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Stall-Flutter in Cascade*

(3rd Report, Arbitrary Mode Vibration of Cascade Blades as Treated by Semi-Actuator Disc Method)

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This paper presents a theoretical treatment of cascade blades vibrating in arbitrary mode, under any condition of cascade — whether stalled or not — on a unified basis, by means of the semi-actuator disc method.

1. Introduction

The 1st and 2nd reports with the same title⁽¹⁾ have shown a theoretical treatment of cascade blades vibrating in translatory mode, under any condition of cascade — whether stalled or not — on a unified basis, by means of the semi-actuator disc method, and have given by some calculations the boundaries of nonstall- and stall-flutter of compressor and turbine cascades of several kinds.

The present paper, extending the method mentioned above, treats the cascade blades vibrating in arbitrary mode by the semi-actuator disc method.

2. Nomenclature

b_a, b_θ : influence factors of inlet state and cascade geometry for outlet state
 c : blade chord
 f, m : unsteady aerodynamic force and moment on blade
 i : $\sqrt{-1}$
 k : reduced frequency $\equiv \omega c/U$
 $P, p e^{i\omega t}$: static pressures, steady and unsteady
 $P_t, p_t e^{i\omega t}$: total pressures, steady and unsteady
 p_{tr} : total pressure relative to blade
 $q e^{i\omega t}$: blade velocity due to vibration
 s : blade spacing
 s' : blade displacement relative to the adjoining one
 t : time
 U, V : mean velocities of flow
 $u e^{i\omega t}, v e^{i\omega t}$: perturbation velocities of flow
 u_r, v_r : flow velocities relative to blade
 V_s : propagating velocity of perturbation along

cascade

 x, y : rectangular co-ordinates z, Z : total pressure-loss coefficients

$z_{a,\theta}, Z_{a,\theta}$: derivatives of total pressure-loss coefficients, with respect to v_{r01}/u_{r01} and $\Theta + \theta e^{i\omega t}$

 α : angle of flow $\zeta e^{i\omega t}$: vorticity

$\Theta, \theta e^{i\omega t}$: stagger angle and angle of torsional vibration

ξ : co-ordinate along blade measured from the leading edge

 ρ : fluid density $\psi e^{i\omega t}$: perturbation stream function ω : angular frequency of vibration $\operatorname{sgn} \omega = \operatorname{sgn} V_s$

σ : phase difference of vibration between two points separated by a chord-length along cascade $\equiv \omega c/V_s$

 τ : constant of boundary-layer time delay φ : angle of vibration

Suffixes

1: upstream of cascade or at the leading edge

2: downstream of cascade or at the trailing edge

01: origin, just upstream of cascade

02: origin, just downstream of cascade

3: inside of cascade

31: just downstream of the leading edge

32: just upstream of the trailing edge

I: upstream of control surface or at the inlet of control surface

II: downstream of control surface or at the outlet of control surface

III: inside of control surface

 N : normal to blade surface T : tangential to blade surface

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Eliminating A_2 between these equations

$$u_{02} + i v_{02} = \frac{\zeta_{02}}{-\frac{\omega}{U} \left(1 - \frac{V_2}{V_s}\right) + i \frac{\omega}{V_s}} \quad \dots\dots\dots (4)$$

Similarly for the upstream flow

$$u_{01} - i v_{01} = \frac{\zeta_{01}}{\frac{\omega}{U} \left(1 - \frac{V_1}{V_s}\right) + i \frac{\omega}{V_s}} \quad \dots\dots\dots (5)$$

Assuming that the upstream flow is irrotational,

$$\zeta_{01} = 0 \quad \dots\dots\dots (6)$$

5. Flow through the cascade

5.1 Fundamental equations

With the aid of an appropriate control surface within the cascade, the fundamental equations for the flow through the cascade are derived.

5.1.1 Control surface

It is assumed that all the blades in the cascade oscillate with the same frequency and amplitude in an identical mode and, as shown in Fig. 2, the blade $\widehat{ab}$ vibrates with a phase angle $\omega s/V_s$ (rad.) ahead of the adjacent blade $\widehat{cd}$, and that the segment $\overline{AB}$ at ξ measured along the blade $\widehat{ab}$ from its leading edge vibrates at a velocity $q(\xi)e^{i\omega t}$ with an angle $\varphi(\xi)$ to the direction of the cascade circumference.

Now, consider a space between these two adjoining blade segments, $AC'CDD'BA$, as a control volume, which moves at a constant speed equal to that at which the segment $\overline{AB}$ travels at that instant.

The displacement of the blade $\widehat{cd}$ relative to the blade $\widehat{ab}$, that is cc' , dd' , CC' , DD' and so on, is given by

$$s' = \frac{q(\xi)}{i\omega} (e^{-i\omega s/V_s} - 1) e^{i\omega t} \doteq -\frac{q(\xi)}{V_s} s e^{i\omega t} \quad \dots\dots (7)$$

The fluid velocities relative to the control surface are given by

$$\left. \begin{aligned} u_r &= U + u e^{i\omega t} + \overline{q e^{i\omega t}} \sin \varphi \\ v_r &= V + v e^{i\omega t} - \overline{q e^{i\omega t}} \cos \varphi \end{aligned} \right\} \quad \dots\dots\dots (8)$$

where $\overline{q e^{i\omega t}}$ indicates the transfer velocity of the control surface which must be taken as a constant.

5.1.2 Equation of continuity

Under the assumption of incompressible fluid, the equation of continuity for a control volume may be represented by

$$\oint_{c.s.} \vec{W} \cdot d\vec{A} = \oint_{c.s.} dw_{out} = 0 \quad \dots\dots\dots (9)$$

where

$\vec{W}$: local instantaneous velocity of flow relative to the control surface

dw_{out} : mass rate of flow outgoing from the element dA of the control surface

$\oint_{c.s.}$: surface integral summed up over the entire

control surface

Now, the excess of the outgoing rate of flow into the field downstream of the control surface over the incoming one from the upstream is

$$\begin{aligned} &\Delta \{u_r(s + s' \cos \varphi) + v_r s' \sin \varphi\} \\ &\doteq \left\{ \Delta u - \frac{U}{V_s} \Delta [q(\cos \varphi + \tan \alpha \sin \varphi)] \right\} e^{i\omega t} \end{aligned}$$

where $\Delta(\) = (\)_{II} - (\)_I$ (subscripts I and II refer to "just upstream" and "just downstream" of the control surface)

$$\tan \alpha = V/U$$

The outgoing rate of flow due to the relative displacement of the blade is

$$\begin{aligned} &(q e^{i\omega(t-s/V_s)} - \overline{q e^{i\omega t}}) \cos(\Theta - \varphi) \Delta \xi \\ &\doteq -i \frac{\omega s}{V_s} q \cos(\Theta - \varphi) \Delta \xi e^{i\omega t} \end{aligned}$$

Substituting these relations into Eq. (9), we get the equation of continuity for the control surface ($AC'CDD'BA$), by neglecting second-order terms,

$$\begin{aligned} &\Delta u - \frac{U}{V_s} \Delta [q(\cos \varphi + \tan \alpha \sin \varphi)] \\ &- i \frac{\omega}{V_s} q \cos(\Theta - \varphi) \Delta \xi = 0 \quad \dots\dots\dots (10) \end{aligned}$$

From Eq. (10), the equation of continuity for the flow at each position in the cascade is definitely obtained as follows: At the position within the cascade where the blade surface is smooth, setting $\alpha = \Theta$ and taking the limit of Eq. (10),

$$\begin{aligned} &\frac{du}{d\xi} - \frac{U}{V_s} \frac{d}{d\xi} [q(\cos \varphi + \tan \Theta \sin \varphi)] \\ &- i \frac{\omega}{V_s} q \cos(\Theta - \varphi) = 0 \quad \dots\dots\dots (11) \end{aligned}$$

At the inlet and outlet of the cascade, and at the position within the cascade where the blade surface is not smooth or the state of flow changes discontinuously, setting $\Delta \xi = 0$ and making q and φ constants in Eq. (10),

$$\Delta u - \frac{U}{V_s} q \sin \varphi \Delta(\tan \alpha) = 0 \quad \dots\dots\dots (12)$$

where α in the cascade must be replaced by Θ .

Using Eqs. (11) and (12), the equation of continuity for the cascade as a whole is given by

$$\begin{aligned} u_{02} &= u_{01} + \frac{U}{V_s} [q(\cos \varphi + \tan \alpha \sin \varphi)]_1^2 \\ &+ \int_1^2 i \frac{\omega}{V_s} q \cos(\Theta - \varphi) d\xi \quad \dots\dots\dots (13) \end{aligned}$$

where suffix 1 refers to the upstream field or the leading edge of the cascade and 2 the downstream field or the trailing edge.

5.1.3 Equation of energy

It is assumed that, while the fluid flows through the control surface ($AC'CDD'BA$), it loses a part of its total pressure only at the entrance of the control surface. This will be written as

$$\Delta_1 p_{tr} = -\frac{\rho}{2} u_{r1}^2 Z \left(\frac{v_{r01}}{u_{r01}}, \Theta + \theta e^{i\omega t}, \omega \right) \quad \dots\dots\dots (14)$$

where

Δ_1 : increment at the entrance of the control surface

p_{tr} : total pressure relative to the control surface

$\frac{\rho}{2} u_{r1}^2$: axial-velocity head of the incoming flow relative to the control surface

Z : total pressure-loss coefficient, depending on the inlet angle of flow relative to the cascade v_{r01}/u_{r01} , the angle of local stagger $\Theta + \theta e^{i\omega t}$, and the oscillating frequency of blade ω .

Expanding Eq. (14) around the steady state and neglecting the terms of second order, the energy equation at the entrance of the control surface is obtained as follows:

$$\begin{aligned} \frac{\Delta_1 p_t}{\rho U} = & -u_1 Z + q[\cos \varphi \Delta(\tan \alpha) - Z \sin \varphi] \\ & + \frac{1}{2} [u_{01} \tan \alpha_1 - v_{01} + q_1(\cos \varphi_1 + \tan \alpha_1 \sin \varphi_1)] Z_\alpha - \frac{1}{2} U \theta Z_\theta, \end{aligned}$$

where $p_t e^{i\omega t}$: unsteady total pressure

$$Z = Z(\tan \alpha_1, \Theta), \quad Z_\alpha = \frac{\partial}{\partial (\tan \alpha_1)} Z(\tan \alpha_1, \Theta, \omega), \quad Z_\theta = \frac{\partial}{\partial \Theta} Z(\tan \alpha_1, \Theta, \omega)$$

At the exit of the control surface, there being no loss of the relative total pressure,

$$\frac{\Delta_1 p_t}{\rho U} = q \cos \varphi \Delta(\tan \alpha)$$

For the flow in the control surface, from the linearized Crocco's equation of motion along the blade

$$\frac{\partial}{\partial t} \{ (u \cos \Theta + v \sin \Theta) e^{i\omega t} \} = -\frac{1}{\rho} \frac{\partial p_t}{\partial \xi} e^{i\omega t},$$

the increase of the total pressure in the control surface is

$$\frac{\Delta_1 p_t}{\rho U} = -i\omega \int_{\xi}^{\xi + \Delta \xi} (u \cos \Theta + v \sin \Theta) d\xi = -i\omega (\bar{u} \cos \Theta + \bar{v} \sin \Theta) \Delta \xi$$

$$\text{where } \overline{(\quad)} = \frac{1}{\Delta \xi} \int_{\xi}^{\xi + \Delta \xi} (\quad) d\xi.$$

Then the equation of energy for the flow through the control surface (AC'CDD'BA) is given by

$$\begin{aligned} \frac{\Delta p_t}{\rho U} = & \sum_{i=1}^N \Delta_i p_t \quad \dots\dots\dots (15) \\ = & -u_1 Z + q[\cos \varphi \Delta(\tan \alpha) - Z \sin \varphi] - i\omega (\bar{u} \cos \Theta + \bar{v} \sin \Theta) \Delta \xi \\ & + \frac{1}{2} [u_{01} \tan \alpha_1 - v_{01} + q_1(\cos \varphi_1 + \tan \alpha_1 \sin \varphi_1)] Z_\alpha - \frac{1}{2} U \theta Z_\theta \end{aligned}$$

Using Eq. (15), the equation of energy for the flow through the cascade is definitely given as follows:

For the flow within the cascade, where the blade surface is smooth, setting $\alpha = \Theta$ and replacing Z , Z_α and Z_θ by $z(\xi) d\xi$, $z_\alpha(\xi) d\xi$ and $z_\theta(\xi) d\xi$ or distributed loss of energy,

$$\begin{aligned} \frac{1}{\rho U} \frac{dp_t}{d\xi} = & -uz + q \left[\cos \varphi \frac{d}{d\xi} (\tan \Theta) - z \sin \varphi \right] - i\omega (u \cos \Theta + v \sin \Theta) \\ & + \frac{1}{2} [u_{01} \tan \alpha_1 - v_{01} + q_1(\cos \varphi_1 + \tan \alpha_1 \sin \varphi_1)] z_\alpha - \frac{1}{2} U \theta z_\theta \quad \dots\dots\dots (16) \end{aligned}$$

At the inlet and outlet of the cascade, and at the position within the cascade where the blade surface is not smooth or the state of flow changes discontinuously, setting $\Delta \xi = 0$,

$$\begin{aligned} \frac{\Delta p_t}{\rho U} = & -u_1 Z + q[\cos \varphi \Delta(\tan \alpha) - Z \sin \varphi] \\ & + \frac{1}{2} [u_{01} \tan \alpha_1 - v_{01} + q_1(\cos \varphi_1 + \tan \alpha_1 \sin \varphi_1)] Z_\alpha - \frac{1}{2} U \theta Z_\theta \quad \dots\dots\dots (17) \end{aligned}$$

where α in the cascade must be replaced by Θ .

