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On the Method of Successive Approximations for Dynamic Optimization of a Nonlinear Control System Subjected to a Stationary Gaussian Random Disturbance*

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In this paper, an analytical method of finding the suboptimal controller for a plant consisting of n -th order servosystem with the control signal $\theta(t)$ constrained by the condition that $|\theta(t)|=1$ and a stationary Gaussian white noise is presented.

The performance criterion considered here is the mathematical expectation of the integrated value of the mean-squared error. By using the dynamic programming method, the problem of this synthesis is reducible to the problem of solving a parabolic partial differential equation for the criterion function. Since, as a rule, it is very difficult actually to carry out the calculations, we use an approximate asymptotic method for the solution of such equations. That is, we assume that solution is expanded to the power series with respect to the dispersion of disturbance, at each approximation, it is determined in such manner as satisfies the necessary condition for optimality.

1. Introduction

Recently, much progress has been made in the theoretical aspects of dynamic optimization of control system owing to the increasing demand for control system with high degree of performance hierarchy.

Design techniques of optimum control system have been developed by various research workers, based on such mathematical approaches as R. Bellman's Dynamic Programming⁽¹⁾, L. S. Pontryagin's Maximum Principle⁽²⁾ and others.

On the other hand, relay control system is widely applied to control industrial plants and guided missiles. In practice, theoretical and practical studies on relay control system are becoming a challenging field in the recent development of control engineering.

However, the fact that the command signal to relay control systems actually encounter is a random time function must be taken into account.

In the statistical optimal problem, the problem of analytically synthesizing an optimal control regulator is equivalent to the problem of solving a nonlinear partial differential equation for the criterion function.

Here, the purpose of the optimal control regu-

lator consists in finding the control function $|\theta(t)|=1$, that will minimize the mathematical expectation of the integrated value of the mean-squared error.

However, in spite of the fact that the dynamic programming method permits an analytically clear formulation of the problem, exact solution of this equation is often impossible. In order to solve it, therefore, it is desirable to use the perturbation method, which is effective when the variance of the noise is small.

In many applications, since the variance of the noise can be considered to be small, such a modification is of considerable importance.

2. Statement of the fundamental equation

We consider a system with feedback as shown in Fig. 1.

The input signal $v(t)$ is a given function of time, the plant which is to be controlled is a linear filter, given by the operator $L(s)=1/\sum_{i=0}^n a_i s^{n-i}$ where $s=d/dt$ is the differential operator.

Fig. 1 Block diagram of a relay servosystem

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And also, the controller has a magnitude constraint of the form

$$|\theta(t)|=1. \quad (1)$$

The error signal $e(t)$ is, of course, given by

$$e(t)=v(t)-x(t)\equiv e_1(t) \quad (2)$$

and the signal $z(t)$ driving the on-off relay element is now expressed by the following function Γ which is called the switching function

$$z(t)=\Gamma(e) \quad (3)$$

where the notation e is the vector variable that consists of the variables $e_1, e_2, \dots, e_n$, and e_i the i th derivative of the error signal.

The output signal $\theta(t)$ of the on-off relay element is therefore, expressed by

$$\theta(t)=\varphi[\Gamma(e)]=\begin{cases} 1 & \text{for } \Gamma(e)>0 \\ -1 & \text{for } \Gamma(e)<0 \end{cases} \quad (4)$$

Furthermore, we assume that $\xi(t)$ can be considered a stationary purely random process and its probability distribution may be taken as the Gaussian type with zero mean and variance given by $E[\xi^2(t)]=N$.

Then, the differential equation of the control system shown in Fig. 1 may be given by

$$\sum_{i=0}^n a_i \frac{d^{n-i}e}{dt^{n-i}} + \theta(t) = \sum_{i=0}^n a_i \frac{d^{n-i}v}{dt^{n-i}} + \xi(t) \quad (5)$$

where we suppose $\sum_{i=0}^n a_i d^{n-i}v/dt^{n-i}=0$.

The above Eq. (5) can be written in the form of a linear system of n first-order equations, and in vector form, such that

$$\frac{de}{dt} = Ae - \Theta + \xi \quad (6)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_n & -a_{n-1} & \dots & -a_1 & 0 \end{bmatrix}, \quad \Theta = (0, 0, \dots, \theta),$$

$$\xi = (0, 0, \dots, \xi)$$

Now, let us consider the following performance index. Because the system is subjected to random disturbance, the performance index will always be an averaged quantity of the integral-error squared for a finite control interval.

Thus the performance index R is

$$R = E \left[\int_0^T e_1^2(t) dt \right] \quad (7)$$

where E denotes the mathematical expectation.

The controller can be determined such as gives a minimum value of the performance index R .

Now, for some simplification of the following calculation, the matrices $[A]$ are transformed to a diagonal matrix. To find the diagonal matrix, it is best to make first a linear transformation of the variables e_i

$$z_i = \sum_{j=1}^n c_{ij} e_j \quad (i, j=1, 2, \dots, n) \quad (8)$$

where the matrix $[C]$ is the matrix which diagonalizes $[A]$.

We select $[C]$ so that the matrix $[D]$ may be diagonal

$$d_{ij} = \lambda_i \delta_{ij} \quad (9)$$

where δ_{ij} is the unit matrix.

As we know, in this case the numbers λ_i on the principal diagonal of the matrix $[D]$ are the roots of the equation

$$\text{Det}(a_{ij} - \lambda \delta_{ij}) = 0 \quad (10)$$

In addition, the matrix $[C]$ is the matrix which diagonalizes $[A]$, so that,

$$\sum_{j=1}^n c_{ij} a_{jl} = \lambda_i c_{il} \quad (11)$$

The result of the linear transformation of Eq. (8) we obtain in the system

$$\frac{dz}{dt} = Dz + C\Theta + C\xi \quad (12)$$

To make the optimization problem more concrete we resort to the well known procedure of dynamic programming.

