539.43.019.2

Effects of Size on the Fatigue Limit and the Branch Point in Rotary Bending Tests of Carbon Steel Specimens*

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The effects of size on the fatigue limits and the branch points of electropolished notched specimens were investigated.

To clarify the effect of work hardened layers due to machining, the specimens having work hardened layers were also investigated.

The main results obtained are as follows.

1) The necessary condition of non-propagating cracks is to take the root radius of a notch under a critical value ρ_0 which is a material constant. ρ_0 is about 0.5 mm for the 0.23 % carbon steel.

2) The repeated maximum stress at the root of a notch at the fatigue limit based on crack initiation is determined only by the stress gradient $\chi = (1/\sigma)(d\sigma/dx)$ at the root of the notch.

3) The size effect of the fatigue limits based on fracture in the range of existing of non-propagating cracks is explained by the fact that the fatigue limits are determined from the stress concentration factor at the branch point $\alpha|_{\rho=\rho_0}$.

1. Introduction

The size effects in rotary bending fatigue tests of metallurgically uniform materials may be considered to be controlled mainly by the following two factors.

- (1) The stress gradient
- (2) The extent of area exposed to the risk of fracture

The factor (1) is concerned with the mechanical cause, and the factor (2) is concerned with the statistical cause.

In some studies, the above two factors are treated without any distinction as the one which concerns the volume exposed to the risk of fracture. According to the author's opinion, however, they must be distinguished.

In this paper, paying attention to the factor (1), the effects of size on the fatigue limit and on the branch point⁽¹⁾ (i.e. the point at which the lines showing the fatigue limit based on crack initiation and the crack strength intersect) are investigated.

In the previous papers^{(2)~(6)}, it has been made clear that the root radius of a notch ρ is almost constant at the branch point, irrespective of the

notch depth t and the diameter of the minimum section d .

This fact will be significant in the examination of the size effect in fatigue and the fatigue limit of defective materials⁽⁷⁾.

2. Materials, specimens, and the testing method

The materials used are rolled round bars of 0.23 % carbon steel. The specimens were turned after annealing the bars for about 60 minutes at the temperature of 890°C. The chemical composition and the mechanical properties of these annealed bars are shown in Table 1.

Some of the turned specimens having the work hardened layers due to the machining were tested without annealing or other treatments (in the following, we will call them the work hardened specimens). The other specimens, after turned, were annealed in vacuum to eliminate the residual stress

Table 1 Chemical composition and mechanical properties of the material used

Chemical composition %						Mechanical properties kg/mm ² , %			
C	Si	Mn	P	S	Cu	σ_{sl}	σ_B	σ_T	ψ
0.23	0.31	0.37	0.011	0.021	0.13	28.5	47.9	89.0	61.6

σ_{sl} : lower yield point,

σ_T : actual stress at fracture,

σ_B : tensile strength,

ψ : area contraction.

* Received 25th August, 1967.

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for 60 minutes at the temperature of 600 °C, and then, the surface layers of those specimens were electropolished to the depth of about 10 microns (in the following, we will call them the electropolished specimens).

The forms of the plain and notched (60° V-groove) specimens are shown in Figs. 1 and 2. The principal dimensions of the notch are shown in

Fig. 1 Plain specimen

Fig. 2 Notched specimen

Fig. 3

Fig. 4

Fig. 5

Table 2 Principal dimensions of specimens and the results of rotary bending tests