On the other hand, from the linearized Crocco's equation

$$\frac{\partial (v e^{i\omega t})}{\partial t} + U \zeta e^{i\omega t} = -\frac{1}{\rho} \frac{\partial p_t}{\partial y} e^{i\omega t} \quad \dots\dots\dots (18)$$

for the flow up- and downstream of the cascade, the equation of energy for the cascade as a whole may also be written as

$$i\frac{U}{\omega}\zeta_{02}=i\frac{U}{\omega}\zeta_{01}-v_{01}+v_{02}-\frac{U}{V_s}\frac{\Delta_0 p_t}{\rho U} \dots\dots\dots(19)$$

where $\Delta_0 p_t \equiv p_{t02} - p_{t01}$ can be obtained from Eqs. (16) and (17).

5.1.4 Boundary and outlet conditions of the cascade.

It is assumed that the fluid in the cascade travels along the blade surface; hence

$$\tan(\Theta + \theta e^{i\omega t}) = v_r/u_r$$

Then, by neglecting the second-order terms, the boundary condition in the cascade is given by

$$v = u \tan \Theta + q(\cos \varphi + \tan \Theta \sin \varphi) + U\theta \sec^2 \Theta \dots\dots\dots(20)$$

The inlet and outlet angles of the flow relative to the oscillating cascade can be expressed by

$$\tan \alpha + \delta(\tan \alpha) e^{i\omega t} = v_r/u_r$$

which leads to

$$\delta(\tan \alpha) = -[u \tan \alpha - v + q(\cos \varphi + \tan \alpha \sin \varphi)]/U$$

Express the deviation angle of flow at the exit of the cascade by

$$\tan \alpha_2 + \delta(\tan \alpha_2) e^{i\omega t} = \tan(\Theta_2 + \theta e^{i\omega t}),$$

and the unsteady part in it will be

$$[\delta(\tan \alpha_2) - \theta \sec^2 \Theta_2] e^{i\omega t}$$

In practice, the deviation angle of the outlet flow depends on the inlet angle and the cascade geometry, but for small perturbations the condition at the outlet of the cascade may be reasonably expressed by

$$\delta(\tan \alpha_2) - \theta \sec^2 \Theta_2 = b_a \delta(\tan \alpha_1) + b_\theta \theta$$

where b_a and b_θ are the influence coefficients of the

inlet state and the cascade geometry for the outlet flow condition and generally the function of the oscillating frequency. From these equations, the condition for the outgoing flow is given by

$$v_{02} = u_{02} \tan \alpha_2 + q_2(\cos \varphi_2 + \tan \alpha_2 \sin \varphi_2) + U\theta \sec^2 \Theta_2 - b_a[u_{01} \tan \alpha_1 - v_{01} + q_1(\cos \varphi_1 + \tan \alpha_1 \sin \varphi_1)] + b_\theta U\theta \dots\dots\dots(21)$$

This completes the enumeration of the fundamental equations necessary to solve the problem of the unsteady flow field under consideration; that is, when the condition of the cascade (Θ , α_1 , α_2), the conditions of the blade vibration (q , ω , V_s , φ) and the stall characteristics (Z , b) are given, there being six fundamental equations (4), (5), (6), (13), (19), and (21) for six unknowns (u_{01} , v_{01} , ζ_{01} , u_{02} , v_{02} , ζ_{02}), we can determine the whole field of flow.

5.2 Aerodynamic force acting on vibrating cascade blades

The momentum theorem for a control volume may be written as

$$\vec{F} = \int_{C.V.} \frac{\partial}{\partial t} (\rho \vec{W}) dv + \oint_{C.S.} \rho (\vec{W} \cdot d\vec{A}) \vec{W} \dots\dots\dots(22)$$

where $\vec{F}$ is a vector sum of external forces acting on the fluid within the control surface and, on the right-hand side, the first term is the time rate of change of the momentum of the fluid within the control surface and the second term represents the excess of outgoing momentum over an incoming one. For convenience, dividing Eq. (22) into x - and y -components

$$\begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \int_{C.V.} \frac{\partial}{\partial t} \begin{Bmatrix} \rho u_r \\ \rho v_r \end{Bmatrix} dv + \oint_{C.S.} \rho \begin{Bmatrix} u_r \\ v_r \end{Bmatrix} dw_{out} \dots\dots\dots(23)$$

Applying these momentum equations to the control surface (AC'CDD'BA), each term on the right-hand side of Eq. (23) is given by

$$\begin{aligned} \int_{C.V.} \frac{\partial}{\partial t} \begin{Bmatrix} u_r \\ v_r \end{Bmatrix} dv &= i\omega s \cos \Theta e^{i\omega t} \int_{\xi}^{\xi+\Delta\xi} \begin{Bmatrix} u \\ v \end{Bmatrix} d\xi = i\omega s \cos \Theta \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} \Delta\xi e^{i\omega t} \\ \oint_{C.S.} \begin{Bmatrix} u_r \\ v_r \end{Bmatrix} dw_{out} &= \Delta \left[\begin{Bmatrix} u_r \\ v_r \end{Bmatrix} [u_r(s+s' \cos \varphi) + v_r s' \sin \varphi] \right] - i\frac{\omega s}{V_s} q \cos(\Theta - \varphi) e^{i\omega t} \int_{\xi}^{\xi+\Delta\xi} \begin{Bmatrix} u_r \\ v_r \end{Bmatrix} d\xi \end{aligned}$$

The force acting on the fluid in the control surface is

$$\begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \begin{Bmatrix} \Delta[-(P + p e^{i\omega t})(s + s' \cos \varphi)] + F_{bx} \\ \Delta[-(P + p e^{i\omega t})s' \sin \varphi] + F_{by} \end{Bmatrix}$$

where P , $p e^{i\omega t}$: static pressures, steady and unsteady, F_b : forces exerted on the fluid in the control surface by the upper surface of the blade segment $\overline{AB}$ and the lower surface of $\overline{CD}$, which are given by

$$\begin{aligned} F_{bx} &= \Delta P s + \bar{P} \Delta[s' \cos \varphi] - \Delta f_x e^{i\omega t} - \frac{i\omega s}{V_s} \sin \Theta \int_{\xi}^{\xi+\Delta\xi} p e^{i\omega t} d\xi \\ F_{by} &= U \Delta V + \bar{P} \Delta[s' \sin \varphi] - \Delta f_y e^{i\omega t} + \frac{i\omega s}{V_s} \cos \Theta \int_{\xi}^{\xi+\Delta\xi} p e^{i\omega t} d\xi \end{aligned}$$

where the first term on the right-hand side of each equation represents the steady aerodynamic force, the second one the incidental fluid force due to the relative displacement of blades, and the sum of the third and last ones is the unsteady force exerted by the blade segments $\overline{AB}$ and $\overline{CD}$, in which $\Delta f e^{i\omega t}$ is an unsteady force acting on the blade segment $\overline{AB}$.

Substituting these relationships into Eq. (23) and neglecting the terms of higher order in small quantities, the unsteady aerodynamic force acting on the blade segment $\overline{AB}$ is given by

$$\begin{aligned} \frac{\Delta f_x}{\rho U s} = & \frac{1}{\rho U} \left\{ -\Delta p_t + \Delta P_t \frac{q}{V_s} \cos \varphi - i \frac{\omega}{V_s} \bar{p}_t \sin \Theta \Delta \xi \right\} - \Delta(u - v \tan \alpha) \\ & + \frac{U}{V_s} \left\{ \Delta[q(\cos \varphi + \tan \alpha \sin \varphi)] - \frac{1}{2} q \cos \varphi \Delta(\tan^2 \alpha) \right\} \\ & + i \frac{\omega}{U} \left\{ -\bar{u} \left(\cos \Theta - \frac{U}{V_s} \sin \Theta \right) + \frac{U}{V_s} \bar{v} \sin \Theta \tan \Theta + \frac{U}{V_s} q \cos(\Theta - \varphi) \right\} \Delta \xi \dots \dots \dots (24) \end{aligned}$$

$$\begin{aligned} \frac{\Delta f_y}{\rho U s} = & \frac{1}{\rho U} \left\{ \Delta P_t \frac{q}{V_s} \sin \varphi + i \frac{\omega}{V_s} \bar{p}_t \cos \Theta \Delta \xi \right\} - \Delta(u \tan \alpha + v) - q \sin \varphi \Delta(\tan \alpha) \\ & + \frac{U}{V_s} \left\{ \Delta[q \tan \alpha (\cos \varphi + \tan \alpha \sin \varphi)] - \frac{1}{2} q \sin \varphi \Delta(\tan^2 \alpha) \right\} \\ & - i \frac{\omega}{U} \left\{ \frac{U}{V_s} \bar{u} \cos \Theta + \bar{v} \cos \Theta \left(1 + \frac{U}{V_s} \tan \Theta \right) - \frac{U}{V_s} q \tan \Theta \cos(\Theta - \varphi) \right\} \Delta \xi \dots \dots \dots (25) \end{aligned}$$

From Eqs. (24) and (25), the equation of the unsteady aerodynamic force acting on each part of the cascade blade is given as follows:

At the position in the cascade where the blade surface is smooth, setting $\alpha = \Theta$,

$$\begin{aligned} \frac{1}{\rho U s} \frac{df_x}{d\xi} = & \frac{1}{\rho U} \left\{ -\frac{dp_t}{d\xi} + \frac{dP_t}{d\xi} \frac{q}{V_s} \cos \varphi - i \frac{\omega}{V_s} p_t \sin \Theta \right\} - \frac{d}{d\xi} (u - v \tan \Theta) \\ & + \frac{U}{V_s} \left\{ \frac{d}{d\xi} [q(\cos \varphi + \tan \Theta \sin \varphi)] - \frac{1}{2} q \cos \varphi \frac{d}{d\xi} (\tan^2 \Theta) \right\} \\ & + i \frac{\omega}{U} \left\{ -u \left(\cos \Theta - \frac{U}{V_s} \sin \Theta \right) + \frac{U}{V_s} v \sin \Theta \tan \Theta + \frac{U}{V_s} q \cos(\Theta - \varphi) \right\} \dots \dots \dots (26) \end{aligned}$$

$$\begin{aligned} \frac{1}{\rho U s} \frac{df_y}{d\xi} = & \frac{1}{\rho U} \left\{ \frac{dP_t}{d\xi} \frac{q}{V_s} \sin \varphi + i \frac{\omega}{V_s} p_t \cos \Theta \right\} - \frac{d}{d\xi} (u \tan \Theta + v) - q \sin \varphi \frac{d}{d\xi} (\tan \Theta) \\ & + \frac{U}{V_s} \left\{ \frac{d}{d\xi} [q \tan \Theta (\cos \varphi + \tan \Theta \sin \varphi)] - \frac{1}{2} q \sin \varphi \frac{d}{d\xi} (\tan^2 \Theta) \right\} \\ & - i \frac{\omega}{U} \left\{ \frac{U}{V_s} u \cos \Theta + v \cos \Theta \left(1 + \frac{U}{V_s} \tan \Theta \right) - \frac{U}{V_s} q \tan \Theta \cos(\Theta - \varphi) \right\} \dots \dots \dots (27) \end{aligned}$$

At the leading and trailing edges, and at the position in the cascade where the blade surface is not smooth or the state of flow changes discontinuously, setting $\Delta \xi = 0$ and making q and φ constants in Eqs. (24) and (25),

$$\frac{\Delta f_x}{\rho U s} = \frac{1}{\rho U} \left\{ -\Delta p_t + \Delta P_t \frac{q}{V_s} \cos \varphi \right\} - \Delta(u - v \tan \alpha) + \frac{U}{V_s} q \left\{ \sin \varphi \Delta(\tan \alpha) - \frac{1}{2} \cos \varphi \Delta(\tan^2 \alpha) \right\} \dots \dots \dots (28)$$

$$\begin{aligned} \frac{\Delta f_y}{\rho U s} = & \frac{\Delta P_t}{\rho U} \frac{q}{V_s} \sin \varphi - \Delta(u \tan \alpha + v) \\ & + q \left\{ -\sin \varphi \Delta(\tan \alpha) + \frac{U}{V_s} \left\{ \Delta[\tan \alpha (\cos \varphi + \tan \alpha \sin \varphi)] - \frac{1}{2} \sin \varphi \Delta(\tan^2 \alpha) \right\} \right\} \dots \dots \dots (29) \end{aligned}$$

where α in the cascade must be replaced by Θ .