Then, let us denote by $\{e(t), \theta(t), \xi(t) | e_0, \theta_0, \xi_0, t_0\}$ the Markov random vector function, generated by the initial conditions $e=e_0, \theta=\theta_0, \xi=\xi_0$, when $t=t_0$.

The expression (7) can now be written out in a more detailed form

$$R = E \left[\int_0^T e_1^2(t) dt | e(0) \right] \quad (13)$$

And then, under the transformation of Eq. (8), the performance index of Eq. (13) is

$$R = E \left[\int_0^T \mu\{z(\tau)\} d\tau | z(0) \right] \quad (14)$$

where

$$z(\tau) = \{z_1(\tau), z_2(\tau), \dots, z_n(\tau)\}$$

$$\mu\{z(\tau)\} = \frac{1}{\Delta^2} (z_1 A_{11} + z_2 A_{21} + \dots + z_n A_{n1})^2$$

$$\Delta = \text{Det}(c_{ij})$$

and also, $A_{i,1}$ is the cofactor of $\text{Det}(c_{i,1})$

We now introduce a loss function

$$F(z, \theta, t) = \min_{\theta} E \left[\int_t^T \mu\{z(\tau)\} d\tau | z(0) \right] \quad (15)$$

Then, by using the method of dynamic programming the equation for the performance index is satisfied⁽³⁾

$$-\frac{\partial F}{\partial t} = \min_{\theta} \left[\mu(z) + \sum_{i=1}^n (\lambda_i z_i - \theta) \frac{\partial F}{\partial z_i} + \frac{N}{2} \sum_{i,j=1}^n c_{in} c_{jn} \frac{\partial^2 F}{\partial z_i \partial z_j} \right] \quad (16)$$

Finding the switching surface Γ is fundamental to our problem, in Eq. (16), the algorithm is determined according to which the optimum controller must operate.

Here, we will consider the stationary mode of

system operation. In order to do so, we let the time T at which operation terminates approach to infinity in Eq. (14). For the stationary case of this kind, this permits us to consider a loss function $f(z, \theta)$ that is not explicitly time-dependent. If we take⁽⁵⁾

$$f(z, \theta) = \lim_{T \rightarrow \infty} \{F(z, \theta, t) - \kappa(T - t)\} \quad (17)$$

then, for the function $f(z, \theta)$, Eq. (17) yields

$$\kappa = \min_{\theta} \left[\mu(z) + \frac{N}{2} \sum_{i,j=1}^n c_{in} c_{jn} \frac{\partial^2 f}{\partial z_i \partial z_j} + \sum_{i=1}^n (\lambda_i z_i - \theta) \frac{\partial f}{\partial z_i} \right] \quad (18)$$

Quantity κ characterizes the stationary average noise per unit time. To this equation we must add additional conditions that must be satisfied by the function $f(z, \theta)$.

Thus, the only solution of Eq. (18) that is of interest to us will be obtained if we require that the function $f(z, \theta)$ have a polynomial growth at infinity.

Nevertheless even in this form it is not amenable to exact solution because of the presence of the nonlinear term $\theta \left| \sum_{i=1}^n \frac{\partial f}{\partial z_i} \right|$.

So the method of successive approximations presented below is convenient in the case when the coefficients in the diffusive terms are small. With some small parameter $\varepsilon = N/2$ at the right hand side of Eq. (18), one may apply the following variant of the method of successive approximations.

The solution $f(z, \theta)$ of Eq. (18) can, therefore, be expanded into the following power series with respect to

$$f(z, \theta) = \sum_{i=0}^{\infty} \varepsilon^i f^{(i)}(z, \theta) \quad (19)$$

Substituting Eq. (19) for $f(z, \theta)$ into Eq. (18) and arranging the coefficient of zeroth order of the power ε of ε , we have

$$\kappa^{(0)} = \min_{\theta} \left[\mu(z) + \sum_{i=1}^n (\lambda_i z_i - \theta) \frac{\partial f^{(0)}}{\partial z_i} \right] \quad (20)$$

Similarly, for the m -th order of the power ε with respect to ε

$$\kappa^{(m)} = \min_{\theta} \left[\sum_{i,j=1}^n c_{in} c_{jn} \frac{\partial^2 f^{(m-1)}}{\partial z_i \partial z_j} + \sum_{i=1}^n (\lambda_i z_i - \theta) \frac{\partial f^{(m)}}{\partial z_i} \right] \quad (21)$$

In Eq. (19), when the sum to m -th order is denoted by

$$f_{(m)}(z, \theta) = \sum_{i=0}^m \varepsilon^i f^{(i)}(z, \theta) \quad (22)$$

Eqs. (20), (21) can be written

$$\kappa_{(m)} = \min_{\theta} \left[g_{0,m}(z) + \sum_{i=1}^n (\lambda_i z_i - \theta) \frac{\partial f_{(m)}}{\partial z_i} \right] \quad (23)$$

where

$$\left. \begin{aligned} g_{0,m}(z) &= \mu(z) + \varepsilon \sum_{i,j=1}^n c_{in} c_{jn} \frac{\partial^2 f_{(m-1)}}{\partial z_i \partial z_j} \\ \kappa_{(m)} &= \sum_{j=0}^m \varepsilon^j \kappa^{(j)} \end{aligned} \right\} \quad (24)$$

and

$$\partial^2 f_{(-1)} / \partial z_i \partial z_j \equiv 0$$

3. Determination of the switching surface

In this chapter, the determination of the switching surface will be evaluated by using Eq. (23).

From Eq. (23) we express the region of $\sum_{i=1}^n \partial f / \partial z_i$

> 0 by R^+ and that of $\sum_{i=1}^n \partial f / \partial z_i < 0$ by R^- . Analogously, the function $f(z, \theta)$ corresponding to the

condition of $\sum_{i=1}^n \partial f / \partial z_i > 0$ is denoted by $f^+(z, \theta)$ and the function $f(z, \theta)$ corresponding to $\sum_{i=1}^n \partial f / \partial z_i < 0$

by $f^-(z, \theta)$.

