equations (36), (37) and (38) must conform with the boundary conditions completely, hence, from Eqs. (7), (9) and (36), (38), we have the following according to each boundary condition 1st, 2nd or 3rd.

On the problems of 1st boundary condition.

$$\left. \begin{aligned} A_0\nu + \frac{B_0}{a^2} + C_0(2\nu-1) &= a_0, & \bar{A}_n + \sum_{s=1}^{\infty} \bar{C}_s P_{sn} &= a_n, \\ A_0\nu + \frac{B_0}{b^2} + C_0(2\nu-1) &= 0, & \bar{B}_n + \sum_{s=1}^{\infty} \bar{C}_s Q_{sn} &= 0, \\ A_0(1-\nu) + C_0 \cdot 2(2-\nu) &= 0, & \bar{C}_s + \sum_{n=1}^{\infty} (\bar{A}_n R_{ns} + \bar{B}_n S_{ns}) &= 0 \end{aligned} \right\} \dots\dots\dots (39)$$

On the problems of 2nd boundary condition.

$$\left. \begin{aligned} A_0\nu + \frac{B_0}{a^2} + C_0(2\nu-1) &= 0, & \bar{A}_n + \sum_{s=1}^{\infty} \bar{C}_s P_{sn} &= 0, \\ A_0\nu + \frac{B_0}{b^2} + C_0(2\nu-1) &= b_0, & \bar{B}_n + \sum_{s=1}^{\infty} \bar{C}_s Q_{sn} &= b_n, \\ A_0(1-\nu) + C_0 \cdot 2(2-\nu) &= 0, & \bar{C}_s + \sum_{n=1}^{\infty} (\bar{A}_n R_{ns} + \bar{B}_n S_{ns}) &= 0 \end{aligned} \right\} \dots\dots\dots (40)$$

On the problems of 3rd boundary condition.

$$\left. \begin{aligned} A_0\nu + \frac{B_0}{a^2} + C_0(2\nu-1) &= 0, & \bar{A}_n + \sum_{s=1}^{\infty} \bar{C}_s P_{sn} &= 0, \\ A_0\nu + \frac{B_0}{b^2} + C_0(2\nu-1) &= 0, & \bar{B}_n + \sum_{s=1}^{\infty} \bar{C}_s Q_{sn} &= 0, \\ A_0(1-\nu) + C_0 \cdot 2(2-\nu) &= h_0, & \bar{C}_s + \sum_{n=1}^{\infty} (\bar{A}_n R_{ns} + \bar{B}_n S_{ns}) &= h_s \end{aligned} \right\} \dots\dots\dots (41)$$

where

$$\begin{aligned} P_{sn} &= \frac{(-1)^n 4\gamma_s h(n\pi)^2 \sinh^2 \gamma_s h C_0(\gamma_s a)}{\Omega_s \{(\gamma_s h)^2 + (n\pi)^2\}^2}, \\ Q_{sn} &= \frac{(-1)^n 4\gamma_s h(n\pi)^2 \sinh^2 \gamma_s h C_0(\gamma_s b)}{\Omega_s \{(\gamma_s h)^2 + (n\pi)^2\}^2}, \\ R_{ns} &= \frac{2(-1)^n}{A_n \{b^2 C_0(\gamma_s b)^2 - a^2 C_0(\gamma_s a)^2\}} \frac{S_A}{(\beta_n^2 + \gamma_s^2)}, \\ S_{ns} &= \frac{2(-1)^n}{A_n \{b^2 C_0(\gamma_s b)^2 - a^2 C_0(\gamma_s a)^2\}} \frac{S_B}{(\beta_n^2 + \gamma_s^2)} \end{aligned}$$

According to given problems, we can obtain the solutions $\bar{A}_n$, $\bar{B}_n$, $\bar{C}_s$ of infinite simultaneous linear equations (39), (40) or (41), then the stresses and displacements are given from Eqs. (27)~(32).