d mm	t mm	rho mm	chi mm ⁻¹	alpha	Electropolished specimen			Work-hardened specimen		
					σ_{w1} kg/mm ²	σ_{w2} kg/mm ²	$\alpha\sigma_{w1}$ kg/mm ²	σ_{w1} kg/mm ²	σ_{w2} kg/mm ²	$\alpha\sigma_{w1}$ kg/mm ²
10	0.2	∞	0.2	1.00	21.5	—	21.5	23.5	—	23.5
		1.0	2.6	1.67	15.0	—	25.1	16.5	—	27.6
		0.6	4.2	1.90	14.0	—	26.6	14.5	—	27.6
		0.3	8.0	2.33	12.5	13.5	29.1	13.0	13.5	30.3
		0.1	22.6	3.39	9.5	13.5	32.2	10.5	14.0	35.6
5	0.1	∞	0.4	1.00	22.5	—	22.5	25.0	—	25.0
		1.0	2.9	1.43	17.0	—	24.8	19.5	—	27.9
		0.6	4.5	1.59	16.0	—	25.4	18.5	—	29.4
		0.3	8.4	1.90	15.0	16.0	28.5	16.0	17.0	30.4
		0.1	23.4	2.66	12.0	15.5	31.9	13.5	17.0	35.9
5	0.5	∞	0.4	1.00	22.5	—	22.5	25.0	—	25.0
		1.0	2.6	1.54	16.0	—	24.6	17.5	—	27.0
		0.3	7.6	2.26	12.0	12.5	27.1	13.0	14.0	29.4
		0.1	21.8	3.46	9.5	12.5	32.9	10.5	14.5	36.3

σ_{w1} : fatigue limit based on crack initiation,

σ_{w2} : crack strength (the limit stress for fracture in the range of non-propagating crack existing)

The machine used is the Ono-type rotary bending fatigue testing machine. The rate of stress repetitions was nearly 2500 cycles per minute throughout the test.

For the specimens which endured the stress repetitions of more than 10^7 cycles, the existence of the non-propagating crack was inspected by observing directly the surface of the root of the notch using

an optical metallurgical microscope ($\times 400$) in the case of the electropolished specimens and by observing the etched longitudinal section of the specimen in the case of the work hardened specimens.

3. Results and discussion

The values of fatigue limit σ_{w1} (the limit stress for crack initiation) and of crack strength σ_{w2} (the limit stress for fracture in the range of non-propagating crack existing) are shown in Table 2. The stresses are all nominal ones, calculated by the formula $\sigma = 32M/(\pi d^3)$ (M : bending moment, d : diameter of the minimum section).

S-N curves in each case are shown in Figs. 3~8.

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

The relations between stress concentration factor α and the fatigue limit σ_{w1} or the crack strength σ_{w2} are shown in Figs. 9 ~ 14. In these Figs., the mark $\bullet$ signifies that the specimens were fractured before 10^7 cycles of the stress repetitions; the mark $\odot$ signifies that they were not fractured till 10^7 cycles of the stress repetitions, but they had non-propagating cracks; and the mark $\circ$ indicates that there appeared neither fracture nor non-propagating crack till 10^7 cycles of the stress repetitions.

The relations between $\alpha\sigma_{w1}$ (the maximum stress repeated at the root of a notch in the fatigue limit based on crack initiation) and χ [the stress gradient at the root of a notch, that is, $(1/\sigma)(d\sigma/dx)$] are shown in Figs. 15 and 16. In this paper, the following approximate expression for χ was used⁽²⁾.

$$\chi = \frac{\left(\frac{3}{2} + 2\sqrt{\frac{t}{\rho}}\right)\frac{2}{\rho} + \left(1 + \sqrt{\frac{t}{\rho}}\right)\left(1 - \sqrt{\frac{t}{\rho} - \frac{2t}{\rho}}\right)\frac{2}{d+2t}}{\left(1 + 2\sqrt{\frac{t}{\rho}}\right) - \left(1 + \sqrt{\frac{t}{\rho}}\right)\frac{2t}{d+2t}}$$

In Figs. 15 and 16, $\odot^*$, Δ^+ and $\circ^*$, Δ^* mean respectively similar specimens.

Figs. 17 and 18 show the relations between the limit stress for fracture and the reciprocal of the root

Fig. 11

Fig. 13

Fig. 12

Fig. 14

radius of a notch.

Figs. 19 and 20 show the surface state of electropolished specimens after 10^7 cycles of the stress repetitions. The value of the repeated stress is the one at the crack initiation limit.

Fig. 21 shows the non-propagating crack of an electropolished specimen (the photograph of the surface of the notch root).