Under these expressions, system (23) takes the form

$$\kappa_{(m)} = g_{0,m}^{\pm}(z) + \sum_{i=1}^n (\lambda_i z_i - \theta) \frac{\partial f_{(m)}^{\pm}}{\partial z_i} \quad (25)$$

Let us put

$$H_{(m)}\{z, \phi(t), \theta(t)\} = I_m(z, \phi) - \theta \varphi_m(z, \phi) \quad (26)$$

where

$$I_m(z, \phi) = \frac{g_{0,m}^-(z) + g_{0,m}^+(z)}{2} + \sum_{i=1}^n g_i \phi_{i,m} \quad (27)_1$$

$$\varphi_m(z, \phi) = \frac{g_{0,m}^-(z) - g_{0,m}^+(z)}{2} + \sum_{i=1}^n \phi_{i,m} \quad (27)_2$$

and also,

$$g_i = \lambda_i z_i, \quad \phi_{i,m} = \frac{\partial f_{(m)}}{\partial z_i}$$

By putting

$$z_{n+1,m}^{\pm} = \int_t^{\infty} g_{0,m}^{\pm}\{z(\tau)\} d\tau \quad (28)$$

The characteristic differential equation of Eq. (25) is

$$\dot{z}_i = \frac{\partial H_m}{\partial \phi_{i,m}}, \quad \dot{\phi}_{i,m} = -\frac{\partial H_m}{\partial z_i} \quad (29)$$

The expression $\varphi_m(z, \phi)$ is called the switching function and is determined by the following two categories.

We have seen that the behavior of the switching function along optimal trajectories can be classified into two categories:

1) the zeros of $\varphi_m(z, \phi)$ along optimal trajectories are isolated in time.

2) the condition $\varphi_m(z, \phi)=0$ is satisfied for positive intervals of time along optimal trajectories.

When the condition (1) is fulfilled, the control variable θ must be chosen as a function of z_i and ϕ_i so that $H_{(m)}$ may be minimized.

Therefore from Eq. (25)

$$\theta = \text{sgn} \left[\sum_{i=1}^n \frac{\partial f_{(m)}}{\partial z_i} \right] \dots \dots \dots (30)$$

Therefore, the solution of differential Eqs. (29), (30), with boundary conditions

$$z(0)=z^0, \quad \phi_m(\infty)=0, \quad \psi_{n+1,m}=1 \dots \dots \dots (31)$$

traces out the optimum trajectory in z phase, space, starting from arbitrary initial conditions z^0 .

The switch points on this trajectory can be recognized as the corners of switch points when z_0 is varied and represents the required switching surface.

On the other hand, when the condition (2) is satisfied the singular switching surface⁽⁶⁾ Γ_s is the set of states z, t at which the optimal switching function and all its derivatives are zero, the interior of the singular switching surface Γ_s is characterized by the condition

$$\varphi_m=0 \dots \dots \dots (32)$$

which implies the simultaneous vanishing of all time derivatives of

$$\dot{\varphi}_m = \ddot{\varphi}_m = \dots \equiv 0 \dots \dots \dots (33)_1$$

and

$$I_m = \kappa_{(m)} \dots \dots \dots (33)_2$$

Finally, let us consider a solution of partial differential equation (25).

For n independent variables $z_1, z_2, \dots, z_n$ we must consider the n -dimensional integral surfaces but also $(n-1)$ -dimensional characteristic manifolds ϕ . An integral surface can be constructed either from an $(n-1)$ parameter family of characteristic curves.

By the theory of partial differential equation⁽⁴⁾, every characteristic curve of Eq. (29) which has one point in common with an integral surface lies entirely on the integral surface. Moreover, every integral surface is generated by a one-parameter family of characteristic curves.

Accordingly, an integral surface of Eq. (25) can be determined on condition of containing an $(n-1)$ -dimensional characteristic manifold ϕ .

This manifold ϕ is determined such that fulfills the given $(n-1)$ -dimensional switching surface Γ .

Thus we have the optimal solution of Eq. (25). Furthermore, for the higher approximations, substi-

tuting this optimal solution into $\sum_{i,j=1}^n c_{in} c_{jn} \frac{\partial^2 f}{\partial z_i \partial z_j}$ of Eq. (25), and using Eq. (22), the succeeding approximations are made taking the diffusion into account.

4. Numerical example

In order to illustrate the use of the general theory of the foregoing chapter, we shall now consider a system in which the main process or plant has a transfer function $1/s^2 + \beta s + \omega_0^2$ between input $\theta(t)$ and output $x(t)$. Then the differential equations for the error signal $e(t)$ are written as a set of first-order equations as follows

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = -\beta e_2 - \omega_0^2 e_1 - \theta + \xi \end{cases} \dots \dots \dots (34)$$

so the transformation matrix $[C]$ for the matrix $[a_{ij}]$ is given by

$$[c_{ij}] = \begin{bmatrix} -\lambda_1 & 1 \\ -\lambda_2 & 1 \end{bmatrix}$$

where system eigen-values $\lambda_{1,2}$ are

$$\lambda_{1,2} = -\beta/2 \pm \sqrt{\beta^2/4 - \omega_0^2} \dots \dots \dots (35)$$

We put $\lambda_1 = -a$, $\lambda_2 = -b$, and it is best to make a linear transformation of e_i by using Eq. (8) such that

$$\begin{cases} z_1 = ae_1 + e_2 \\ z_2 = be_1 + e_2 \end{cases} \dots \dots \dots (36)$$

Then, from Eq. (25), we can now derive

$$\kappa_{(0)} = \mu(z_1, z_2) - (bz_1 + \theta) \frac{\partial f_{(0)}^\pm}{\partial z_1} - (az_2 + \theta) \frac{\partial f_{(0)}^\pm}{\partial z_2} \dots \dots \dots (37)_1$$

$$\begin{aligned} \kappa_{(m)} = & \mu(z_1, z_2) + \varepsilon \left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right)^2 f_{(m-1)}^\pm(z_1, z_2) \\ & - (bz_1 + \theta) \frac{\partial f_{(m)}^\pm}{\partial z_1} - (az_2 + \theta) \frac{\partial f_{(m)}^\pm}{\partial z_2} \dots \dots \dots (37)_2 \end{aligned}$$

where

$$\mu(z_1, z_2) = \frac{1}{\beta} (z_1 - z_2)^2, \quad \varepsilon = \frac{N}{2}.$$

The problem is to select a control law for which $f(z_1, z_2)_{(m)}^\pm$ is a minimum. These equations can be used to compute a control law which is optimal.