In the same way, we can resolve all the problems with the other boundary conditions 4th~12th.

5. Numerical examples

Now using the preceding method, we calculate the stresses and displacements of the finite hollow cylinders of which the inner or outer surfaces are subjected to a band of uniform pressure, and also each end surface to a band of axial pressure annularly. Taking $a/b=0.25$, $h/b=1.0$ and Poisson's ratio ν as 0.3, all above numerical examples are calculated by making use of the digital computer of Kanazawa University (NEAC-2230) and the flow-chart is shown in Fig. 3.

5.1 On the cylinder with a band of uniform pressure over the inner surface

As the first example, we consider the case of inner band pressure with $2d$ width as shown in Fig. 4. The boundary condition is as follows by using the Fourier transformation.

Fig. 3 Flow-chart for the numerical calculations

Fig. 4 Finite cylinder with a band of uniform pressure over the inner surface

$$\left. \begin{aligned} \sigma_r &= p \left[d/h + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi d}{h} \cos \frac{n\pi}{h} z \right], \\ \tau_{rz} &= 0 \text{ for } r=a \\ \sigma_r &= \tau_{rz} = 0 \text{ for } r=b \\ \sigma_z &= \tau_{rz} = 0 \text{ for } z=\pm h \end{aligned} \right\} \dots (42)$$

As this is the 1st boundary condition, from the expression (39) we find

$$\left. \begin{aligned} A_0 \nu + \frac{B_0}{a^2} + C_0(2\nu-1) &= -\frac{pd}{h} \\ A_0 \nu + \frac{B_0}{b^2} + C_0(2\nu-1) &= 0 \\ A_0(1-\nu) + 2C_0(2-\nu) &= 0 \\ \bar{A}_n + \sum_{s=1}^{\infty} \bar{C}_s P_{sn} &= \frac{2p}{\pi n} \sin \frac{n\pi d}{h} \\ \bar{B}_n + \sum_{s=1}^{\infty} \bar{C}_s Q_{sn} &= 0 \\ \bar{C}_s + \sum_{n=1}^{\infty} (\bar{A}_n R_{ns} + \bar{B}_n S_{ns}) &= 0 \end{aligned} \right\} \dots (43)$$

Table 3 Values of a_n , $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_s$ (in unit of p)

n, s	a_n	$\bar{A}_n$	$\bar{B}_n$	$\bar{C}_s$
0	0.2	-0.03487	0.2	0.00718
1	0.3742	0.3896	-0.00885	0.09224
2	0.3027	0.2869	0.00968	-0.00089
3	0.2018	0.2133	-0.00743	-0.04527
4	0.09355	0.08503	0.00572	0.01027
5	0	0.00668	-0.00462	-0.00757
6	-0.06237	-0.06783	0.00384	-0.00536
7	-0.08649	-0.08189	-0.00327	0.03434
8	-0.07568	-0.07965	0.00283	-0.00107
9	-0.04158	-0.03810	-0.00248	-0.02257
10	0	-0.00308	0.00221	0.00406
11	0.03402	0.03678	-0.00197	-0.00566
12	0.05046	0.04797	0.00178	-0.00268
13	0.04657	0.04883	-0.00162	0.02108
14	0.02673	0.02467	0.00147	-0.00074
15	0	0.00189	-0.00135	-0.01382

Fig. 5 Distribution of radial stress σ_r

and so A_0 , B_0 and C_0 are obtained at once, and using these values and applying the method of successive approximation, we can get $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_s$. Hence the stresses and displacement components are obtained from Eqs. (27)~(32).

Now, corresponding to the width of pressure band $d/h=0.2$ taken, the values of A_0 , B_0 , C_0 , $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_s$ are shown in Table 3. The distributions of the stresses σ_r , σ_θ , σ_z and τ_{rz} are given in Figs. 5~8, taking r/b as the parameter, z/h as the trans-

Fig. 6 Distribution of tangential stress σ_θ

Fig. 7 Distribution of axial stress σ_z

Fig. 8 Distribution of shear stress τ_{rz}

verse.