From Eqs. (26) ~ (29), we can obtain the unsteady aerodynamic force and moment acting on the cascade vibrating in arbitrary mode, by using the results on the unsteady flow field as stated in the preceding section.

In the next chapter, however, a cascade of flat-plate blades will be treated in detail.

6. Cascade of flat-plate blades

On the cascade of flat-plate blades, Θ and φ being constants, the oscillating mode shape of the blade generally consists of three fundamental modes; that is, translatory or normal-to-chord motion, pitching motion about the leading edge, and tangential-to-chord motion.

Hence

$$\vec{q}(\xi) e^{i\omega t} = [\vec{B} \ \vec{P} \ \vec{T}] e^{i\omega t} \quad 0 \leq \xi \leq c \dots \dots \dots (30)$$

where B , $P(=i\omega\theta)$, and T are the velocity amplitudes of translatory, pitching, and tangential motions

respectively. The angles of vibration φ of each mode are equal to Θ , Θ , and $(\Theta - \pi/2)$,

Subscripts 3, 31, and 32 used in this chapter represent the values in the cascade—within the cascade, just downstream of the inlet, and just upstream of the outlet respectively.

6.1 Equations for the flow through the cascade

6.1.1 Equation of continuity

From Eqs. (11) and (12), the equation of continuity between the point just upstream of the cascade and an arbitrary point in the cascade may be written in a matrix form as follows:

$$\begin{aligned} u_3 = & u_{01} + [BPT] \\ & \times \left\{ \begin{array}{l} -\frac{U}{V_s} \sin \Theta (\tan \alpha_1 - \tan \Theta) + i \frac{\omega}{V_s} \xi \\ \frac{U}{V_s} \sec \Theta \xi + \frac{1}{2} i \frac{\omega}{V_s} \xi^2 \\ \frac{U}{V_s} \cos \Theta (\tan \alpha_1 - \tan \Theta) \end{array} \right\} \dots (31) \end{aligned}$$

From Eq. (13), the equation of continuity for

the cascade as a whole is also written as

$$u_{02} = u_{01} + [B P_c T] \begin{bmatrix} E_B \\ E_P \\ E_T \end{bmatrix} \dots\dots\dots (32)$$

where

$$\begin{aligned} E_B &= -\frac{U}{V_s} \sin \theta (\tan \alpha_1 - \tan \alpha_2) + i \frac{\omega c}{V_s} \\ E_P &= \frac{U}{V_s} (\cos \theta + \tan \alpha_2 \sin \theta) + \frac{1}{2} i \frac{\omega c}{V_s} \\ E_T &= \frac{U}{V_s} \cos \theta (\tan \alpha_1 - \tan \alpha_2) \end{aligned}$$

6.1.2 Boundary and outlet conditions of the cascade

From Eq. (20), the boundary condition for the flow in the cascade is expressed by

$$v_3 = u_3 \tan \theta + [B P T] \begin{bmatrix} \sec \theta \\ \left(-i \frac{U}{\omega} + \xi \cos \theta \right) \sec^2 \theta \\ 0 \end{bmatrix} \dots\dots\dots (33)$$

Rewriting Eq. (21) by using Eqs. (31) and (32), the outlet condition of flow for this case is

$$v_{02} = u_{01} (\tan \alpha_2 - b_a \tan \alpha_1) + v_{01} b_a + [B P_c T] \begin{bmatrix} F_B \\ F_P \\ F_T \end{bmatrix} \dots\dots\dots (34)$$

where

$$\begin{aligned} F_B &= \cos \theta + \tan \alpha_2 \sin \theta + \tan \alpha_2 \left[-\frac{U}{V_s} \sin \theta (\tan \alpha_1 - \tan \alpha_2) + i \frac{\omega c}{V_s} \right] - b_a (\cos \theta + \tan \alpha_1 \sin \theta) \\ F_P &= \cos \theta + \tan \alpha_2 \sin \theta - i \frac{U}{\omega c} \sec^2 \theta + \tan \alpha_2 \left[\frac{U}{V_s} (\cos \theta + \tan \alpha_2 \sin \theta) + \frac{1}{2} i \frac{\omega c}{V_s} \right] - i \frac{U}{\omega c} b_a \\ F_T &= -\cos \theta (\tan \alpha_2 - \tan \theta) + \frac{U}{V_s} \cos \theta \tan \alpha_2 (\tan \alpha_1 - \tan \alpha_2) + b_a \cos \theta (\tan \alpha_1 - \tan \theta) \end{aligned}$$

6.1.3 Equation of energy

The equation of energy for the flow through the cascade is obtained from Eqs. (16) and (17) with the help of Eqs. (31) ~ (33) as follows:

At the inlet of the cascade,

$$\begin{aligned} \frac{\Delta_1 p_i}{\rho U} &= \frac{p_{t31} - p_{t01}}{\rho U} \dots\dots\dots (35) \\ &= -u_{01} \left(Z_1 - \frac{1}{2} Z_{a1} \tan \alpha_1 \right) - \frac{1}{2} Z_{a1} v_{01} \\ &\quad + [B P_c T] \begin{bmatrix} -(\tan \alpha_1 - \tan \theta) \cos \theta - Z_1 \sin \theta + \frac{1}{2} Z_{a1} (\cos \theta + \tan \alpha_1 \sin \theta) \\ \frac{1}{2} i \frac{U}{\omega c} Z_{a1} \\ -(\tan \alpha_1 - \tan \theta) \sin \theta + Z_1 \cos \theta - \frac{1}{2} Z_{a1} \cos \theta (\tan \alpha_1 - \tan \theta) \end{bmatrix} \end{aligned}$$

At the outlet of the cascade,

$$\begin{aligned} \frac{\Delta_2 p_i}{\rho U} &= \frac{p_{t02} - p_{t32}}{\rho U} \dots\dots\dots (36) \\ &= -u_{01} \left(Z_2 - \frac{1}{2} Z_{a2} \tan \alpha_1 \right) - \frac{1}{2} Z_{a2} v_{01} \\ &\quad + [B P_c T] \begin{bmatrix} (\tan \alpha_2 - \tan \theta) \cos \theta - Z_2 \left[\sin \theta \left\{ 1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right\} + i \frac{\omega c}{V_s} \right] \\ + \frac{1}{2} Z_{a2} (\cos \theta + \tan \alpha_1 \sin \theta) \\ (\tan \alpha_2 - \tan \theta) \cos \theta - Z_2 \left[\sin \theta + \frac{U}{V_s} \sec \theta + \frac{1}{2} i \frac{\omega c}{V_s} \right] + \frac{1}{2} i \frac{U}{\omega c} Z_{a2} \\ (\tan \alpha_2 - \tan \theta) \sin \theta + Z_2 \cos \theta \left[1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right] \\ - \frac{1}{2} Z_{a2} \cos \theta (\tan \alpha_1 - \tan \theta) \end{bmatrix} \end{aligned}$$

For the flow within the cascade,

$$\frac{p_{t3}}{\rho U} = \frac{p_{t31}}{\rho U} - u_{01} \left(i \frac{\omega \xi}{U} \sec \theta + [z_3]_{\xi} - \frac{1}{2} [z_{a3}]_{\xi} \tan \alpha_1 \right) - \frac{1}{2} [z_{a3}]_{\xi} v_{01} \dots (37)$$

$$- [BPT] \left[\begin{aligned} & i \frac{\omega}{U} \left\{ \tan \theta \left[1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right] \xi + \frac{1}{2} i \frac{\omega}{V_s} \xi^2 \sec \theta \right\} \\ & + [z_3]_{\xi} \sin \theta \left[1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right] + i \frac{\omega}{V_s} [\xi z_3]_{\xi} \\ & - \frac{1}{2} [z_{a3}]_{\xi} (\cos \theta + \tan \alpha_1 \sin \theta) \\ & \sec \theta \left\{ \xi \tan \theta + \frac{1}{2} i \frac{\omega}{U} \left(\sin \theta + \frac{U}{V_s} \sec \theta \right) \xi^2 - \frac{1}{6} \frac{\omega}{U} \frac{\omega}{V_s} \xi^3 \right\} \\ & + [\xi z_3]_{\xi} \left(\sin \theta + \frac{U}{V_s} \sec \theta \right) + \frac{1}{2} i \frac{\omega}{V_s} [\xi^2 z_3]_{\xi} - \frac{1}{2} i \frac{U}{\omega} [z_{\theta 3}]_{\xi} \\ & i \frac{\omega}{V_s} (\tan \alpha_1 - \tan \theta) \xi - [z_3]_{\xi} \cos \theta \left\{ 1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right\} \\ & + \frac{1}{2} [z_{a3}]_{\xi} \cos \theta (\tan \alpha_1 - \tan \theta) \end{aligned} \right]$$

where $[z]_{\xi} = \int^{\xi} z(\xi) d\xi$ and so on.

From Eq. (18), p_{t31} in Eq. (37) can be expressed by

$$\frac{p_{t31}}{\rho U} = \frac{p_{t01} + \Delta_1 p_t}{\rho U} = \frac{V_s}{U} \left(v_{01} - i \frac{U}{\omega} \zeta_{01} \right) + \frac{\Delta_1 p_t}{\rho U} \dots (38)$$

Then the total increase of the total pressure of the flow through the cascade is

$$\Delta_0 p_t = \sum_{i=1}^3 \Delta_i p_t \dots (39)$$

where $\Delta_3 p_t = p_{t32} - p_{t31}$

Eliminating $\Delta_0 p_t$ between Eqs. (19) and (39), the energy equation for the cascade as a whole is finally given by

$$i \frac{U}{\omega} \zeta_{02} = i \frac{U}{\omega} \zeta_{01} + C u_{01} + D v_{01} + [BPT] \begin{bmatrix} G_B \\ G_P \\ G_T \end{bmatrix} \dots (40)$$

where

$$C = \tan \alpha_2 - b_a \tan \alpha_1 + \frac{U}{V_s} \left[i \frac{\omega c}{U} + \sum_{i=1}^3 \left(Z_i - \frac{1}{2} Z_{at} \tan \alpha_1 \right) \right]$$

$$D = -1 + b_a + \frac{1}{2} \frac{U}{V_s} \sum_{i=1}^3 Z_{at}$$

$$G_B = \cos \theta + \tan \alpha_2 \sin \theta + \frac{U}{V_s} (\tan \alpha_1 - \tan \alpha_2) (\cos \theta - \tan \alpha_2 \sin \theta) + i \frac{\omega c}{V_s} (\tan \theta + \tan \alpha_2)$$

$$- \left(\frac{\omega c}{V_s} \right)^2 \left[\frac{1}{2} \sec \theta + i \frac{U}{\omega c} \tan \theta (\tan \alpha_1 - \tan \theta) \right] - b_a (\cos \theta + \tan \alpha_1 \sin \theta)$$

$$+ \frac{U}{V_s} \left[Z_1 \sin \theta + Z_2 \left[\sin \theta \left\{ 1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right\} + i \frac{\omega c}{V_s} \right] \right]$$

$$+ [z_3]_c \sin \theta \left[1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right] + i \frac{\omega c}{V_s} \left[\frac{\xi}{c} z_3 \right]_c - \frac{1}{2} (\cos \theta + \tan \alpha_1 \sin \theta) \sum_{i=1}^3 Z_{at}$$

$$G_P = \cos \theta + \tan \alpha_2 \sin \theta - i \frac{U}{\omega c} \sec^2 \theta + \frac{U}{V_s} \sin \theta (\sec^2 \theta + \sec^2 \alpha_2)$$

$$+ \frac{1}{2} i \frac{\omega c}{V_s} (\tan \theta + \tan \alpha_2) + \frac{1}{6} \frac{U}{V_s} \frac{\omega c}{V_s} \sec \theta \left(3i \sec \theta - \frac{\omega c}{U} \right) - i \frac{U}{\omega c} b_a$$

$$+ \frac{U}{V_s} \left\{ Z_2 \left[\sin \theta + \frac{U}{V_s} \sec \theta + \frac{1}{2} i \frac{\omega c}{V_s} \right] + \left[\frac{\xi}{c} z_3 \right]_c \left(\sin \theta + \frac{U}{V_s} \sec \theta \right) + \frac{1}{2} i \frac{\omega c}{V_s} \left[\frac{\xi^2}{c^2} z_3 \right]_c - \frac{1}{2} i \frac{U}{\omega c} \sum_{i=1}^3 z_{\theta i} \right\}$$

$$G_T = \cos \theta (\tan \theta - \tan \alpha_2) + \frac{U}{V_s} (\tan \alpha_1 - \tan \alpha_2) (\sin \theta + \tan \alpha_2 \cos \theta)$$

$$+ (\tan \alpha_1 - \tan \theta) \left(i \frac{U}{V_s} \frac{\omega c}{V_s} + b_a \cos \theta \right) - \frac{U}{V_s} \left\{ Z_1 \cos \theta + Z_2 \cos \theta \left[1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right] \right\}$$

$$+ [z_3]_c \cos \theta \left[1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right] - \frac{1}{2} \cos \theta (\tan \alpha_1 - \tan \theta) \sum_{i=1}^3 Z_{at}$$

with $Z_{\alpha\beta, \theta\beta} = \int_0^c z_{\alpha\beta, \theta\beta} d\xi$ in them.