For the case which satisfies the relation of $\varphi_m = 0$, we obtain the system

$$\varphi_m(z, \phi) = 0 \dots \dots \dots (38)$$

with boundary condition of Eq. (31)

And also, for the case which satisfies the relation $\varphi_m = 0$ for positive interval of time, the first time derivative of the switching function φ_m along solution of Eq. (29), is given by

$$\dot{\varphi}_m = \frac{1}{2} (\langle \nabla g_{(m-1)}^-, g \rangle - \langle \nabla g_{(m-1)}^+, g \rangle) - \langle \dot{\phi}, 1 \rangle \dots \dots \dots (39)$$

From Eqs. (27)₁ and (27)₂, we obtain

$$\dot{\phi} = W_1^{-1}(z) W_2(z) \dots \dots \dots (40)$$

where $W_1(z)$ is a 2×2 matrix defined by

$$W_1(z) = \begin{bmatrix} 1 \\ g^T \end{bmatrix}$$

the column vector $W_2(z)$ is given by

$$W_2(z) = \left\{ \frac{1}{2}(g^{+}_{(m-1)} - g^{-}_{(m-1)}), -\frac{1}{2}(g^{-}_{(m-1)} + g^{+}_{(m-1)}) - \kappa_{(m)} \right\}$$

and $\nabla g_{(m-1)} = (\partial g_{(m-1)}/\partial z_1, \partial g_{(m-1)}/\partial z_2)$, $\langle a, b \rangle$ is the inner product of a and b . It is assumed that $\text{Det}(W_1) \neq 0$. Upon substituting Eq. (40) into Eq. (27), we get the singular switching surface of $\Gamma_s(z)$ as

$$\frac{1}{2}(\langle \nabla g^{-}_{(m-1)}, g \rangle - \langle \nabla g^{+}_{(m-1)}, g \rangle) - \frac{1}{2}(\nabla g^{+}_{(m-1)} + \nabla g^{-}_{(m-1)}) + \langle W_1^{-1}W_2, g \rangle = 0 \quad (41)$$

In the following, we consider two cases in which λ is a real and complex numbers.

4.1 The case of which the eigen-value is a real

We consider that the eigen-values of the control equation are $\lambda_1 = -\beta$, $\lambda_2 = 0$. Then, the characteristic equation corresponding to Eq. (29) is expressed as

$$\frac{dz_1}{dt} = -\theta, \quad \frac{dz_2}{dt} = -(\beta + \theta), \quad \frac{d\psi_{1,m}}{dt} = -\frac{\partial g_{0,m}}{\partial z_1}, \quad \frac{d\psi_{2,m}}{dt} = -\frac{\partial g_{0,m}}{\partial z_2} - \beta\psi_{2,m} \quad (42)$$

where

$$g_{0,m} = \mu(z_1, z_2) + \varepsilon \sum_{i,j=1}^2 c_{in} c_{jm} \frac{\partial^2 f_{(m-1)}}{\partial z_i \partial z_j}$$

Now, from Eq. (36) the switching function for the singular trajectory can be expressed as

$$\frac{\partial g_{0,m}^-}{\partial z_2}(\beta z_2 + 1) - \frac{\partial g_{0,m}^+}{\partial z_2}(\beta z_2 - 1) + \frac{\partial g_{0,m}^+}{\partial z_1} + \frac{\partial g_{0,m}^-}{\partial z_1} - \frac{1}{z_2}(g_{0,m}^+ + g_{0,m}^-) = 0 \quad (43)$$

and, for the zeroth approximation is given by

$$z_1 = z_2 \quad (44)$$

However, Eq. (44) does not satisfy the optimal trajectory. Therefore, we must determine by using an optimal condition that the zeros of φ_m along optimal trajectories are isolated in time.

In responding from any initial conditions, it is reasonable to suppose that the optimum system will eventually settle to the desired final condition $z_1 = z_2 = 0$. This means that if we compute backwards in time, the optimal solution of system will obey

$$\left. \begin{aligned} z_1(t) &= \eta t + C_1, & z_2 &= -\frac{\eta}{\beta} + C_2 e^{\beta t}, & \psi_{1,0}(t) &= -\frac{2}{\beta^2} \left(-\frac{\eta}{\beta} t + \frac{C_1}{\beta} e^{\beta t} - \frac{1}{2} \eta t^2 - C_2 t \right) + C_3 \\ \psi_{2,0}(t) &= \frac{2\eta}{\beta^3} \left(t - \frac{1}{\beta} \right) + \frac{2}{\beta^2} C_2 + \frac{2}{\beta^4} \eta - \frac{C_1}{\beta^3} e^{\beta t} + C_4 e^{-\beta t} \end{aligned} \right\} \quad (45)$$

where $\eta = 1$ for the region R^+

$\eta = -1$ for the region R^-

$C_i (i=1, 2, 3, 4)$ are arbitrary integral constants.

In Eq. (45), by introducing differential small initial conditions

$$z_1(0) = z_2(0) = [O(10^{-6})], \quad \psi_{1,0} = \psi_{2,0} = 0$$

in the computer to start the system from rest a whole range of optimum responses may be computed. And on the optimum switching boundary, an integral constant C_i is calculated by the method which is sketched in Fig. 2. Thus, the optimum switching boundary is then obtained by drawing a curve through all the switch points.

By the above method, the switching line is plotted in Fig. 3 for $\beta = 0.1, 1, 2$, respectively.