From these calculated results, we can find as the characteristics of the three-dimensional analysis that σ_r is compressive in the range of $z/h=0\sim0.3$ and diminishes rapidly with the distance from the inner surface, but tensile in the range of $z/h>0.3$, although its magnitude is small. The distribution of σ_θ is almost tensile in most parts of the cylinder, but only in near portion of the end surface, that is, for $z/h>0.8$ it is compressive. In this distribution, it is noteworthy that the value at the boundary between the loaded range and free surface is maximum beyond the value at the middle section $z/h=0$. Next, we can also recognize that σ_z at the inner surface suddenly changes to the tensile stress from compressive at the above same section $z/h=0.2$.

Lastly, consider the shear stress τ_{zr} (Fig. 8). This stress is smaller than the other stresses, but is a very important factor for the destruction of materials, and so we must pay attention to this stress. As this shear stress shows similar tendency to the other stresses σ_θ , σ_z that give maximum values at the boundary section of $z/h=0.2$, the most dangerous portion of the cylinder is $z/h=0.2$.

Following the stress distributions mentioned above, the displacement components u , w of radial and axial directions are calculated and given concretely in Table 4; and by using these values we

Fig. 9 Deformation

Table 4 Displacements at the surfaces of the finite cylinder under the inner pressure
(in unit of p/E)

z/h or $(r-a)/(b-a)$		0	0.05	0.10	0.15	0.2	0.4	0.6	0.8	1.0
Inner surface $r=a$	u/b	-0.225	-0.218	-0.209	-0.190	-0.143	-0.050	-0.032	-0.018	-0.006
	w/h	0	0.015	0.027	0.040	0.050	0.044	0.039	0.035	0.035
Outer surface $r=b$	u/b	-0.051	-0.051	-0.050	-0.048	-0.046	-0.033	-0.020	-0.008	0.001
	w/h	0	0	-0.002	-0.005	-0.008	-0.011	-0.011	-0.010	-0.009
End surfaces $z=\pm h$	u/b	-0.006	—	—	—	-0.001	0.001	0.002	0.002	0.001
	w/h	0.035	—	—	—	0.026	0.017	0.008	-0.001	-0.009

showed the deformation of the cylinder in Fig. 9.

5.2 On the cylinder with a band of uniform pressure over the outer surface

As a second example, the cylinder under the outer band pressure of the $2d$ -width (Fig. 10) is taken, the boundary condition being

$$\left. \begin{aligned} \sigma_r = \tau_{zr} = 0 & \text{ for } r=a \\ \sigma_r = p \left[\frac{d}{h} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi d}{h} \cos \frac{n\pi}{h} z \right], \\ \tau_{zr} = 0 & \text{ for } r=b \\ \sigma_z = \tau_{zr} = 0 & \text{ for } z=\pm h \end{aligned} \right\} \dots\dots(44)$$

From this condition and the expression (40), we have

$$\left. \begin{aligned} A_0\nu + \frac{B_0}{a^2} + C_0(2\nu-1) &= 0 \\ A_0\nu + \frac{B_0}{b^2} + C_0(2\nu-1) &= \frac{pd}{h} \\ A_0(1-\nu) + C_0 \cdot 2(2-\nu) &= 0 \\ \bar{A}_n + \sum_{s=1}^{\infty} \bar{C}_s P_{sn} &= 0 \\ \bar{B}_n + \sum_{s=1}^{\infty} \bar{C}_s Q_{sn} &= \frac{2p}{\pi n} \sin \frac{n\pi d}{h} \\ \bar{C}_s + \sum_{n=1}^{\infty} (\bar{A}_n R_{ns} + \bar{B}_n S_{ns}) &= 0 \end{aligned} \right\} \dots\dots\dots(45)$$

The values A_0 , B_0 , C_0 , $\bar{A}_n$, $\bar{B}_n$, and $\bar{C}_s$ calculated from these equations are given in Table 5, and by using these values we get the stresses and displacement components, that is, the stresses σ_r , σ_θ , σ_z and τ_{zr} are shown in Figs. 11~14 and the deformation of the cylinder in Fig. 15.