Fig. 22 shows the non-propagating cracks of

Fig. 15 The relations between the maximum stress $\alpha\sigma_{wl}$ repeated at the root of a notch in crack initiation limit and the stress gradient χ (electropolished specimens)

Fig. 16 The relations between the maximum stress $\alpha\sigma_{wl}$ repeated at the root of a notch in crack initiation limit and the stress gradient χ (work-hardened specimens)

work hardened specimens (the photographs of the longitudinal section of the notched part).

The experimental results are summarised as follows:

(1) In the experiment of both electropolished and work hardened specimens, with a constant diameter of the minimum section d and notch depth t ,

Fig. 17 The relations between the branch point and the root radius of a notch (electropolished specimens)

Fig. 18 The relations between the branch point and the root radius of a notch (work-hardened specimens)

(a) Surface state of virgin plain specimen
(b) $\rho=\infty$, $\chi=0.4 \text{ mm}^{-1}$, $\alpha\sigma_{wl}=23 \text{ kg/mm}^2$
(c) $\rho=1.0 \text{ mm}$, $\chi=2.9 \text{ mm}^{-1}$, $\alpha\sigma_{wl}=25 \text{ kg/mm}^2$
(d) $\rho=0.3 \text{ mm}$, $\chi=8.4 \text{ mm}^{-1}$, $\alpha\sigma_{wl}=28 \text{ kg/mm}^2$
(e) $\rho=0.1 \text{ mm}$, $\chi=23.4 \text{ mm}^{-1}$, $\alpha\sigma_{wl}=32 \text{ kg/mm}^2$

Fig. 19 The surface states of the root of notches after 10^7 cycles of σ_{wl} (electropolished specimens, $d=5 \text{ mm}$, $t=0.1 \text{ mm}$, $\times 320$, $\leftrightarrow$ axial direction of specimens)

but a varied radius of a notch ρ , the value ρ at the branch point was found to be almost constant, namely 0.5 mm, irrespective of d and t or of the presence of work hardened layers (see Figs. 17 and 18).

It has been confirmed^{(2)~(6)} for specimens of various materials with work hardened layers that ρ is almost constant at the branch point, irrespective of t , provided that $t \geq 0.1$ mm. It was made clear in our experiment, however, that this fact also holds good irrespective of the existence of work hardened layers. The values of the root radius of notches ρ_0 at the branch point are about 0.4 mm for 0.35 % annealed carbon steel⁽²⁾ ($\sigma_{st}=34.3$ kg/mm², $\sigma_B=61.2$ kg/mm²), about 0.5 mm for 0.25 % annealed carbon steel^{(5) (6)} ($\sigma_{st}=30.3$ kg/mm², $\sigma_B=50.4$ kg/mm²), about 0.4 mm for annealed pure copper⁽³⁾ ($\sigma_{0.2}=4.0$ kg/mm², $\sigma_B=22.1$ kg/mm²) and about 0.4 mm for heat-treated Ni-Cr steel⁽⁴⁾ ($\sigma_{0.2}=86.5$ kg/mm², $\sigma_B=97.1$ kg/mm²). As above, ρ_0 is 0.4 ~ 0.5 mm for fairly various kinds of materials, but it becomes much smaller than

0.4 ~ 0.5 mm, if the tensile strength becomes much higher than those materials. For example, ρ_0 is less than 0.1 mm for heat-treated Ni-Cr-Mo steel ($\sigma_{0.2}=116.3$ kg/mm², $\sigma_B=141.7$ kg/mm²).

The fact that ρ is constant at the branch point, gives a ground for estimation of the fatigue limit of defective materials, while the fact that ρ_0 becomes small when the tensile strength is high, seems to have a significant meaning in the examination of notch sensitivity.