Next, in Eq. (37), when the time interval T is not fixed, it follows that⁽⁷⁾

$$\kappa_0 = 0 \quad (46)$$

Now, we consider the integral surface $f_{10}^*(z_1, z_2)$ of the differential equation. In the solution of Eq. (37), the initial condition is given by $z_1(0) = S$, $z_2(0) = 0$. Then, by eliminating arbitrary integrating constants C_1, C_2 , we have

Fig. 2 Signal flow chart for the calculation of switching function

$$t = \frac{1}{\beta} \log \left| \frac{\beta}{\eta} z_2 + 1 \right|, \quad S = z_1 - \frac{\eta}{\beta} \log \left| \frac{\beta}{\eta} z_2 + 1 \right| \quad (47)$$

Furthermore, calculating $z_3(t) = f_{(0)}^+(t)$ using Eq. (28) and eliminating the time variable t , we have

$$f_{(0)}^+(z_1, z_2) = \frac{1}{\beta^3} \left\{ \frac{1}{3\beta^2} \log^3 \left| \frac{\beta}{\eta} z_2 + 1 \right| - \frac{1}{\beta} \left(\eta z_1 + \frac{1}{\beta} \right) \log^2 \left| \frac{\beta}{\eta} z_2 + 1 \right| \right. \\ \left. + \left(z_1^2 + \frac{2\eta}{\beta} z_1 - \frac{1}{\beta^2} \right) \log \left| \frac{\beta}{\eta} z_2 + 1 \right| - \frac{2z_1 z_2}{\beta} + \frac{1}{2\beta} z_2 + \frac{\eta}{\beta^2} z_2 \right\} + \Phi_0(S) \quad (48)$$

where $\Phi_0(S)$ is an unknown initial curve. If for $\Phi_0(S)$ we choose the characteristic curve

$$f_{(0)}^+(S, 0) = \Phi_0(S) = a_{1,0} S^{5/2} = a_{2,0} S^{6/2} + a_{3,0} S^{7/2} \quad (49)$$

then, the equation that must satisfy the switching function is given by

$$\psi_{1,0} + \psi_{2,0} = -\frac{\eta}{\beta} z_2 \log^2 \left| \frac{\beta}{\eta} z_2 + 1 \right| + \frac{2}{\beta^3} z_2 \left(z_1 + \frac{\eta}{\beta} \right) \log \left| \frac{\beta}{\eta} z_2 + 1 \right| \\ - \frac{1}{\beta^3} (z_2^2 + 2z_1 z_2 - z_1^2) + \frac{5}{2} a_{1,0} z_2 S^{3/2} + 3a_{2,0} z_2 S^2 + \frac{7}{2} a_{3,0} z_2 S^{5/2} = 0 \quad (50)$$

where $a_{j,0}$ ($j=1, 2, 3$) is an unknown constant and its values can be determined by the zeroth switching function which is obtained in Fig. 3.

Table 1 indicates the value of $a_{j,0}$ obtained by the above method and Fig. 4 indicates the result of digital calculation.

Let us now consider further approximate solution. If we know the solution corresponding to the $(m-1)$ -th approximation, then m -th approximation is given by Eq. (37)₂.

For this purpose it is necessary to calculate $\sum_{i,j=1}^2 c_{in} c_{jn} \frac{\partial^2 f_{(m-1)}}{\partial z_i \partial z_j}$ according to Eq. (37)₁. Therefore, in the same manner as when we have the zeroth approximate solution, first approximate solution is given by

$$f_{(1)}^+(z_1, z_2) = f_{(0)}^+(z_1, z_2) + \epsilon f_{(1)}^+(z_1, z_2) \quad (51)$$

where

$$f_{(1)}^+(z_1, z_2) = -\left(\frac{3}{\beta^4} - \frac{4\eta}{\beta^4(\beta z_2 + \eta)} + \frac{2}{\beta^4(\beta z_2 + \eta)} + \frac{\eta z_1}{\beta^3(\beta z_2 + \eta)^2} \right) \log \left| \frac{\beta}{\eta} z_2 + 1 \right| \\ + \frac{1}{\beta^4} \left(1 + \frac{1}{2(\beta z_2 + \eta)^2} \right) \log^2 \left| \frac{\beta}{\eta} z_2 + 1 \right| - \frac{4}{\beta z_2 + \eta} \left(\frac{z_1}{\beta^3} + \frac{\eta}{\beta^4} \right) + \frac{1}{(\beta z_2 + \eta)^2} \left(\frac{2\eta z_1}{\beta^3} + \frac{1}{2\beta^4} + \frac{z_1^2}{2\beta^2} \right) \\ + \left(\frac{5}{2} a_{1,0} S^{3/2} + 3a_{2,0} S + \frac{7}{2} a_{3,0} S^{5/2} \right) \left(\frac{\beta^2}{2\eta} - \frac{\beta^2}{2\eta(\beta z_2 + \eta)^2} \right) + \left(\frac{15}{4} a_{1,0} S^{1/2} + 6a_{2,0} S + \frac{35}{4} a_{3,0} S^{3/2} \right) \\ \times \left(\frac{\beta}{2} - \frac{2}{\eta} - \frac{\beta}{2(\beta z_2 + \eta)^2} + \frac{2}{\beta z_2 + \eta} + \frac{1}{\beta} \log \left| \frac{\beta}{\eta} z_2 + 1 \right| \right) + \frac{2\eta}{\beta^3} S - \frac{S^2}{2\beta^2} + \Phi_1(S) \quad (52)$$

And also, we suppose the unknown function

Fig. 3 Scheme of 0th, 1st, 2nd and 3rd order switching function for the eigen-value $\lambda_1 = -\beta$, $\lambda_2 = 0$

Fig. 4 Numerical plot of the characteristic manifold obtained by the digital calculation

$$\phi_1(S) = a_{1,1}S^{3/2} + a_{2,1}S^2 + a_{3,1}S^{5/2} \dots\dots\dots(53)$$

and $a_{j,1}$ can be determined uniquely from the first approximation switching function

$$\phi_{(1)}(z, \phi) = \frac{g_{0,1}^- - g_{0,1}^+}{2} + \frac{\partial f_{(1)}^+}{\partial z_1} + \frac{\partial f_{(1)}^-}{\partial z_2} = 0 \dots\dots\dots(54)$$

The results of above discussion can be summed up in Table 1.