Fig. 10 Finite cylinder with a band of uniform pressure over the outer surface

From these results, we know that σ_r and σ_θ are compressive for the most parts of the cylinder, but tensile for the part of $z/h=0.8\sim 1.0$. These stresses in the middle section of a pressure band are smaller than the values calculated from the equations for the two-dimensional theory of elasticity,

$$\sigma_r/p = \{b^2/(b^2-a^2)\}(1-a^2/r^2),$$

$$\sigma_\theta/p = \{b^2/(b^2-a^2)\}(1+a^2/r^2),$$

especially for the stress σ_θ .

Next, the distribution of stress σ_z is similar to that for the inner band pressure, and on the outer cylindrical surface it turns into a high tensile stress from the compressive stress at the boundary line of the band pressure, and the tensile stress produced at $z/h=0$ of the inner surface increases to about 50% of the compressive stress on the outer surface. In the case of shearing stress τ_{zr} , we recognize that this is the tensile stress at almost any portion of the cylinder, but tends to turn compressive in both end

Table 5 Values of b_n , $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_n$ (in unit of p)

n, s	b_n	$\bar{A}_n$	$\bar{B}_n$	$\bar{C}_n$
0	0.2	0.5579	-0.2	-0.1149
1	0.3742	-0.05508	0.4073	-0.3384
2	0.3027	0.05532	0.2659	0.00585
3	0.2018	-0.03873	0.2305	0.1622
4	0.09355	0.02796	0.07139	-0.05048
5	0	-0.02148	0.01799	0.02569
6	-0.06237	0.01730	-0.07730	0.03199
7	-0.08649	-0.01440	-0.07382	-0.1130
8	-0.07568	0.01228	-0.08664	0.00707
9	-0.04158	-0.01065	-0.03198	0.07320
10	0	0.00937	-0.00850	-0.02734
11	0.03402	-0.00833	0.04161	0.01826
12	0.05046	0.00747	0.04362	0.01767
13	0.04657	-0.00675	0.05277	-0.06792
14	0.02673	0.00613	0.02109	0.00467
15	0	-0.00560	0.00516	0.04458

Fig. 11 Distribution of radial stress σ_r

Fig. 12 Distribution of tangential stress σ_θ

Fig. 13 Distribution of axial stress σ_z

Fig. 14 Distribution of shear stress τ_{zr}

parts.

Lastly, on the deformation of the cylinder we know the following facts from Table 6 and Fig. 15. That is, the displacement u at $z/h=0$ of inner surface is taken as 46% in the loaded part of outer, on the other hand the upheaval of inner periphery at the end surface is about 50%.

5.3 On the cylinder with a circular band of axial pressure over each end surface

As a third example, we consider the cylinder loaded by the circular band ($c-a$) in width of axial pressure around the hole (Fig. 16). The boundary condition is

$$\left. \begin{aligned} \sigma_r = \tau_{rz} = 0 \text{ for } r=a \\ \sigma_r = \tau_{rz} = 0 \text{ for } r=b \\ \sigma_z = p \left[\frac{c^2 - a^2}{b^2 - a^2} + 2ac \sum_{s=1}^{\infty} \frac{C_1(r_s c)}{\lambda_s \{b^2 C_0(r_s b)^2 - a^2 C_0(r_s a)^2\}} C_0(r_s r) \right], \quad \tau_{rz} = 0 \text{ for } z = \pm h \end{aligned} \right\} \dots\dots\dots(46)$$

Since this is the 3rd boundary condition, we have the following relations.