6.2 Solution of the flow field

The fundamental equations for the flow through the cascade of flat-plate blades, Eqs. (32), (34), and (40), may be arranged in a matrix form⁽²⁾ as follows:

$$\begin{bmatrix} u_{02} \\ v_{02} \\ i \frac{U}{\omega} \zeta_{02} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \tan \alpha_2 - b_a \tan \alpha_1 & b_a & 0 \\ C & D & 1 \end{bmatrix} \begin{bmatrix} u_{01} \\ v_{01} \\ i \frac{U}{\omega} \zeta_{01} \end{bmatrix} + \begin{bmatrix} E_B & E_P & E_T \\ F_B & F_P & F_T \\ G_B & G_P & G_T \end{bmatrix} \begin{bmatrix} B \\ P_C \\ T \end{bmatrix} \dots\dots\dots (41)$$

Now, assuming the irrotational flow upstream of the cascade, the equations for the upstream flow, Eqs. (5) and (6), are also arranged in a matrix form; that is

$$\begin{bmatrix} u_{01} \\ v_{01} \\ i \frac{U}{\omega} \zeta_{01} \end{bmatrix} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} u_{01} \dots\dots\dots (42)$$

Putting Eq. (42) into Eq. (41)

$$\begin{bmatrix} u_{02} \\ v_{02} \\ i \frac{U}{\omega} \zeta_{02} \end{bmatrix} = \begin{bmatrix} 1 \\ \tan \alpha_2 - b_a (\tan \alpha_1 + i) \\ C - iD \end{bmatrix} u_{01} + \begin{bmatrix} E_B & E_P & E_T \\ F_B & F_P & F_T \\ G_B & G_P & G_T \end{bmatrix} \begin{bmatrix} B \\ P_C \\ T \end{bmatrix} \dots\dots\dots (43)$$

while, in the flow field downstream of the cascade, there must be satisfied Eq. (4) or

$$i \frac{U}{\omega} \zeta_{02} + (u_{02} + i v_{02}) \left\{ \frac{U}{V_s} + i \left(1 - \frac{U}{V_s} \tan \alpha_2 \right) \right\} = 0 \dots\dots\dots (4)'$$

Then, substituting Eq. (43) into Eq. (4) and eliminating u_{02} , v_{02} , and ζ_{02} , we can obtain the perturbation velocity upstream of the cascade, u_{01} , induced by the vibration of the cascade blades (B , P , T).

Putting u_{01} thus obtained into Eqs. (42) and (43), the flow fields up- and down-stream of the cascade are completely determined, and in consequence the flow in the cascade is known.

6.3 Aerodynamic force acting on the vibrating cascade blades

The unsteady aerodynamic force acting on the oscillating cascade of flat-plate blades is deduced from Eqs. (26) ~ (29). In this section, for convenience, there will be given the equations of the unsteady aerodynamic force for the directions normal and tangential to the blade chord.

From Eqs. (28) and (29), the unsteady force acting on the leading edge is given by

$$\begin{aligned} \frac{f_{N1}}{\rho U s} &= \frac{\Delta_1 p_t}{\rho U} \sin \theta + u_{01} [\cos \theta (\tan \alpha_1 - \tan \theta) - \tan \theta \sec \theta] + v_{01} (\cos \theta + \tan \alpha_1 \sin \theta) \dots\dots\dots (44) \\ &+ [B P_C T] \left[\begin{aligned} &-\sec^2 \theta + (\tan \alpha_1 - \tan \theta) \left\{ \sin \theta \cos \theta + \frac{U}{V_s} [\tan^2 \theta - \cos \theta (\cos \theta + \tan \alpha_1 \sin \theta)] \right\} \\ &i \frac{U}{\omega c} \sec^3 \theta \\ &-(\tan \alpha_1 - \tan \theta) \left\{ \cos^2 \theta + \frac{U}{V_s} \left[2 \tan \theta - \frac{1}{2} (\tan \alpha_1 - \tan \theta) \cos 2\theta \right] \right\} - \frac{1}{2} \frac{U}{V_s} Z_1 \end{aligned} \right] \end{aligned}$$

$$\begin{aligned} \frac{f_{T1}}{\rho U s} &= -\frac{\Delta_1 p_t}{\rho U} \cos \theta + (u_{01} \sin \theta - v_{01} \cos \theta) (\tan \alpha_1 - \tan \theta) \dots\dots\dots (45) \\ &+ [B P_C T] \left[\begin{aligned} &\sin^2 \theta (\tan \alpha_1 - \tan \theta) + \frac{U}{V_s} \left[1 + \tan \alpha_1 \tan \theta + \frac{1}{2} (\tan^2 \alpha_1 - \tan^2 \theta) \right] \\ &-(\cos \theta + \tan \alpha_1 \sin \theta)^2 - \frac{1}{2} Z_1 \\ &0 \\ &\sin \theta \cos \theta (\tan \alpha_1 - \tan \theta) \left[-1 + \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right] \end{aligned} \right] \end{aligned}$$

where $\Delta_1 p_t$ is given by Eq. (35).

Similarly on the trailing edge,

$$\frac{f_{N2}}{\rho U s} = \frac{\Delta_2 p_t}{\rho U} \sin \theta + u_{02} [\cos \theta (\tan \theta - \tan \alpha_2) + \sec \theta \tan \theta] - v_{02} (\cos \theta + \tan \alpha_2 \sin \theta) \dots (46)$$

$$+ [B P_G T] \left[\begin{aligned} & \sec^2 \theta - (\tan \alpha_2 - \tan \theta) \left\{ \sin \theta \cos \theta + \frac{U}{V_s} [\tan^2 \theta - \cos \theta (\cos \theta + \tan \alpha_2 \sin \theta)] \right\} \\ & \sec^2 \theta \left(\cos \theta - i \frac{U}{\omega c} \right) - (\tan \alpha_2 - \tan \theta) \left\{ \sin \theta \cos \theta \right. \\ & \quad \left. + \frac{U}{V_s} [\tan^2 \theta - \cos \theta (\cos \theta + \tan \alpha_2 \sin \theta)] \right\} \\ & (\tan \alpha_2 - \tan \theta) \left[\cos^2 \theta + \frac{U}{V_s} \left\{ \frac{1}{2} (\tan \alpha_2 + 3 \tan \theta) - \cos^2 \theta (\tan \alpha_2 - \tan \theta) \right\} \right] \\ & \quad + \frac{1}{2} \frac{U}{V_s} Z_2 \end{aligned} \right]$$

$$\frac{f_{T2}}{\rho U s} = -\frac{\Delta_2 p_t}{\rho U} \cos \theta + (-u_{02} \sin \theta + v_{02} \cos \theta) (\tan \alpha_2 - \tan \theta) \dots (47)$$

$$+ [(B + P_G) T] \left[\begin{aligned} & -\sin^2 \theta (\tan \alpha_2 - \tan \theta) - \frac{U}{V_s} \left[1 + \tan \alpha_2 \tan \theta + \frac{1}{2} (\tan^2 \alpha_2 - \tan^2 \theta) \right. \\ & \quad \left. - (\cos \theta + \tan \alpha_2 \sin \theta)^2 + \frac{1}{2} Z_2 \right] \\ & \sin \theta \cos \theta (\tan \alpha_2 - \tan \theta) \left[1 - \frac{U}{V_s} (\tan \alpha_2 - \tan \theta) \right] \end{aligned} \right]$$

where $\Delta_2 p_t$ is given by Eq. (36). Especially for the smoothly outgoing flow at the exit of the cascade, $f_{N2} = f_{T2} = 0$.

From Eqs. (26) and (27), the unsteady force acting on the blade segment between the leading and trailing edges is given by

$$\frac{1}{\rho U s} \frac{df_{N3}}{d\xi} = -i \frac{\omega}{V_s} u_{01} \sec^2 \theta + \left(i \frac{\omega}{V_s} + \sin \theta \frac{d}{d\xi} \right) \frac{p_{t3}}{\rho U} \dots (48)$$

$$- [B P T] \left[\begin{aligned} & \sec^2 \theta \left\{ i \frac{\omega}{U} \cos^3 \theta + i \frac{\omega}{V_s} \left[2 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) - \left(\frac{\omega}{V_s} \right)^2 \xi \right] \right\} \\ & \sec^2 \theta \left\{ 1 + \cos^2 \theta + 2 \frac{U}{V_s} \tan \theta + \left[i \frac{\omega}{U} \cos^3 \theta + i \frac{\omega}{V_s} \left(2 \sin \theta \right. \right. \right. \\ & \quad \left. \left. + \frac{U}{V_s} \sec \theta \right) \right] \xi - \frac{1}{2} \left(\frac{\omega}{V_s} \right)^2 \xi^2 \right\} \\ & \frac{U}{V_s} \left\{ i \frac{\omega}{V_s} \sec \theta (\tan \alpha_1 - \tan \theta) - \frac{1}{2} z_3 \right\} \end{aligned} \right]$$

$$\frac{1}{\rho U s} \frac{df_{T3}}{d\xi} = -i \frac{\omega}{U} u_{01} - \frac{1}{\rho U} \frac{dp_{t3}}{d\xi} \cos \theta \dots (49)$$

$$- [B P T] \left[\begin{aligned} & i \frac{\omega}{U} \left\{ \sin \theta \left[1 - \frac{U}{V_s} (\tan \alpha_1 - \tan \theta) \right] + i \frac{\omega}{V_s} \xi \right\} + \frac{1}{2} \frac{U}{V_s} z_3 \\ & \tan \theta + i \frac{\omega}{U} \left\{ \left(\sin \theta + \frac{U}{V_s} \sec \theta \right) \xi + \frac{1}{2} i \frac{\omega}{V_s} \xi^2 \right\} + \frac{1}{2} \frac{U}{V_s} \xi z_3 \\ & i \frac{\omega}{V_s} \cos \theta (\tan \alpha_1 - \tan \theta) \end{aligned} \right]$$

where p_{t3} is given by Eq. (37). Especially for the flow without loss, $f_{T3} = 0$.

Using the equations mentioned above, the unsteady aerodynamic force and moment about the leading edge can be calculated by the following equations:

$$f_N = f_{N1} + f_{N2} + \int_0^c \frac{df_{N3}}{d\xi} d\xi, \quad f_T = f_{T1} + f_{T2} + \int_0^c \frac{df_{T3}}{d\xi} d\xi, \quad m = f_{N2} c + \int_0^c \xi \frac{df_{N3}}{d\xi} d\xi \dots (50)$$

6.4 Numerical results

In this section, we will execute some numerical calculations on a special case, where the total pressure loss of the flow occurs only at the inlet of the cascade but with some time delay against the blade motion and the fluid always flows out the cascade tangentially without loss; hence

$$Z_2 = z_3(\xi) = 0, \quad b_a = b_g = 0, \quad \alpha_2 = \theta \dots (51)$$

which are good approximations for any cascade with high solidity, whether stalled or not.

The unsteady aerodynamic force and moment acting on the cascade blades for this case are expressed by

$$\left. \begin{aligned} f_N &= f_{N1} + \int_0^c \frac{df_{N3}}{d\xi} d\xi, \quad f_T = f_{T1} \\ m &= \int_0^c \xi \frac{df_{N3}}{d\xi} d\xi \end{aligned} \right\} \dots\dots\dots (50)'$$

Assuming that the total pressure-loss occurs with an exponential time delay behind the step-change of the inlet flow angle and the stagger angle, the dynamic derivative of the total pressure-loss coefficient at the inlet of the cascade may be expressed by

$$Z_{a1, \theta 1} = \frac{Z_{a1, \theta 1}^0}{1 + i\omega\tau} \dots\dots\dots (52)$$

where $Z_{a1, \theta 1}^0$: static slope of the total pressure-loss coefficient

τ : constant of time delay

For the case with no time delay or $\tau=0$, we can easily derive the equations of the unsteady aerodynamic force and moment for each mode in the explicit form, which are summarized in appendix. Söhngen⁽³⁾ has also given a theoretical treatment of the cascade blade vibration in non-stall state by the wing theory under the identical assumptions as used in this paper. In the non-stall state, of course, the results of the present theory in appendix coincide with the Söhngen's except a few disagreements.