Furthermore, the higher approximate solution is given by

$$\begin{aligned} f^{(2)\pm}(z_1, z_2) = & \frac{6}{\beta^3 \left(\frac{\beta}{\eta} z_2 + 1\right)^2} - \frac{1}{\beta^3 \left(\frac{\beta}{\eta} z_2 + 1\right)^3} \left\{ \frac{47}{3} + 4 \log \left| \frac{\beta}{\eta} z_2 + 1 \right| + 4\eta\beta S \right\} \\ & + \frac{1}{\beta \left(\frac{\beta}{\eta} z_2 + 1\right)^4} \left\{ \frac{141}{32\beta^2} + \frac{29\eta}{8\beta} S + \frac{3}{4} S^2 + \frac{37}{8\beta^2} \log \left| \frac{\beta}{\eta} z_2 + 1 \right| + \frac{3\eta S}{2\beta} \log \left| \frac{\beta}{\eta} z_2 + 1 \right| \right. \\ & + \frac{3}{4\beta^2} \log^2 \left| \frac{\beta}{\eta} z_2 + 1 \right| - \frac{3\beta}{4\eta \left(\frac{\beta}{\eta} z_2 + 1\right)^4} \left(\frac{5}{2} a_{1,0} S^{3/2} + 3a_{2,0} S^2 + \frac{7}{2} a_{3,0} S^{5/2} \right) \Big\} \\ & - \left(\frac{15}{4} a_{1,0} S^{1/2} + 6a_{2,0} S + \frac{35}{4} a_{3,0} S^{3/2} \right) \left\{ \left(\frac{\beta}{\eta} z_2 + 1 \right)^4 - \frac{4}{3\eta} \left(\frac{\beta}{\eta} z_2 + 1 \right)^3 + \eta \left(\frac{\beta}{\eta} z_2 + 1 \right) \right\} \\ & - \frac{1}{\beta} \left(\frac{15}{8} a_{1,0} S^{-1/2} + 6a_{2,0} + \frac{105}{8} a_{3,0} S^{1/2} \right) \left\{ \frac{1}{4} \left(\frac{\beta}{\eta} z_2 + 1 \right)^4 - \frac{2}{3\eta} \left(\frac{\beta}{\eta} z_2 + 1 \right)^3 + \frac{1}{2} \left(\frac{\beta}{\eta} z_2 + 1 \right)^2 \right\} \\ & + \left(\frac{3}{2} a_{1,1} S^{1/2} + 2a_{2,1} S + \frac{5}{2} a_{3,1} S^{3/2} \right) \left(\frac{\beta^2}{2\eta} - \frac{\beta^2}{2\eta(\beta z_2 + \eta)^2} \right) + \left(\frac{3}{4} a_{1,1} S^{-1/2} + 2a_{2,1} + \frac{15}{4} a_{3,1} S^{1/2} \right) \\ & \times \left(\frac{\beta}{2} - \frac{\eta}{2} - \frac{\beta}{2(\beta z_2 + \eta)^2} + \frac{2}{\beta z_2 + \eta} + \frac{1}{\beta} \log \left| \frac{\beta}{\eta} z_2 + 1 \right| \right) + \phi_2(S) \dots\dots\dots(55) \end{aligned}$$

Table 1

	$\beta=0.1$	$\beta=1$	$\beta=2$	$\lambda_1 = -i, \lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$	$\lambda_2 = i, \lambda_1 = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$
a_1^0	240.903	0.76257	0.1341	-0.0577	0.0253
a_2^0	13.554	0.09516	0.0252	0.4835	1.5232
a_3^0	-7.352	0.02875	0.0094	0.1428	0.3426
a_1^1	175.405	0.4326	0.0926	-0.0645	0.0341
a_2^1	8.745	0.1208	0.0453	0.3648	1.2381
a_3^1	0.002	0.1649	0.0082	0.0976	0.2825
a_1^2	156.322	0.4132	0.1091	0.0015	0.0188
a_2^2	7.862	0.0807	0.0326	0.3269	1.1725
a_3^2	0.016	0.0534	0.0076	0.0857	0.2542

Fig. 5 Value of performance index when system starts from $S=1$

we have already discussed in some detail the optimal switching function; these results are plotted in Fig. 3.

To show that these results satisfy the optimum condition, we assume that the switching curve of Fig. 3 can be written

$$z_2 = \alpha(1 - e^{-(1/\alpha)z_1}) \quad (\alpha: \text{parameter}) \dots\dots\dots(56)$$

Then, we calculate the value of

$$\int_0^\infty g_{0,m}^\pm \{z_1(\tau), z_2(\tau)\} d\tau$$

with initial condition $z_1(0)=1, z_2(0)=0$. These results are shown in Fig. 5 and we know that these results satisfy the optimum condition for each

Fig. 6 Switching function of e_1 - e_2 plane for the eigen-value $\lambda_1 = -\beta, \lambda_2 = 0$

approximation. The switching curve of e_1 - e_2 plane that we are seeking is shown in Fig. 6.