$$\left. \begin{aligned} A_0 \nu + \frac{B_0}{a^2} + C_0(2\nu - 1) = 0, \quad \bar{A}_n + \sum_{s=1}^{\infty} \bar{C}_s P_{sn} = 0 \\ A_0 \nu + \frac{B_0}{b^2} + C_0(2\nu - 1) = 0, \quad \bar{B}_n + \sum_{s=1}^{\infty} \bar{C}_s Q_{sn} = 0 \\ A_0(1 - \nu) + 2C_0(2 - \nu) = \frac{c^2 - a^2}{b^2 - a^2} p, \quad \bar{C}_s + \sum_{n=1}^{\infty} (\bar{A}_n R_{ns} + \bar{B}_n S_{ns}) = \frac{2ac C_1(r_s c) p}{\lambda_s \{b^2 C_0(r_s b)^2 - a^2 C_0(r_s a)^2\}} \end{aligned} \right\} \dots\dots\dots(47)$$

The values of A_0 , B_0 , C_0 , A_n , B_n and $\bar{C}_s$ calculated from the above equations for $c/a=1.6$ are given in Table 7 and also the stress distributions of σ_z , σ_θ , σ_r and τ_{rz} are shown in Figs. 17~20 by taking z/h as the parameter, and r/b as the transverse.

In these figures, we are interested in the following points; the stress σ_z takes a high compressive value in the whole part near the hole but in the outer end surface a small tensile stress. Next, the stress σ_θ is compressive in the range of $z/h > 0.5$, and on the other hand in the middle zone of the cylinder ($z/h < 0.5$) it is tensile, and its distribution is peculiar.

Now, let us consider on the distribution of σ_r , though this stress is lower than σ_z and σ_θ , we must note that the tensile stress produced at the boundary between the free and loaded surface is not small. From Fig. 21, we can recognize the following tendency on the deformation, that is, the corner parts of the hole in the end surfaces move to the centre and the deformations in $z/h=0$ of the outer and inner surfaces are very small.

Table 6 Displacements at the surfaces of the finite cylinder under the outer pressure (in unit of p/E)

z/h or $(r-a)/(b-a)$		0	0.05	0.10	0.15	0.2	0.4	0.6	0.8	1.0
Inner surface $r=a$	u/b	0.206	0.204	0.200	0.193	0.184	0.134	0.080	0.034	-0.004
	w/h	0	-0.027	-0.048	-0.068	-0.104	-0.175	-0.209	-0.222	-0.222
Outer surface $r=b$	u/b	0.445	0.436	0.421	0.388	0.319	0.153	0.088	0.044	0.008
	w/h	0	-0.032	-0.020	-0.005	0.025	-0.013	-0.040	-0.056	-0.065
End surfaces $z=\pm h$	u/b	-0.004	—	—	—	-0.007	-0.007	-0.006	-0.001	0.008
	w/h	-0.222	—	—	—	-0.194	-0.163	-0.129	-0.094	-0.065

Fig. 15 Deformation

Fig. 16 Finite cylinder with an annular axial pressure over the end surfaces

Table 7 Values of h_s , $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_s$
(in unit of p)

n, s	h_s	$\bar{A}_n$	$\bar{B}_n$	$\bar{C}_s$
0	0.104	0.032	0	0.024
1	0.4586	0.1251	-0.04773	0.6005
2	0.03994	-0.1610	0.04187	0.04152
3	-0.4547	0.1422	-0.02554	-0.6407
4	-0.2893	-0.1184	0.01642	-0.3490
5	-0.00124	0.09737	-0.01286	-0.03808
6	-0.1840	-0.08016	0.01037	-0.1308
7	-0.3501	0.06648	-0.00870	-0.1737
8	-0.07129	-0.05572	0.00750	-0.05759
9	0.1517	0.04728	-0.00657	0.03219
10	0.00248	-0.04063	0.00582	-0.05544
11	-0.03975	0.03534	-0.00521	-0.07039
12	0.2073	-0.03110	0.00470	0.2470
13	0.2610	0.02766	-0.00427	0.3771
14	0.03730	-0.02482	0.00389	0.04822
15	-0.00104	0.02246	-0.00357	-0.07822