Now, consider a compressor cascade with the angle of stagger $\theta=30^\circ$, the stall characteristic of which is expressed by

$$Z_1 = 0.02 + 0.1(\tan \alpha_1 - 0.75)^2 + 2.0(\tan \alpha_1 - 0.75)^4 \dots\dots\dots (53)$$

Hence

$$Z_{a1}^0 = 0.2(\tan \alpha_1 - 0.75) + 8(\tan \alpha_1 - 0.75)^3 \dots\dots\dots (54)$$

Fig. 3 shows the stall-characteristic curves of Eqs. (53) and (54).

On the other hand, for the pitching motion about the leading edge, it is assumed that

$$Z_{\theta 1}^0 = Z_{a1}^0 \sec^2 \alpha_1 \dots\dots\dots (55)$$

by supposing the change of the inlet flow angle

equivalent to the variation of the stagger angle owing to the pitching motion.

In the calculations, there will be used the following parameters;

$$k \equiv \frac{\omega c}{U}: \text{reduced frequency } \text{sgn } k = \text{sgn } V_s \dots\dots (56)$$

$$\sigma \equiv \frac{\omega c}{V_s}: \text{phase difference between two points}$$

separated by a chord-length in the circumference of the cascade(57)

6.4.1 Aerodynamic damping force and moment

a) Translatory oscillation The calculated results of the aerodynamic damping force of the cascade blades vibrating in translatory mode with the phase angle $\sigma=1$ [real part of $(-\frac{f_N}{\rho U s(B/|k|)})$] are shown in Fig. 4.

When the blades vibrate in advance of the concave-sided blades ($V_s < 0$), as shown by full lines in Fig. 4 (a), for the larger reduced frequency ($|k|$) the aerodynamic damping force decreases significantly as the cascade falls into stall, say for $\tan \alpha_1 > 1.0$, and for the smaller reduced frequency it may be negative even in the non-stall state, making a slight increase before the sharp drop in the deep stall region.

Now, consider the effect of the boundary-layer time delay on the aerodynamic damping force.

Although there has been no established affirmation on the boundary-layer time delay, we would make the calculation by taking $\tau/(c/U)=1.0$ that is presumed to be reasonable from the Stenning's experiment⁽⁴⁾ on the rotating stall in cascade. As shown by broken lines in Fig. 4 (a), the boundary-layer time delay acts to increase the aerodynamic damping force for the larger $|k|$ and to decrease it in the stall-inception region for the smaller $|k|$.

Next, when the blades vibrate behind the concave-sided blades [$V_s > 0$; see Fig. 4 (b)] without

Fig. 3 Stall characteristics

Fig. 4 Aerodynamic damping force in translatory mode

the boundary-layer time delay, as the angle of attack increases, the aerodynamic damping force increases more steeply in the slightly stalled state than in the non-stall state, especially for the larger reduced frequency (k), but immediately after, it decreases abruptly (to be negative for the larger k) before it recovers again in the deep stall region.

The effect of the boundary-layer time delay appears, for the larger k , to suppress the change of the aerodynamic damping force due to the stall, and, for the intermediate k , to increase it remarkably in the stall-inception region. On the other hand, for the smaller k , there is no significant effect of the boundary layer time delay on the aerodynamic damping force.

b) Pitching oscillation about the leading edge
The calculated results of the aerodynamic damping moment of the cascade blades oscillating about its leading edge with the phase angle $\sigma=1$ [imaginary part of $(-\frac{m}{\rho U^2 s c \theta})$] are illustrated in Fig. 5.

When the blades vibrate in advance of the concave-sided blades [$V_s < 0$, Fig. 5 (a)], the aerodynamic damping moment increases as the cascade falls into stall.

On the other hand, the aerodynamic damping moment for the non-stall oscillation is independent of the angle of attack as shown by chain lines in the figures.

The boundary-layer time delay seems to keep down the rate of increase of the aerodynamic damping moment.

When the blades vibrate behind the concave-sided blades without the boundary-layer time delay [full lines in Fig. 5 (b)], the larger the reduced frequency, the larger the drop of the aerodynamic damping moment in the slightly stalled state is, but we can see its significant recovery in the deep stall region.

The effect of the boundary-layer time delay is most remarkable for the intermediate k in the negative sense, and, for the larger and smaller k , it appears to suppress the change of the aerodynamic damping moment owing to the stall.

6.4.2 Flutter velocity

The critical flutter velocity of a cascade is defined at each inlet angle as the lowest reciprocal of the reduced frequency ($|k|$) giving neutral stability or null aerodynamic damping for every inter-blade phase angle.

In this paper, however, considering only the case of $\sigma=1$, the flutter velocity is taken as the smaller reciprocal of $|k|$ at neutral stability, either for $V_s < 0$ or for $V_s > 0$, in Figs. 4 and 5 respectively.

The results are shown in Fig. 6.

In the transitory oscillation, the non-stall flutter velocity of the cascade decreases monotonically as the inlet angle increases, as shown by chain line in Fig. 6. When the cascade is accompanied by stall, the flutter velocity becomes higher than the non-stall one at the inception of stall, being followed by an abrupt drop in the deep stall. The boundary-layer time delay acts to restrain the change of the flutter velocity owing to the stall.

The same descriptions as mentioned above may be applied to the pitching oscillation about the leading edge.

In Fig. 6, supposing the cascade with a pitching/translation frequency ratio equal to 5, the ordinate of the pitching flutter is stretched five times as large as that of the transitory flutter. Then, the cascade would be liable to fall into a pitching flutter in the non-stall state, but, for the higher inlet angle or in the stalled state, the transitory flutter would become more dangerous than the pitching one.

Fig. 5 Aerodynamic damping moment in pitching mode

Fig. 6 Flutter velocity

Applying the semi-actuator disc method to "rotating stall" in cascade, it may be prescribed that the rotating stall occurs under the following condition⁽¹⁾:

$$Z_a = 2(Z + \sec^2 \theta) / \tan \alpha_1$$

Hence, in this case, the rotating stall would occur at $\tan \alpha_1 = 1.44$. Then, as shown in Fig. 6, the stall flutter precedes the rotating stall.

7. Conclusions

This paper gives a theoretical treatment of the cascade blades vibrating in arbitrary mode, under any condition of cascade — whether stalled or not —

on a unified basis, by means of the semi-actuator disc method: that is, firstly, general treatment of the cascade with camber and distributed loss of energy, and secondly, the definite fundamental equations for the cascade of flat-plate blades vibrating in the three fundamental modes.

Then, some calculations are made for a compressor cascade of flat-plate blades under the assumption that the loss of energy is concentrated at the entrance of the cascade but the fluid flows through and out of the cascade without loss of energy.

The calculated results indicate firstly that, both in translatory mode and in pitching mode about the leading edge, the stall-flutter velocity of the compressor cascade is higher in the slightly stalled state and lower in the deep stall region than the non-stall one, and secondly that the effect of the boundary-layer time delay appears to suppress the deviation of the stall-flutter velocity from the non-stall one.

As mentioned above, we can readily obtain several interesting informations about the stall flutter in cascade by this theory. Although there is yet no affirmation on the extent to which these results would hold for a cascade with finite spacing, the present method may be applied not only to a cascade with high solidity but also to a cascade with finite spacing by taking into consideration the blade camber, relaxed outlet condition, generalized stall characteristics and so on.

Appendix Unsteady aerodynamic force and moment acting on vibrating flat-plate blades in cascade

In the following, there will be treated a special case, where the total pressure loss of flow occurs only at the inlet of the cascade without boundary-layer time delay and the fluid flows out the cascade smoothly. [cf. Eqs. (51) and (50)']

The following symbols are used:

$$k = \frac{\omega c}{U} \quad \sigma = \frac{\omega c}{V}$$

$$C_a = \sin(\alpha_1 - \theta) / \cos \alpha_1 : \text{factor of steady circulation about a blade}^{(1)}$$

$$N = \sigma^2 \left\{ 1 + Z \cos^2 \theta - \frac{1}{2} Z_a (C_a + \sin \theta) \cos \theta \right\}^2 + \left\{ k(\sigma + 2 \cos \theta) - \frac{1}{2} Z_a \sigma \cos \theta \right\}^2 \cos^2 \theta$$

$$A_j = A_{jr} + i A_{ji} \text{ etc.}$$

Append.-1 Translatory vibration:

$$\frac{f_N}{\partial U B s} = [\text{cf. Eq. (35) in the 1st report}]$$

$$\frac{m}{\partial U s c(iB)} = \frac{1}{N} \left\{ \sum_{j=0}^1 A_j C_a^j + Z \left(\sum_{j=0}^1 B_j C_a^j \right) + Z^2 \left(\sum_{j=0}^1 C_j C_a^j \right) + Z_a \left(\sum_{j=0}^2 D_j C_a^j \right) + Z_a^2 \left(\sum_{j=0}^3 E_j C_a^j \right) + Z Z_a \left(\sum_{j=0}^2 F_j C_a^j \right) \right\}$$

$$\text{where } A_{0r} = \frac{1}{24 \cos^2 \theta} \{ -12 \sigma^3 \sin \theta + k \sigma^2 \cos \theta (\sigma^2 + 4 \sigma \cos \theta) - 2 k^2 \sigma \sin \theta \cos^2 \theta (3 \sigma^2 + 16 \sigma \cos \theta + 24 \cos^2 \theta) - k^3 \cos^3 \theta (\sigma + 2 \cos \theta) (\sigma^3 + 8 \sigma^2 \cos \theta + 24 \sigma \cos^2 \theta + 24 \cos^3 \theta) \}$$

$$A_{0i} = \frac{1}{12 \cos^2 \theta} \{ 2 \sigma^3 (\sigma + 3 \cos \theta) + 2 k \sigma^3 \sin \theta \cos \theta + k^2 \sigma \cos^2 \theta (\sigma^3 + 7 \sigma^2 \cos \theta + 16 \sigma \cos^2 \theta + 12 \cos \theta \cos 2 \theta) - k^3 \sigma \sin \theta \cos^3 \theta (\sigma + 2 \cos \theta) (\sigma + 4 \cos \theta) \}$$