4.2 The case for which the eigen-value is a complex number

When the character of the control equation is written for the underdamped case, the eigen-value of system equation is a complex number. We put these complex numbers $a = \alpha + i\gamma$, $b = \alpha - i\gamma$

The solution of differential Eq. (29) is given by

$$z_1(t) = x(t) + iy(t), \quad z_2(t) = x(t) - iy(t) \quad \dots\dots\dots (57)$$

where

$$x(t) = \frac{-\eta\alpha}{\alpha^2 + \gamma^2} + C_1 e^{-\alpha t} \cos \gamma t - C_2 e^{-\alpha t} \sin \gamma t, \quad y(t) = \frac{-\eta\gamma}{\alpha^2 + \gamma^2} + C_2 e^{-\alpha t} \cos \gamma t + C_1 e^{-\alpha t} \sin \gamma t \quad \dots\dots\dots (58)$$

Now, for any differentiable complex-valued function $f(z_1, z_2)$, let us introduce the differentiation operations.

$$\left. \begin{aligned} \frac{\partial f^{(m)\pm}}{\partial z_1} &= \frac{1}{2} \left(\frac{\partial f^{(m)\pm}}{\partial x} + \frac{1}{i} \frac{\partial f^{(m)\pm}}{\partial y} \right), & \frac{\partial f^{(m)\pm}}{\partial z_2} &= \frac{1}{2} \left(\frac{\partial f^{(m)\pm}}{\partial x} - \frac{1}{i} \frac{\partial f^{(m)\pm}}{\partial y} \right), \\ \left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right)^2 f^{(m)\pm} &= \frac{\partial^2 f^{(m)\pm}}{\partial x^2} \end{aligned} \right\} \quad \dots\dots\dots (59)$$

as the result of applying successive transformations, we obtain

$$\frac{y^2}{\mu} + \varepsilon \frac{\partial^2 f^{(m)\pm}}{\partial x^2} - (\alpha x + \gamma y + \theta) \frac{\partial f^{(m)\pm}}{\partial x} - (\alpha y - \gamma x) \frac{\partial f^{(m)\pm}}{\partial y} = \kappa_{(m)} \quad \dots\dots\dots (60)$$

where

$$\mu = \beta^2 - 4\omega_0^2/4$$

Then, the switching function corresponding to Eq. (41) can be written in the form

$$\begin{aligned} & \frac{\partial g_{0,m}^+}{\partial x} \{(\alpha - \gamma)x + (\alpha + \gamma)y - 1\} - \frac{\partial g_{0,m}^-}{\partial x} \{(\alpha - \gamma)x + (\alpha + \gamma)y + 1\} \\ & + \frac{1}{\alpha y - \gamma x} [g_{0,m}^+ \{(\alpha^2 + \gamma^2)y - \gamma\} - g_{0,m}^- \{(\alpha^2 + \gamma^2)y + \gamma\} + 2\gamma \kappa_{(m)}] = 0 \quad \dots\dots\dots (61) \end{aligned}$$

Therefore the switching line for the zeroth approximation is given by

$$y = 0 \quad \dots\dots\dots (62)$$

However, from this equation does not fulfil an optimal condition, and we must solve the characteristic equation for $\phi(t)$

$$\begin{bmatrix} \frac{d\psi_{1,0}}{dt} \\ \frac{d\psi_{2,0}}{dt} \end{bmatrix} + \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \psi_{1,0} \\ \psi_{2,0} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2y}{\mu} \end{bmatrix} \quad \dots\dots\dots (63)$$

with boundary conditions of Eq. (31)

The solution of Eq. (63) can be written

$$\begin{aligned} \psi_{1,0}(t) &= \frac{8\gamma^2\eta}{\mu(\alpha^2 + \gamma^2)^2} - \frac{2\gamma}{\mu\alpha(\alpha^2 + \gamma^2)} (C_1\gamma + C_2\alpha) e^{\alpha t} \cos \gamma t \\ &+ \frac{2\gamma}{\mu\alpha(\alpha^2 + \beta^2)} (C_1\alpha - C_2\gamma) e^{\alpha t} \sin \gamma t + A e^{-\alpha t} \cos \gamma t - B e^{-\alpha t} \sin \gamma t \quad \dots\dots\dots (64) \end{aligned}$$

where the integral constants C_1 , C_2 , A , B are found in similar manner to Chapter 4.1 using the flow chart of Fig. 2.

Eq. (64) allows the subsequent switch time to be determined by control law $\psi_{1,0} = 0$.

In Figs. 7, 8, the dotted line represents the switching curve of zeroth approximation for $\alpha = 0$, $\gamma = 1$, $\alpha = 1/2$, $\gamma = -\frac{\sqrt{3}}{2}$ respectively.

Next, for the zeroth approximation solution in Eq. (60), the characteristic equation is given by

$$\frac{df^{(0)\pm}}{dt} = \frac{y^2}{\mu} \quad \dots\dots\dots (65)$$

Substituting Eq. (58) into Eq. (65), and eliminating the integral constant and time variable, we have

$$f^{(0)\pm}(x, y) = A_1 x^2 + A_2 y^2 + 2A_3 xy + A_4 \eta x + A_5 \eta y + \frac{\mu^4}{\gamma} \Omega^2 \tan^{-1} \frac{\alpha y - \gamma x}{\alpha x + \gamma y + \eta} + \Phi_0(S) \quad \dots\dots\dots (66)$$

where

$$\left. \begin{aligned} A_1 &= -\frac{\mu\gamma^4}{(\alpha^2 + \gamma^2)^3} \left(\alpha^3 + 2\alpha^2\gamma^4 + \frac{\gamma^4}{2} \right), & A_2 &= -\frac{\mu \left(5\alpha^3\gamma^2 + 4\alpha\gamma^4 + 2\alpha^5 + \frac{\gamma^6}{\alpha} \right)}{(\alpha^2 + \gamma^2)^3}, & A_3 &= -\frac{\mu\gamma}{(\alpha^2 + \gamma^2)^3} (2\alpha^2\gamma^2 + \alpha^4 + \gamma^4), \\ A_4 &= -\frac{\mu^4\gamma^2}{(\alpha^2 + \gamma^2)^2}, & A_5 &= \frac{-2\gamma\mu}{(\alpha^2 + \beta^2)^3} \left(\frac{\gamma^4}{\alpha} - \alpha^3 \right), & \Omega^2 &= \left(\frac{\gamma\eta}{\alpha^2 + \gamma^2} \right)^2 \end{aligned} \right\} \quad \dots\dots\dots (67)$$

and

$$S = \left[\exp \left\{ \frac{\alpha}{\gamma} \tan^{-1} \frac{\alpha y - \gamma x}{\alpha x + \gamma y + \eta} \right\} \sqrt{(\alpha^2 + \gamma^2)(x^2 + y^2) + 2\alpha\eta x + 2\gamma\eta y + 1} - \eta \right] \frac{1}{\alpha^2 + \gamma^2} \quad (68)$$