Fig. 17 Distribution of axial stress σ_z Fig. 18 Distribution of tangential stress σ_θ Fig. 19 Distribution of radial stress σ_r Fig. 20 Distribution of shear stress τ_{rz}

Fig. 21 Deformation

6. Conclusion

The problems on the finite hollow cylinder have never been investigated until the present but have recently come to be regarded as very important for the design of dies in the cold working and the other elements. In this paper we considered the theoretical solution on the cylinder under the axial symmetrical loads by the three-dimensional theory of elasticity, and using this method the stress distributions and

Table 8 Displacements at the surfaces of the finite cylinder under the axial pressure

(in unit of p/E)

z/h or $(r-a)/(b-a)$		0	1/15	2/15	0.2	0.4	0.6	0.8	1.0
Inner surface $r=a$	u/b	-0.029	—	—	-0.029	-0.029	-0.020	0.008	0.108
	w/h	0	—	—	0.043	0.089	0.151	0.252	0.377
Outer surface $r=b$	u/b	-0.051	—	—	-0.050	-0.045	-0.033	-0.010	0.017
	w/h	0	—	—	0.012	0.021	0.025	0.026	0.031
End surfaces $z=\pm h$	u/b	0.108	0.100	0.093	0.089	0.051	0.032	0.023	0.017
	w/h	0.377	0.348	0.311	0.241	0.119	0.077	0.051	0.031

deformations of the finite hollow cylinder are calculated by the digital computer of Kanazawa University (NEAC-2230).

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Discussion

I. NAKAHARA: (1) In Fig. 6, I doubt whether the distribution of σ_θ on the inner surface $r/b=0.25$ is correct or not. From the results shown in Fig. 12 on the cylinder subjected to the outer pressure, I think it should be as shown in Append.-Fig. 1. But if the distribution of σ_θ as shown in Fig. 6 is correct, why does it distribute discontinuously?

(2) In the distribution of σ_r (Fig. 19), noting that the values in the end parts of inner surface are not zero, I think that the boundary condition is not perfectly satisfied because the stress is not reduced to zero continuously at the end planes.

T. KOIZUMI: (3) It seems that the technique to decide the stress functions is very important in the theoretical analyses. With this in mind, I want to ask about the relations between the part $F(r, z)$ and the terms of series in Eq. (10), that is, the stress function which the authors adopted in this paper.

Append.-Fig. 1

(4) I think that a line load condition is more fundamental than the present one mathematically, and then what is the significance of the authors' load conditions?

T. MATSUMOTO: (5) Were the values of $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_s$ shown in Table 3 calculated as $n, s=15$? If the authors did so, the parts of discontinuity will arise in the calculated results due to the bad convergence of series.

K. IMAI: (6) If I am allowed, I feel like adding brief comments on this study. In the introduction of the paper, you stated "this is a very difficult problem and only a few discussions have been made on this matter." To my knowledge, however, quite a few problems concerning the axial symmetric case have been studied, though there might be some differences observed in the way to approach the subject. I understand that many problems of this kind have already been solved.

However, in order to increase understanding on the strain and stress distribution for the actual case, many numerical examples would be valuable.

If my report^(a) "Experimental Study for the Thick-Walled Cylinder under the Axial Compressive Load" will be of any service to you, I shall be very

(a) K. Imai: *Reports of Committee for Technical Research of Stress Measurements and Analyses*, No. 5 (1955).

happy.

Authors' closure

(1) The distributions of σ_r and τ_{rz} on the cylinder with a band of inner or outer pressure are shown in Append.-Fig. 2~5. From these figures and Fig. 5, 8, 11, and 14 in Chapter 5, we can calculate the values of $\partial\sigma_r/\partial r$ and $\partial\tau_{rz}/\partial z$ at many points of the cylinder. According to the results shown in Append.-Table 1, those values satisfy perfectly the equations of equilibrium. On the other hand, no

discontinuity is shown in the distribution of strain. Considering these facts, we believe that the result of Fig. 6 is correct.