$$\begin{aligned}
A_{1r} &= \frac{1}{6} \{k\sigma^2 \sin \theta (\sigma + 6 \cos \theta) - k^2 \sigma (\sigma + 2 \cos \theta) (\sigma \cos \theta + 3 \cos 2\theta) \cos \theta\} \\
A_{1i} &= -\frac{1}{6} \{k\sigma^2 (\sigma \cos \theta + 3 \cos 2\theta) + k^2 \sigma \sin \theta \cos \theta (\sigma + 2 \cos \theta) (\sigma + 6 \cos \theta)\} \\
B_{0r} &= -\frac{1}{12} \{12\sigma^3 \sin \theta + k(\sigma^4 \cos \theta + 4\sigma^3 \cos^2 \theta + 6\sigma^2 \cos^3 \theta) + k^2 \sin \theta \cos^2 \theta (3\sigma^3 + 10\sigma^2 \cos \theta + 12\sigma \cos^2 \theta)\} \\
B_{0i} &= \frac{1}{12} \{4\sigma^3 (\sigma + 3 \cos \theta) - 2k\sigma^2 \sin \theta \cos \theta (\sigma + 3 \cos \theta) + k^2 \cos^2 \theta (\sigma^4 + 5\sigma^3 \cos \theta \\
&\quad + 6\sigma^2 \cos^2 \theta - 12\sigma \sin^2 \theta \cos \theta)\} \\
B_{1r} &= \frac{1}{6} \{3\sigma^3 \sin^2 \theta - k\sigma^3 \sin \theta \cos \theta (\sigma + 4 \cos \theta)\} \\
B_{1i} &= \frac{1}{6} \{-\sigma^3 \sin \theta (\sigma + 3 \cos \theta) - k\sigma^2 \cos \theta [3 \cos \theta + \sigma (2 \sin^2 \theta + 1)]\} \\
C_{0r} &= -\frac{1}{8} \sigma^2 \cos^2 \theta \{4\sigma \sin \theta + k \cos \theta (\sigma + 2 \cos \theta)^2\}, & C_{0i} &= \frac{\sigma^3}{6} \cos^2 \theta \{(\sigma + 3 \cos \theta) - k \sin \theta \cos \theta\} \\
C_{1r} &= \frac{1}{2} \sigma^3 \sin^2 \theta \cos^2 \theta, & C_{1i} &= -\frac{1}{6} \sigma^3 \sin \theta \cos^2 \theta (\sigma + 3 \cos \theta) \\
D_{0r} &= \frac{1}{24 \cos \theta} \{12\sigma^3 \sin^2 \theta + k \sin \theta \cos \theta (\sigma^4 + 12\sigma^3 \cos \theta + 24\sigma^2 \cos^2 \theta) \\
&\quad + k^2 [2\sigma^4 \cos^2 \theta + 15\sigma^3 \cos^3 \theta + \sigma^2 (24 \cos^4 \theta - 3 \cos^3 \theta + 16 \cos^2 \theta + 3) + 36\sigma \cos^3 \theta]\} \\
D_{0i} &= \frac{1}{24 \cos \theta} \{-4\sigma^3 \sin \theta (\sigma + 3 \cos \theta) + 2k \cos \theta [-\sigma^4 \cos \theta + \sigma^3 (1 + 14 \cos^2 \theta) - 6 \cos \theta \cos 2\theta] \\
&\quad - k^2 \sin \theta \cos^2 \theta (\sigma^4 + 2\sigma^3 \cos \theta - 8\sigma^2 \cos^2 \theta - 12\sigma \cos \theta)\} \\
D_{1r} &= \frac{1}{24 \cos \theta} \{2\sigma^3 (3 - \sigma \cos \theta) \sin \theta + k \cos \theta [\sigma^4 (3 - 2 \cos^2 \theta) + 6\sigma^3 \cos \theta + 12\sigma^2 \cos^2 \theta \cos 2\theta] \\
&\quad + k^2 \sin \theta \cos^2 \theta (3\sigma^3 + 10\sigma^2 \cos \theta + 12\sigma \cos^2 \theta)\} \\
D_{1i} &= \frac{1}{24 \cos \theta} \{2\sigma^4 (\sin^2 \theta - 2) - 12\sigma^3 \cos \theta + k \cos \theta [12\sigma^2 \sin \theta \cos \theta (1 + 2 \cos^2 \theta) \\
&\quad + \sigma^3 (2 \sin \theta + 6 \sin^3 \theta + 14 \sin \theta \cos^2 \theta) - 2\sigma^4 \sin \theta \cos \theta] \\
&\quad + k^2 \cos^2 \theta [12\sigma \sin^2 \theta \cos \theta - \sigma^2 (\sigma^2 + 4\sigma \cos \theta + 6 \cos^2 \theta)]\} \\
D_{2r} &= \frac{1}{24 \cos \theta} \{-6\sigma^3 \sin^2 \theta + 2k \cos \theta [-6\sigma \cos^3 \theta + \sigma^2 \sin \theta (\sigma^2 + 4\sigma \cos \theta + 3 \cos^2 \theta)]\} \\
D_{2i} &= \frac{1}{24 \cos \theta} \{2\sigma^4 \sin \theta + 6\sigma^3 \sin \theta \cos \theta + k \cos \theta [6\sigma^2 \cos \theta + \sigma^3 (6 \sin^2 \theta + 2 \cos^2 \theta)] \\
&\quad + k^2 \cos^2 \theta [12 \sin^2 \theta \cos \theta - \sigma^2 (\sigma^2 + 4\sigma \cos \theta + 6 \cos^2 \theta)]\} \\
E_{0r} &= -\frac{1}{96} \{12\sigma^3 \sin \theta - k \cos \theta (24\sigma^2 \cos^2 \theta + 16\sigma^3 \cos \theta + 3\sigma^4)\} \\
E_{0i} &= \frac{1}{24} \{\sigma^3 (\sigma + 3 \cos \theta) - k \cos \theta \sin \theta (6\sigma^2 \cos \theta + 2\sigma^3)\} \\
E_{1r} &= -\frac{1}{96} \{12\sigma^3 \sin^2 \theta + k \sin \theta \cos \theta (24 \cos^2 \theta + 28\sigma^3 \cos \theta + 6\sigma^4)\} \\
E_{1i} &= -\frac{1}{96} \{4\sigma^4 \sin \theta + 12\sigma^3 \sin \theta \cos \theta + k \cos \theta [4\sigma^3 (1 + 3 \sin^2 \theta) + 12\sigma^2 \cos \theta (1 + 2 \sin^2 \theta)] \\
&\quad - k^2 \sin \theta \cos^2 \theta (12 \cos^2 \theta + 12\sigma \cos \theta + 3\sigma^2)\} \\
E_{2r} &= \frac{1}{96} \{4\sigma^4 \cos 2\theta + 12\sigma^3 \cos \theta \cos 2\theta - 4k\sigma^2 \sin \theta \cos \theta (2\sigma + 3 \cos \theta)\} \\
E_{2i} &= \frac{1}{96} \{4\sigma^4 \cos 2\theta + 12\sigma^3 \cos \theta \cos 2\theta - 4k\sigma^2 \sin \theta \cos \theta (2\sigma + 3 \cos \theta)\} \\
E_{3r} &= \frac{1}{8} \sigma^3 \sin^2 \theta, & E_{3i} &= -\frac{1}{24} \sigma^3 \sin \theta (\sigma + 3 \cos \theta) \\
F_{0r} &= \frac{1}{24} \{12\sigma^3 \sin^2 \theta \cos \theta + k \cos^2 \theta (12\sigma^2 \sin \theta \cos^2 \theta + 14\sigma^3 \sin \theta \cos \theta + 3\sigma^4 \sin \theta)\} \\
F_{0i} &= \frac{1}{12} \{-2\sigma^3 \sin \theta \cos \theta (\sigma + 3 \cos \theta) + k \cos^2 \theta [3\sigma^2 \cos \theta (2 \sin^2 \theta + 1) + \sigma^3 (3 \sin^2 \theta + 1)]\} \\
F_{1r} &= \frac{1}{24} \{12\sigma^3 \sin \theta \cos^3 \theta + k\sigma^2 \cos^2 \theta (12 \cos^2 \theta + 12\sigma \cos \theta + 3\sigma^2)\}
\end{aligned}$$

$$F_{1t} = \frac{1}{6} \sigma^2 \cos \theta \{ -\sigma(\sigma + 3 \cos \theta) \cos^2 \theta + k \sin \theta \cos \theta (2\sigma + 3 \cos \theta) \}$$

$$F_{2r} = -\frac{1}{2} \sigma^3 \sin^2 \theta \cos \theta, \quad F_{2t} = \frac{1}{6} \sigma \sin \theta \cos \theta (\sigma + 3 \cos \theta)$$

Append.-2 Pitching vibration about the leading edge:

$$\frac{f_N}{\rho U^2 s \theta} = \frac{-1}{N} \left\{ \sum_{j=0}^1 A_j C_{a^j} + Z \left(\sum_{j=0}^1 B_j C_{a^j} \right) + Z^2 C + Z_{\alpha} \left(\sum_{j=1}^1 D_j C_{a^j} \right) + Z_{\alpha}^2 \left(\sum_{j=0}^2 E_j C_{a^j} \right) \right. \\ \left. + Z_{\theta} \left(\sum_{j=0}^1 F_j C_{a^j} \right) + Z Z_{\alpha} \left(\sum_{j=0}^1 G_j C_{a^j} \right) + Z Z_{\theta} \left(\sum_{j=0}^1 H_j C_{a^j} \right) + Z_{\alpha} Z_{\theta} \left(\sum_{j=0}^2 I_j C_{a^j} \right) \right\}$$

where

$$A_{0r} = \frac{1}{24 \cos^3 \theta} \{ 12 \sigma^2 (\sigma + 2 \cos \theta)^2 + 12 k \sigma \sin \theta \cos \theta (\sigma^2 - 8 \cos^2 \theta) + k^2 \cos^2 \theta [96 \cos^2 \theta \\ + 48 \sigma \cos \theta (2 + 3 \cos 2\theta) + 192 \sigma^2 \cos^2 \theta + 68 \sigma^3 \cos \theta + 9 \sigma^4] + k^3 \sigma \sin \theta \cos^3 \theta (48 \cos^2 \theta \\ + 32 \sigma \cos \theta + 6 \sigma^2) - k^4 \cos^4 \theta (\sigma + 2 \cos \theta) (\sigma^3 + 8 \sigma^2 \cos \theta + 24 \sigma \cos^2 \theta + 24 \cos \theta) \}$$

$$A_{0t} = \frac{1}{24 \cos^3 \theta} \{ -48 \sigma^2 \sin \theta \cos \theta + 4 k \sigma \cos \theta (-12 \cos 2\theta \cos \theta + 3 \sigma^2 \cos \theta + \sigma^3) \\ - 4 k^2 \sigma \sin \theta \cos^2 \theta (24 \cos^2 \theta + 12 \sigma \cos \theta + \sigma^2) + 2 k^3 \cos^3 \theta [96 \cos^2 \theta + 12 \sigma \cos \theta (8 + 5 \cos 2\theta) \\ + 96 \sigma^2 \cos^2 \theta + 27 \sigma^3 \cos \theta + 3 \sigma^4] + 2 k^4 \sigma \sin \theta \cos^4 \theta (\sigma + 2 \cos \theta) (\sigma + 4 \cos \theta) \}$$

$$A_{1r} = \frac{1}{12 \cos^3 \theta} \{ 3 \sigma^2 \sin 2\theta (\sigma + 2 \cos \theta) + 12 k \sigma \cos^2 \theta (\sigma + 2 \cos \theta \cos 2\theta) + 2 k^2 \sigma \sin \theta \cos^3 \theta (48 \cos^2 \theta \\ + 30 \sigma \cos \theta + 5 \sigma^2) - 2 k^3 \sigma \cos^4 \theta (\sigma + 2 \cos \theta) (3 \cos 2\theta + \sigma \cos \theta) \}$$

$$A_{1t} = \frac{1}{12 \cos^3 \theta} \{ 12 \sigma^2 \cos \theta (\sigma \cos \theta + \cos 2\theta) - 48 k \sigma \sin \theta \cos^4 \theta + k^2 \sigma \cos^3 \theta [48 \cos \theta \cos 2\theta \\ + \sigma (12 + 30 \cos 2\theta) + 10 \sigma^2 \cos \theta] + 2 k^3 \sigma \sin \theta \cos^4 \theta (\sigma + 2 \cos \theta) (\sigma + 6 \cos \theta) \}$$

$$B_{0r} = \frac{1}{12 \cos \theta} \{ 12 \sigma^2 (3 \cos^2 \theta + 3 \sigma \cos \theta + \sigma^2) + 6 k \sigma \sin \theta \cos \theta (-4 \cos^2 \theta + 6 \sigma \cos \theta + 3 \sigma^2) \\ + k^2 \sigma \cos^2 \theta [-48 \sin^2 \theta \cos \theta - 6 \sigma (1 + \sin^2 \theta) + 8 \sigma^2 \cos \theta + 3 \sigma^3] \\ + k^3 \sigma \sin \theta \cos^3 \theta (12 \cos^2 \theta + 10 \sigma \cos \theta + 3 \sigma^2) \}$$

$$B_{0t} = \frac{1}{12 \cos \theta} \{ -36 \sigma^2 \sin \theta \cos \theta + 2 k \sigma \cos \theta [12 \sin^2 \theta \cos \theta + 6 \sigma (3 + 2 \cos 2\theta - \sin^2 \theta) \\ + 21 \sigma^2 \cos \theta + 5 \sigma^3] - 2 k^2 \sigma \sin \theta \cos^2 \theta (24 \cos^2 \theta + 15 \sigma \cos \theta + 11 \sigma^2) \\ + k^3 \sigma \cos^3 \theta (-12 \sin^2 \theta \cos \theta + 6 \sigma \cos^2 \theta + 5 \sigma^2 \cos \theta + \sigma^3) \}$$

$$B_{1r} = \frac{1}{6} \sigma^2 \{ 6 \sin \theta (\sigma + 2 \cos \theta) - 6 k \cos \theta (2 \cos 2\theta + \sigma \cos \theta) - k^2 \sin \theta \cos^2 \theta (\sigma + 6 \cos \theta) \}$$

$$B_{1t} = \frac{1}{6} \sigma^2 \{ 6 (\cos^2 \theta + \sigma \cos \theta) + 6 k \cos \theta \sin \theta (4 \cos \theta + \sigma) - k^2 \cos^2 \theta (3 \cos 2\theta + \sigma \cos \theta) \}$$

$$C_r = \frac{1}{24} \sigma^2 \cos \theta \{ 12 (2 \cos^2 \theta + 2 \sigma \cos \theta + \sigma^2) + 24 k \cos \theta \sin \theta (\sigma + 2 \cos \theta) \\ - k^2 \cos^2 \theta (12 \cos^2 \theta + 12 \sigma \cos \theta + 3 \sigma^2) \}$$

$$C_t = \frac{1}{24} \sigma^2 \cos \theta \{ -24 \sin \theta \cos \theta + 8 k \cos \theta (6 \cos^2 \theta + 6 \sigma \cos \theta + 2 \sigma^2) + k^2 \cos^2 \theta \sin \theta (12 \cos \theta + 8 \sigma) \}$$

$$D_{0r} = \frac{1}{24 \cos^2 \theta} \{ -12 \sigma^2 \sin \theta (2 \cos^2 \theta + 3 \sigma \cos \theta + \sigma^2) + 2 k \sigma \cos \theta [24 \cos \theta (\sin^2 \theta - \cos \theta) \\ + 6 \sigma \cos \theta (3 + 4 \cos^2 \theta) - 3 \sigma^2 (3 + 4 \cos^2 \theta) - 3 \sigma^3 \cos \theta] - k^2 \sigma \cos^2 \theta \sin \theta (48 \cos^2 \theta \cos 2\theta + 60 \sigma \cos^2 \theta \\ + 22 \sigma^2 \cos \theta + 3 \sigma^3) + k^3 \sigma \cos^3 \theta [24 \cos^4 \theta + \sigma \cos \theta (40 \cos^2 \theta - 10) + \sigma^2 (16 \cos^2 \theta - 3) + 2 \sigma^3 \cos \theta] \}$$