On the straight line $y = \frac{\gamma}{\alpha}x$, the initial curve is chosen

$$\Phi_0(S) = a_{1,0}S + a_{2,0}S^2 + a_{3,0}S^3 \quad (69)$$

where $a_{j,0}$ is determined by the condition of $\psi_{1,0}(t) = 0$. These values are shown in Table 1. Namely, the equation of switching function can be written as

$$\begin{aligned} \psi_{1,0} = & 2A_1x + 2A_2y + A_3\eta - \frac{4}{\gamma\mu} \Omega^2 \frac{(\alpha^2 + \gamma^2)y + \gamma\eta}{(\alpha^2 + \gamma^2)(x^2 + y^2) + 2\alpha\eta x + 2\gamma\eta y + 1} \\ & + \left\{ a_{1,0} + 2a_{2,0}S + 3a_{3,0}S^2 + 2 \frac{\gamma^5(5\alpha^2 + \gamma^2)}{\mu(\alpha^2 + \gamma^2)} S - \frac{2\alpha\gamma^2}{\mu(\alpha^2 + \gamma^2)} \left(3 - \frac{\gamma^2}{\alpha^2} \right) \eta \right\} \\ & \times \frac{\left(x - \frac{\alpha}{\gamma}y \right)}{\sqrt{(\alpha^2 + \gamma^2)(x^2 + y^2) + 2\alpha\eta x + 2\gamma\eta y + 1}} \exp \left\{ \frac{\alpha}{\gamma} \tan^{-1} \frac{\alpha y - \gamma x}{\alpha x + \gamma y + \eta} \right\} = 0 \quad (70) \end{aligned}$$

Subsequently, the first approximate switching curve for the singular trajectory using Eq. (66) can be determined as the dotted line of Figs. 7, 8. Furthermore, the first approximate solution can be expressed as

$$\begin{aligned} f^{(1)}(x, y) = & \frac{\alpha^2 + \gamma^2}{4[(\alpha^2 + \gamma^2)x + \alpha\eta]^2 + \{(\alpha^2 + \gamma^2)y + \gamma\eta\}^2} \left\{ 2\alpha \left(x + \frac{\alpha\eta}{\alpha^2 + \gamma^2} \right) \left(y + \frac{\gamma\eta}{\alpha^2 + \gamma^2} \right) \right. \\ & + \gamma \left(x + \frac{\alpha\eta}{\alpha^2 + \gamma^2} \right)^2 - \gamma \left(y + \frac{\gamma\eta}{\alpha^2 + \gamma^2} \right)^2 \left. + \frac{4\alpha\gamma^2}{\alpha^2 + \gamma^2} \left\{ \left(x + \frac{\alpha\eta}{\alpha^2 + \gamma^2} \right)^2 + \left(y + \frac{\gamma\eta}{\alpha^2 + \gamma^2} \right)^2 \right\} \right. \\ & - \frac{\alpha}{4(\alpha^2 + \gamma^2)} \left(y + \frac{\gamma}{\alpha}x + \frac{\eta}{\gamma} \right)^2 + \frac{1}{4(\alpha^2 + \gamma^2)} \left\{ 2\gamma + \alpha \exp \left(\frac{2\alpha}{\gamma} \tan^{-1} \frac{\alpha y - \gamma x}{\alpha x + \gamma y + \eta} \right) \right\} \left(x - \frac{\alpha}{\gamma}y \right)^2 \\ & + \frac{(\alpha^2 + \gamma^2) \exp \left(\frac{\alpha}{\gamma} \tan^{-1} \frac{\alpha y - \gamma x}{\alpha x + \gamma y + \eta} \right)}{\gamma^2 \{ (\alpha^2 + \gamma^2)(x^2 + y^2 + 2\alpha\eta x + 2\gamma\eta y + 1) \}^{3/2}} \left[\frac{1}{2\gamma} \left\{ \left(x + \frac{\alpha\eta}{\alpha^2 + \gamma^2} \right)^2 + \left(y + \frac{\gamma\eta}{\alpha^2 + \gamma^2} \right)^2 \right\} \right. \\ & \left. \times \tan^{-1} \frac{\alpha y - \gamma x}{\alpha x + \gamma y + \eta} + \frac{1}{2\gamma} \left(y + \frac{\gamma\eta}{\alpha^2 + \gamma^2} \right) \left(x + \frac{\alpha\eta}{\alpha^2 + \gamma^2} \right) \right] + \Phi_1(S) \quad (71) \end{aligned}$$

And also, the constant right hand side of Eq. (60) is given by

$$\kappa_{(0)} = \kappa_{(1)} = \kappa_{(2)} = 0 \quad (72)$$

Finally, an approximate optimum switching curve in $e_1 - e_2$ plane is shown in Fig. 9.

5. Conclusions

An analytical method of finding the optimum switching function of a relay servosystem subjected

Fig. 7 Scheme of 0th, 1st order switching function for the eigen-value $\lambda_1 = i$, $\lambda_2 = -i$

to a stationary Gaussian random noise is established by using the method of perturbation.

Although the performance criterion considered here is the mathematical expectation of the integrat-

Fig. 8 Scheme of 0th, 1st and 2nd order switching function for the eigen-value

$$\lambda_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

ed value of squared error response, the method proposed here is, of course, applicable to another performance criterion.

Fig. 9 Switching function of e_1 - e_2 plane for the eigenvalue $\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

The results can now be summarized as follows.

1) In the stochastic problem, the switching function satisfies the optimum condition to switch more rapidly than switching of the deterministic case.

2) In the pure oscillation system, the switching line of time optimal becomes optimum.

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