That the radius of a notch ρ becomes constant at the branch point is, as shown in the following chapter, due to the fact that the stress distributions near the roots of notches become nearly equal to each other to the same depth as ρ , if ρ is constant,

(a) $t=0.1$ mm
 $\alpha\sigma_{wl}=28$ kg/mm²
(b) $t=0.5$ mm
 $\alpha\sigma_{wl}=27$ kg/mm²
Fig. 20 The surface state of the root of notch after 10^7 cycles of σ_{wl} (electropolished specimens, $d=5$ mm, $\rho=0.3$ mm, $\times 320$, $\leftrightarrow$ axial direction of specimens)

$d=5$ mm $t=0.1$ mm
 $\rho=0.1$ mm $\sigma=13$ kg/mm²
Fig. 21 Non-propagating crack of an electropolished specimen ($\times 400$, $\leftrightarrow$ axial direction of specimen)

(a) $\rho=0.3$ mm
 $\sigma=17$ kg/mm²
(b) $\rho=0.1$ mm
 $\sigma=16$ kg/mm²
(c) $\rho=0.05$ mm
 $\sigma=14$ kg/mm²
Fig. 22 Non-propagating cracks of work-hardened specimens (longitudinal section, $d=5$ mm, $t=0.1$ mm $\times 320$)

irrespective of t and d , except for the case when t is extremely small in comparison with ρ and for the case when the diameter of the minimum section d is the same as, or less than, 2ρ .

(2) The maximum stress at the root of a notch in the crack initiation limit $\alpha\sigma_{w1}$ becomes almost constant in each of electropolished and work hardened specimens, irrespective of the size of the test piece, d and t , if the stress gradient χ is constant (Figs. 15 and 16). Therefore, it is permissible to say that $\alpha\sigma_{w1}$ is approximately determined only by χ and the material properties (here the material properties mean the properties of the surface layer which is concerned with the fatigue). When t is extremely small in comparison with ρ , however, $\alpha\sigma_{w1}$ will be concerned not only with χ but also with t/ρ (see Fig. 23).

From the above, the size effect of the fatigue limit σ_{w1} (the mechanical factor) seems to appear mainly through χ .

The fact that $\alpha\sigma_{w1}$ becomes almost constant, when χ is equal (since χ is determined for the most part by ρ , ρ is also said to be almost equal), irrespective of d and t , is, as in the case of (1), due to the fact that the stress distribution near the root of a notch is almost determined by the value of ρ alone (Figs. 23 and 24).

The fact that the higher the stress gradient χ is, the higher the true stress $\alpha\sigma_{w1}$ at the root of a notch at the crack initiation limit becomes, is considered to be due to the following reasons: The fatigue damage is a function of the amount of slip in the crystal grains on the surface of the notch root (that is, when the amount of slip is equal, the damage is also nearly equal), and the amount of slip is related not only to the maximum stress repeated at the root of a notch but also to the stress distribution. Fig. 19 shows that the conditions on the surface of the notch root are similar to each other at the crack initiation limit, irrespective of stress repeated on the surface of the notch root, and supports the above inference indirectly.

According to Neuber, ⁽⁸⁾

$$\sigma_1(x) = p + \frac{p \cosh u_0}{2 \sinh^2 u} \left[e^{u_0} (e^{2u_0} - 3) \left(1 + \frac{1}{2} \coth u \right) e^{-2u} + \cosh u_0 \coth u \right] \dots\dots\dots (1)$$

where

$$\coth u_0 = \sqrt{\frac{t}{\rho}}, \quad u = \cosh^{-1} \left\{ \left(1 + \frac{x}{t} \right) \cosh u_0 \right\}$$

Notations are defined in Fig. 23.

We need the stress distribution in the case when the maximum stress at the root of a notch is equal to 1.

Then,

$$\sigma_1(x) = \frac{m^4 \xi^3 + m^2(m^2 - m - 3)\xi + (m + 1)}{(2m + 1)(m - 1)(m^2 - 1)} \dots\dots\dots (2)$$

where

(3) The crack strength σ_{w2} is determined by the diameter of the minimum section d and the notch depth t , and becomes almost constant, irrespective of ρ , if d and t are constant (Figs. 17 and 18).

(4) Taking the above facts (1) ~ (3) into consideration, it is possible to obtain the crack strength σ_{w2} of an arbitrary notched specimen that has a sharper notch (namely ρ is small) than the one at the branch point. Thus, σ_{w2} is obtained from the following equation.

$$\alpha|_{\rho=\rho_0} \sigma_{w2} = \sigma_0$$

where ρ_0 is the notch radius at the branch point; $\alpha|_{\rho=\rho_0}$ is the stress concentration factor of a notch that is equal in depth and in diameter, and $\rho = \rho_0$; and σ_0 is the maximum stress (true stress) at the root of a notch in the crack initiation limit, when $\rho = \rho_0$. When the notch is sharp, χ is mainly determined by ρ . Therefore, σ_0 is the value — the material constant — that is almost determined by the material properties alone, as is known from (2).