$$D_{0t} = \frac{1}{24 \cos^2 \theta} \{ 12 \sigma^2 \cos \theta (1 + 2 \sin^2 \theta + \sigma \cos \theta) + 2 k \sigma \sin \theta \cos \theta (-12 \cos 2\theta + 12 \sigma \cos^2 \theta \\ - 48 \sigma^2 \cos \theta - 5 \sigma^3) + 2 k^2 \sigma \cos^2 \theta [-48 \cos^5 \theta + 3 \sigma \cos \theta (5 - 26 \cos^2 \theta) + \sigma^2 (2 - 31 \cos^2 \theta) - 5 \sigma^3 \cos \theta] \\ - k^3 \sigma \cos^3 \theta \sin \theta (12 \cos \theta \cos 2\theta + 24 \sigma \cos^2 \theta + 10 \sigma^2 \cos \theta + \sigma^3) \}$$

$$D_{1r} = \frac{1}{24 \cos^2 \theta} \{ -12 \sigma^2 (\sigma + 2 \cos \theta)^2 + 2 k \sigma \cos \theta \sin \theta (12 \cos^2 \theta - 30 \sigma^2 \cos \theta - 9 \sigma^2) \\ + k^2 \sigma \cos^2 \theta (48 \sin^2 \theta \cos \theta - 8 \sigma^2 \cos \theta - 3 \sigma^3) - k^3 \sigma \cos^3 \theta \sin \theta (12 \cos^2 \theta + 10 \sigma \cos \theta + 3 \sigma^2) \}$$

$$D_{1t} = \frac{1}{24 \cos^2 \theta} \{ 48 \sigma^2 \sin \theta \cos \theta - 2 k \sigma \cos \theta (24 \sin^2 \theta \cos \theta + 42 \sigma \cos^2 \theta + 9 \sigma^2 \cos \theta + 5 \sigma^3) \\ + 2 k^2 \sigma \sin \theta \cos^2 \theta (24 \cos^2 \theta + 12 \sigma \cos \theta + 2 \sigma^2) - k^3 \sigma \cos^3 \theta (-12 \sin^2 \theta \cos \theta + 6 \sigma \cos^2 \theta + 5 \sigma^2 \cos \theta + \sigma^3) \}$$

$$\begin{aligned}
E_{0r} &= \frac{\sigma^2}{96 \cos \theta} \{12(2 \cos^2 \theta + 2\sigma \cos \theta + \sigma^2) + 24k \cos \theta \sin \theta (\sigma + 2 \cos \theta) \\
&\quad - k^2 \cos^2 \theta (12 \cos^2 \theta + 12\sigma \cos \theta + 3\sigma^2)\} \\
E_{0i} &= \frac{\sigma^2}{96 \cos \theta} \{-24 \sin \theta \cos \theta + 4k \cos \theta (12 \cos^2 \theta + 12\sigma \cos \theta + 4\sigma^2) + k^2 \sin \theta \cos^2 \theta (12 \cos \theta + 8\sigma)\} \\
E_{1r} &= 2 \sin \theta E_{0r}, \quad E_{1i} = 2 \sin \theta E_{0i}, \quad E_{2r} = E_{0r}, \quad E_{2i} = E_{0i} \\
F_{0r} &= \frac{1}{4} k \cos \theta (\sigma + 2 \cos \theta) \{\sigma (\sigma + 2 \cos \theta) - 4k \sin \theta \cos^2 \theta\} \\
F_{0i} &= \frac{1}{4} \sigma \cos \theta \{2\sigma - 4k \sin \theta \cos \theta - k^2 \cos \theta (\sigma + 2 \cos \theta)^2\} \\
F_{1r} &= \frac{1}{2} \sigma \cos^2 \theta \{\sigma - k \sin \theta (\sigma + 2 \cos \theta)\}, \quad F_{1i} = -\frac{1}{2} \sigma \cos \theta \{\sigma \sin \theta + k \cos^2 \theta (\sigma + 2 \cos \theta)\} \\
G_{0r} &= \frac{1}{24} \sigma^2 \sin \theta \{-12(2 \cos^2 \theta + 2\sigma \cos \theta + \sigma^2) - 24k \sin \theta \cos \theta (\sigma + 2 \cos \theta) \\
&\quad + k^2 \cos^2 \theta (12 \cos^2 \theta + 12\sigma \cos \theta + 3\sigma^2)\} \\
G_{0i} &= \frac{1}{24} \sigma^2 \sin \theta \cos \theta \{24 \sin \theta - 8k(6 \cos^2 \theta + 6\sigma \cos \theta + 2\sigma^2) - k^2 \sin \theta \cos \theta (12 \cos \theta + 8\sigma)\} \\
G_{1r} &= G_{0r} / \sin \theta, \quad G_{1i} = G_{0i} / \sin \theta \\
H_{0r} &= \frac{1}{4} k \sigma^2 \cos^3 \theta (\sigma + 2 \cos \theta), \quad H_{0i} = \frac{\sigma}{2} \cos^3 \theta (\sigma - 2k \sin \theta \cos \theta), \quad H_{1r} = \frac{\sigma^2}{2} \cos^4 \theta \\
H_{1i} &= -\frac{\sigma^2}{2} \cos^3 \theta \sin \theta, \quad I_{0r} = -\frac{\sigma}{8} \cos^2 \theta \{2\sigma \cos \theta + k \sin \theta (\sigma^2 + 2\sigma \cos \theta - 4 \cos^2 \theta)\} \\
I_{0i} &= \frac{\sigma}{8} \cos^2 \theta \{-2\sigma \sin \theta + k \cos \theta (\sigma^2 + 2\sigma \cos \theta + 4 \sin^2 \theta)\}, \quad I_{1r} = -\frac{1}{8} k \sigma^2 \cos^2 \theta (\sigma + 2 \cos \theta) \\
I_{1i} &= \frac{\sigma}{2} \cos^2 \theta (-\sigma + k \sin \theta \cos \theta), \quad I_{2r} = \frac{\sigma^2}{4} \cos^3 \theta, \quad I_{2i} = -\frac{\sigma^2}{4} \cos^2 \theta \sin \theta \\
\frac{m}{\rho U^2 s c \theta} &= \frac{-1}{N} \{A + BZ + CZ^2 + Z_\alpha (\sum_{j=0}^1 D_j C_\alpha^j) + Z_\alpha^2 (\sum_{j=0}^2 E_j C_\alpha^j) + FZ_\theta + ZZ_\alpha (\sum_{j=0}^1 G_j C_\alpha^j) \\
&\quad + ZZ_\theta H + Z_\alpha Z_\theta (\sum_{j=0}^1 I_j C_\alpha^j)\}
\end{aligned}$$

where

$$\begin{aligned}
A_r &= \frac{1}{180 \cos^3 \theta} \{30\sigma^3 (\sigma + 3 \cos \theta) + k^2 \cos^2 \theta [21\sigma^4 + 135\sigma^3 \cos \theta + (30 + 240 \cos^2 \theta)\sigma^2 + 180\sigma \cos \theta \cos 2\theta] \\
&\quad - 30k^3 \cos^3 \theta \sin \theta [\sigma^3 + 6\sigma \cos \theta + 8\sigma \cos^2 \theta] - k^4 \cos^4 \theta (\sigma + 2 \cos \theta) [4\sigma^3 + 33\sigma^2 \cos \theta \\
&\quad + \sigma(60 + 45 \cos 2\theta) + 120 \cos \theta]\} \\
A_i &= \frac{1}{180 \cos^3 \theta} \{90\sigma^3 \sin \theta + 15k\sigma^2 \cos \theta (\sigma^2 + 4\sigma \cos \theta + 6) + 60k^2 \sigma \sin \theta \cos^2 \theta (\sigma^2 + 4\sigma \cos \theta + 6 \cos^2 \theta) \\
&\quad + k^3 \cos^3 \theta [20\sigma^4 + 180\sigma^3 \cos \theta + \sigma^2(45 + 600 \cos^2 \theta) + 360\sigma \cos \theta (2 + \cos 2\theta) + 720 \cos^2 \theta]\} \\
B_r &= \frac{1}{30 \cos \theta} \{10\sigma^3 (\sigma + 3 \cos \theta) - 10k\sigma^3 \sin \theta \cos \theta + k^2 \sigma^2 \cos^2 \theta (10 + 10\sigma \cos \theta + 2\sigma^2)\} \\
B_i &= \frac{1}{36 \cos \theta} \{36\sigma^3 \sin \theta + 6k\sigma^2 \cos \theta (9 + 8\sigma \cos \theta + 2\sigma^2) + 6k^2 \sigma^3 \sin \theta \cos^2 \theta + k^3 \sigma^2 \cos^3 \theta (9 + 6\sigma \cos \theta + \sigma^2)\} \\
C_r &= \frac{1}{60} \sigma^2 \cos \theta \{10\sigma (\sigma + 3 \cos \theta) - 20k\sigma \sin \theta \cos \theta - k^2 \cos^2 \theta (20 + 15\sigma \cos \theta + 3\sigma^2)\} \\
C_i &= \frac{1}{60} \sigma^2 \cos \theta \{30\sigma \sin \theta + 5k \cos \theta (12 + 12\sigma \cos \theta + 3\sigma^2)\} \\
D_{0r} &= \frac{1}{360 \cos^2 \theta} \{-60\sigma^3 \sin \theta (\sigma + 3 \cos \theta) + 30k\sigma^2 \cos \theta [3 \cos \theta (1 - 4 \cos^2 \theta) + \sigma(2 - 8 \cos^2 \theta) - \sigma^2 \cos \theta] \\
&\quad + 6k^2 \sigma \cos^2 \theta [-10 \sin \theta (1 - 4 \cos^2 \theta) + 5\sigma^2 \sin \theta \cos \theta - 2\sigma^3 \sin \theta] \\
&\quad + k^3 \sigma \cos^3 \theta [240 \cos^2 \theta + 3\sigma \cos \theta (25 + 60 \cos^2 \theta) + 96\sigma^2 \cos^2 \theta + 13\sigma^3 \cos \theta]\} \\
D_{0i} &= \frac{1}{360 \cos^2 \theta} \{-180\sigma^3 \sin^2 \theta - 30k\sigma^2 \cos \theta \sin \theta (9 + 12 \cos^2 \theta + 12\sigma \cos \theta + 2\sigma^2) \\
&\quad - 6k^2 \sigma \cos^2 \theta [120 \cos^2 \theta + 30\sigma \cos \theta (4 \cos^2 \theta + 1) + \sigma^2(5 + 60 \cos^2 \theta) + 10\sigma^3 \cos \theta] \\
&\quad - 5k^3 \sigma^2 \cos^3 \theta \sin \theta (9 + 6\sigma \cos \theta + \sigma^2)\}
\end{aligned}$$

$$D_{1r} = -\frac{\sigma^2}{60 \cos^2 \Theta} \{10\sigma(\sigma + 3 \cos \Theta) - 10k\sigma \sin \Theta \cos \Theta + k^2 \cos^2 \Theta (10 + 10\sigma \cos \Theta + 2\sigma^2)\}$$

$$D_{1i} = -\frac{\sigma^2}{60 \cos^2 \Theta} \{30\sigma \sin \Theta + 5k \cos \Theta (9 + 8\sigma \cos \Theta + 2\sigma^2) + 5k^2 \sigma \cos^2 \Theta \sin \Theta\}$$

$$E_{0r} = \frac{\sigma^2}{240 \cos \Theta} \{10\sigma(\sigma + 3 \cos \Theta) - 20k\sigma \cos \Theta \sin \Theta - k^2 \cos^2 \Theta (20 + 15\sigma \cos \Theta + 3\sigma^2)\}$$

$$E_{0i} = \frac{\sigma^2}{240 \cos \Theta} \{30\sigma \sin \Theta + 5k \cos \Theta (12 + 12\sigma \cos \Theta + 3\sigma^2)\}$$

$$E_{1r} = 2 \sin \Theta E_{0r}, \quad E_{1i} = 2 \sin \Theta E_{0i}, \quad E_{2r} = E_{0r}, \quad E_{2i} = E_{0i}$$

$$F_r = \frac{1}{12} k\sigma \cos \Theta \{\sigma(\sigma + 3 \cos \Theta) + 3k \sin \Theta \cos \Theta (\sigma + 2 \cos \Theta)\}$$

$$F_i = \frac{1}{12} k\sigma \cos \Theta \{3\sigma \sin \Theta - k \cos \Theta (\sigma + 2 \cos \Theta) (\sigma + 3 \cos \Theta)\}$$

$$G_{0r} = -4 \cos \Theta \sin \Theta E_{0r}$$

$$G_{0i} = -4 \cos \Theta \sin \Theta E_{0i}$$

$$G_{1r} = -4 \cos \Theta E_{0r}$$

$$G_{1i} = -4 \cos \Theta E_{0i}$$

$$H_r = \frac{1}{12} k\sigma^2 \cos^3 \Theta (\sigma + 3 \sin \Theta),$$

$$H_i = \frac{1}{4} k\sigma^2 \cos^3 \Theta \sin \Theta$$

$$I_{0r} = -\frac{1}{24} k\sigma^2 \cos^2 \Theta \sin \Theta (\sigma + 6 \cos \Theta),$$

$$I_{0i} = \frac{1}{24} k\sigma^2 \cos^2 \Theta (\sigma + 3 \cos 2\Theta)$$

$$I_{1r} = -\frac{1}{24} k\sigma^2 \cos^2 \Theta (\sigma + 3 \cos \Theta),$$

$$I_{1i} = -\frac{1}{8} k\sigma^2 \cos^2 \Theta \sin \Theta$$

References

- (1) Y. Tanida and T. Okazaki: *Bulletin of JSME*, Vol. 6, No. 24 (1963), p. 744, 753.
- (2) H. Takata: *Bull. Aeron. Res. Inst., Univ. of Tokyo*, Vol. 2, No. 6 (1961), p. 305.
- (3) H. Söhngen: *Z. AMP*, Bd. 4 (1953), S. 267.
- (4) A. H. Stenning et al.: *NACA TN 3580* (1956).