One of the reasons why it distributes discontinuously is that the distribution of σ_r influences σ_θ in r -direction as shown from Append.-Figs. 2 and 4. That is, the distribution of the inner pressure is changed more rapidly than that of the outer pres-

Append.-Fig. 2 Radial stress σ_r for the inner pressure

Append.-Fig. 3 Shear stress τ_{rz} for the inner pressure

Append.-Fig. 4 Radial stress σ_r for the outer pressure

Append.-Fig. 5 Shear stress τ_{rz} for the outer pressure

Append.-Table 1 Calculations for examining the stress distribution at the inner surface

Equilibrium equations	$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$				$\frac{\partial\tau_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0$			
	$\frac{\partial\sigma_r}{\partial r}$	$\frac{\partial\tau_{rz}}{\partial z}$	$\frac{\sigma_r - \sigma_\theta}{r}$	Σ	$\frac{\partial\tau_{rz}}{\partial r}$	$\frac{\partial\sigma_z}{\partial z}$	$\frac{\tau_{rz}}{r}$	Σ
Terms								
Calculated points								
$z/h=0$ $r/b=0.25$	$-6.48p/b$	0	$6.48p/b$	0	0	0	0	0
$z/h=0.1$ $r/b=0.25$	$-7.1p/b$	$0.9p/h$	$6.2p/b$	0	$-0.2p/b$	$0.2p/h$	0	0
$z/h=0.3$ $r/b=0.25$	$-0.8p/b$	$-0.68p/h$	$1.48p/b$	0	$-1.0p/b$	$1.0p/h$	0	0

sure.

Moreover, the similar tendency in the stress σ_θ is obtained in our reports^{(b) (8)} on the infinite cylinder and the infinite plate under a band of inner pressure and also in the results⁽⁴⁾ of applying the relaxation method by Kudō and Matsubara.

(2) In the case where the axial pressure acts on the corner parts of the cylinder (the discontinuous parts of elastic body), we consider that this stress σ_r takes zero discontinuously.

The fact of discontinuity in the stress distributions is shown in Saito's report^(c) on the problem of a rectangular plate compressed by rough rigid planes.

(3) It is well known that the stress function must be decided by considering the form of boundary conditions. In this paper, therefore, as the boundary conditions are given in Fourier and Bessel series, the authors considered the stress function containing the function $F(r, z)$ for the constant terms, $\sin \beta_n z$ for the trigonometric series, and $C_0(\gamma_s r)$ for the Bessel series.

(4) Originally, the theoretical research discussed in this paper has been made in order to analyse the thick cylinder fitted by a taper pin. Con-

sequently the boundary conditions are shown in the form of series to explain the problems under the continuous pressure and deformation.

The conditions of line load as the fundamental load systems are of course very important, but we consider that the authorial method is more useful for the practical problems.

On the other hand, the problems of the line load systems may be solved by means of Pan's theory^(d) on the infinite cylinder under the similar condition.

(5) In the numerical calculations, the terms n and s are taken as a number 15, but the values of $\bar{A}_n$, B_n and $\bar{C}_s$ do not influence the convergence of series. That is, the terms of series in the formulas of stresses and deformations converge more rapidly than the above coefficients.

In some loaded parts of the cylinder, therefore, we cannot find any discontinuity of the result at all although the errors of numerical calculations happen to be $\pm 3\%$ at most.

(6) The authors agree with your opinion. But we suppose nobody has reported precisely so far on the general solution based on the three-dimensional theory for the problem of each surface of the finite hollow cylinder being subjected to any axially symmetrical load.

We are intending to experiment our theory described in the present paper. We are sure that your research will be a great help for our future study.

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