Discussion

J. SHIOIRI: (1) It is supposed that such a cascade with infinitesimal spacing as assumed in the semi-actuator disc method would impose harder restrictions on the flow through it in comparison with a cascade with finite spacing. Hence, with respect to the aerodynamic moment which depends upon the shape of the pressure distribution, there might be some discrepancy between the present results and the exact solution, especially when the moment axis is taken near to the aerodynamic centre.

Then I would like to ask the authors about the extent to which the present theory may be applicable for the cascade with finite spacing. For practical use, it would be useful if the results of this theory could be compared with those of the existing theories for a cascade with finite spacing.

T. IKUI: (2) This paper presents an interesting theoretical analysis of stall flutter in the cascade. However, the assumptions used in this paper, that the total pressure losses occur only in the entrance of the cascade and the fluids which flow through the cascade without losses discharge along the blade chord, looks unreasonable for the cascade with infinitesimal spacing, which is a fundamental assumption in the semi-actuator disc theory. Hence, it seems that the stall- and the unstall-flutter

may not be calculated under the same theoretic consideration.

The theoretical development considering the effect of the solidity on the flutter of the cascade will be expected hereafter.

M. MATSUKI and S. FUJII: (3) For such a complicated problem as stall-flutter treated in this paper, it would be necessary to treat the problem under some bold assumptions. In this paper, for example, the cascade with "infinitesimal spacing" is treated on the basis of "finite control surface" which is at first sight contradictory to the former.

However, the discussors think that the present theory would be justified because the numerical results in this paper offer the informations which seem to be reasonable in quality, as seen in Fig. 4 and the others.

(4) Now, discussors would like to ask the authors to explain the derivation of $\Delta_1 p_i / \rho U$ under Eq. (14) in more detail; e.g., what does the neglect of small terms mean concretely?

(5) In the discussors' opinion, the assumption of Eq. (20) in this paper, which makes the fluid flow along the blade surface, is insufficient, because in practice the fluid does not always flow along the blade surface especially when the cascade is accom-

panied by stall.

Then, discussors want to ask for the authors' opinion about the above-mentioned, in connection with the concentration of the energy loss at the inlet of the cascade assumed in the numerical calculation.

Authors' closure

(1) It is clear that there is limitation to the applicability of the present theory on the cascade with finite spacing, because of the assumption of "infinitesimal spacing" of cascade.

To examine this theory, the results calculated by the present method will be compared with those of a potential theory^(a) which has been recently developed on non-stall binary flutter of the cascade with finite spacing.

Translatory oscillation: Append.-Fig. 1 illustrates the aerodynamic damping force acting on the blades vibrating in translatory mode against the inter-blade phase angle ($\sigma = |\beta|(c/s)$, β : inter-blade phase angle of cascade with finite spacing, $\text{sgn } \beta = -\text{sgn } V_s$), in the case of $\Theta = 30^\circ$ and $|k| = 0.35$. From this figure, the larger the inter-blade phase angle ($|\beta|$), the larger the difference between the semi-actuator disc ($c/s = \infty$) and the cascade with finite spacing becomes significant. In other words, imposing a certain limit upon the ratio of the difference against the absolute value, the smaller the solidity, the narrower the extent of the applicability of the semi-actuator disc theory becomes.

Append.-Fig. 1 Aerodynamic damping force in translatory mode

(a) Y. Hanamura and H. Tanaka: *Bulletin of JSME*, Vol. 10, No. 40 (1967), p. 647.

Append.-Fig. 2 shows the calculated critical flutter frequency ($k_{f, \text{crit}}$) of a cascade with the stagger angle $\Theta = 0^\circ$ against the inlet angle of flow; the result of the semi-actuator disc method is drawn as a straight line, $k_{f, \text{crit}} = \frac{1}{4} \tan \alpha_1$. This figure shows

that the semi-actuator disc method is apt to underestimate the translatory flutter velocity against the other but give a relatively good approximation of the cascade of solidity 2.

The relatively good applicability of this theory beyond expectation will be attributed to the fact that, as known from the theory for the cascade with finite spacing or from the experiments, the flutter velocity of the cascade with finite spacing becomes critical or lowest in many cases in the region of relatively small inter-blade phase angle.

Pitching oscillation about the leading edge: As described in the question, it is supposed that the aerodynamic moment calculated by the present method would be considerably different from those on the cascade with finite spacing, because it depends on the pressure distribution.

Append.-Fig. 3 illustrates the calculated aerodynamic damping moment on the cascade blades oscillating about the leading edge, in the case of $\Theta = 30^\circ$ and $|k| = 0.35$. The aerodynamic damping moment calculated by the semi-actuator disc method, which is independent of the inlet angle, does vary more significantly against the inter-blade phase angle than on the cascade with finite spacing, but the difference between them is not so large as we expected.

The calculated results of the critical flutter frequency ($k_{f, \text{crit}}$) of the same compressor cascade are shown against the inlet angle in Append.-Fig. 4, in which P.C. indicates the distance of pitching center from the leading edge divided by the chord.

Append.-Fig. 2 Translatory flutter frequency

length. From this figure, with respect to the critical flutter frequency in pitching mode, the semi-actuator disc corresponds to the cascade with finite spacing which has torsional axis between the leading edge and the point of quarter chord-length.

From these results, to apply the semi-actuator disc theory on the cascade with finite spacing, it would be desirable to choose the position of the elastic-axis and the exit-flow condition appropriately and also to take into account the blade camber, corresponding to the actual pressure distribution.

Stall flutter: For lack of any reliable theory existing on stall-flutter in cascade, the present theory will be compared with the experiments hitherto executed.

Append.-Fig. 5 illustrates the experimental results^(b) of the aerodynamic damping force acting on

Append.-Fig. 3 Aerodynamic damping moment in pitching mode

Append.-Fig. 4 Pitching flutter frequency

the compressor cascade oscillating in translatory mode under the conditions of $\theta = 30^\circ$, $c/s = 1$, and $|\beta| = 60^\circ$, corresponding to Fig. 4 in this paper.

Comparing Fig. 4 with Append.-Fig. 5, there may be recognized qualitative agreement in their characteristics; e.g., as the cascade falls into stall, the aerodynamic damping force for $V_s < 0$ or $\beta = 60^\circ$ decreases abruptly and becomes negative.

On the other hand, stall-flutter velocity has been obtained experimentally only once in turbine cascade^(c). Append.-Fig. 6 shows the comparison between this experimental result and that calculated by the present theory on nearly the same cascade condition, in translatory mode (no scale of ordinate for the experimental results). Both of them give the common tendency of flutter velocity to increase as the cascade stalls.

(2) This paper does not treat a viscous flow in a narrow passage but a flow through the cascade operating at high Reynolds number, and then an inviscid fluid flow is assumed.

In the numerical example, the assumptions that "the fluid loses a part of its energy at the entrance

Append.-Fig. 5 Aerodynamic damping force in translatory mode (experiment)

(b) Y. Tanida et al.: *Bulletin of JSME*, Vol. 6, No. 24 (1963), p. 736.

(c) J. Shioiri and S. Shibata: *Rep. Gov. Mech. Lab. Japan*, No. 26 (1956), p. 59.

of the cascade..." are used for simplicity, but, as stated in this paper, it is possible to assume a distributed energy loss in the cascade and a relaxed outlet condition. For the cascade of flat-plate blades with high solidity, however, the assumptions used in this paper seem to be sufficiently reasonable.

Finally, the authors would like to make a remark on "the boundary of the theoretical treatment of nonstall- and stall-flutter in cascade on a unified basis". As known from the non-stall flutter theory on cascade, the mutual interference between the cascading blades has a dominative effect on the blade oscillation and enables a single-degree-of-freedom flutter to occur, which does never happen on an isolated blade.

Although there have been several investigations on stall flutter in cascade, they are mostly similar to those on an isolated wing, being discussions without necessity of cascading blades.

The authors are convinced that, in future, stall flutter in cascade should continue to be discussed on the same basis with non-stall one by taking into consideration the mutual interference of cascading blades, whether by semi-actuator disc method or not.

(3) The authors should like to make a remark on "finite control surface" described in the question.

In this theory the control surface, considered a space between the adjacent blades in the cascade with infinitesimal spacing, has a finite length equal to the chord-length but an infinitesimal width equal to the blade spacing. Then, the control surface is never inconsistent with the assumption of "infinitesimal spacing of cascade".

$$(4) \quad \Delta_1 p_{tr} = -\frac{\rho}{2} u_{r1}^2 Z \left(\frac{v_{r01}}{u_{r01}}, \Theta + \theta e^{i\omega t}, \omega \right) \dots \dots \dots (14)$$

where

$$\begin{cases} u_{r1} = U + (u_1 + q \sin \varphi) e^{i\omega t} \\ \frac{v_{r01}}{u_{r01}} = \frac{V_1 + (v_{01} - q_1 \cos \varphi_1) e^{i\omega t}}{U + (u_{01} + q_1 \sin \varphi_1) e^{i\omega t}} = \tan \alpha_1 + \delta(\tan \alpha_1) e^{i\omega t} \\ \delta(\tan \alpha_1) = -\frac{1}{U} [u_{01} \tan \alpha_1 - v_{01} + q_1 (\cos \varphi_1 + \tan \alpha_1 \sin \varphi_1)] \end{cases}$$

Expanding the right-hand side of Eq.(14) around the steady state,

$$\begin{aligned} \Delta_1 p_{tr} = & -\frac{\rho}{2} [U^2 + 2U(u_1 + q \sin \varphi) e^{i\omega t} + O(q^2)] \\ & \times [Z(\tan \alpha_1, \Theta) + Z_\alpha(\tan \alpha_1, \Theta, \omega) \delta(\tan \alpha_1) e^{i\omega t} + Z_\theta(\tan \alpha_1, \Theta, \omega) \theta e^{i\omega t} + O(q^2)] \end{aligned}$$

Similarly the left-hand side is

$$\begin{aligned} \Delta_1 p_{tr} = & \Delta_1 \left[P + p e^{i\omega t} + \frac{\rho}{2} \{ [U + (u + q \sin \varphi) e^{i\omega t}]^2 + [V + (v - q \cos \varphi) e^{i\omega t}]^2 \} \right] \\ = & \Delta_1 [P_i + p_i e^{i\omega t} + \rho(U \sin \varphi - V \cos \varphi) q e^{i\omega t} + O(q^2)] \end{aligned}$$

Eliminating the steady parts between both sides and neglecting the terms of higher order of small quantities or $O(q^2)$,

$$\begin{aligned} \frac{\Delta_1 p_i}{\rho U} = & -u_1 Z + q [\cos \varphi \Delta(\tan \alpha) - Z \sin \varphi] \\ & + \frac{1}{2} [u_{01} \tan \alpha_1 - v_{01} + q_1 (\cos \varphi_1 + \tan \alpha_1 \sin \varphi_1)] Z_\alpha - \frac{1}{2} U \theta Z_\theta \end{aligned}$$

(5) The assumption of the infinitesimal spacing in the semi-actuator disc method permits the magnitude of the flow velocity to be induced from the blade oscillation $O(q)$ and that of the flow not parallel to the blade surface $O[i\omega s/V_s q]$, higher than $O(q)$, even if the stall could occur. Then, for the cascade of flatplate blades with infinitesimal spacing, it may be assumed that the fluid flows along the blades within the cascade and the energy loss is concentrated at the entrance of the cascade.

In the cascade with finite spacing, however, the flow does not always run and go out parallel to the blade surface and also the energy loss may be generated within the cascade. Then, for such a case, it will be necessary to consider the distributed energy loss in the cascade and the deviation of flow at the outlet.

Append.-Fig. 6 Flutter velocity of turbine