In our experiment, $\rho_0 = 0.5$ mm, $\sigma_0 = 27.0$ kg/mm² in electropolished specimens and $\rho_0 = 0.5$ mm, $\sigma_0 = 28.5$ kg/mm² in work hardened specimens.

This means that the size effect of the crack strength σ_{w2} appears through the stress concentration factor $\alpha|_{\rho=\rho_0}$ of the notch that is equal in depth and in diameter, and $\rho = \rho_0$.

4. Effect of the root radius of a notch on the stress distribution near the root of the notch

The stress distribution near the root of a notch is controlled mainly by the root radius of the notch ρ . This fact is the reason why ρ at the branch point is independent of the notch depth t and the diameter of the minimum section d , and also the maximum stress at the crack initiation limit becomes nearly constant when χ is constant.

4.1 Stress distribution in the case of tension of an infinite plate having an elliptic hole

The stress distribution in this case corresponds to the one in the case of bending a round bar having a shallow circumferential groove.

$$m = \sqrt{\frac{t}{\rho}}, \quad \xi = \frac{x+t}{\sqrt{m^2(x^2+2tx)+t^2}}$$

When $t \gg \rho$, then $m \rightarrow \infty$ and $\xi \rightarrow 1/\sqrt{1+(2x/\rho)}$.

Therefore,

$$\sigma_1(x) = \frac{1}{2\sqrt{1+\frac{2x}{\rho}}} \left(1 + \frac{1}{1+\frac{2x}{\rho}} \right) \dots\dots\dots (2)'$$

4.2 Stress distribution in the case of bending an infinite body having a deep hyperboloidal notch

The stress distribution in this case corresponds to the one in the case of bending a round bar having a deep circumferential groove.

According to Neuber⁽⁹⁾,

$$\sigma_1(x) = \tan v \left[2\{B - (1+\alpha)C + \left(2 - \frac{\alpha}{2}\right)D\} + \sec^2 v \left\{ A + \frac{2}{3}B + 2C \cos^2 v + D \sin^2 v \right\} \right] \dots\dots\dots (3)$$

where

$$\begin{aligned} A &= -2\alpha(1-\alpha)(1+\cos v_0)^2 \frac{D}{N_1}, & C &= \left\{ -3\alpha + 2(1-\alpha) \cos v_0 + (4-\alpha) \cos^2 v_0 \right\} \frac{D}{N_1}, \\ B &= -(1-\alpha)(C+D), & N_1 &= 2 \cos v_0 \{ 2(1-\alpha) + (4-\alpha) \cos v_0 \}, \\ \frac{D}{N_1} &= \frac{3M \sin^3 v_0}{2\pi a^3 \alpha (1-\cos v_0)^2 \{ 6 + 4(4-\alpha) \cos v_0 + 2(5-2\alpha) \cos^2 v_0 + (4-\alpha) \cos^3 v_0 \}}, \\ \alpha &= 2(1-\nu), & \tan v_0 &= \sqrt{\frac{a}{\rho}}, & v &= \sin^{-1} \left\{ \left(1 - \frac{x}{a} \right) \sin v_0 \right\} \end{aligned}$$

Notations are defined in Fig. 24.

We need the stress distribution in the case when the maximum stress at the root of a notch is equal to 1. Then,

$$\sigma_1(x) = \frac{\{ 3(1+n^2) - 2(1-2\nu)\sqrt{1+n^2} + (5+2\nu) \} \eta + 3n^2 \eta^3}{3(1+2n^2) - 2(1-2\nu)\sqrt{1+n^2} + (5+2\nu)} \dots\dots\dots (4)$$

Where

$$n = \sqrt{\frac{a}{\rho}}, \quad \eta = \frac{a-x}{\sqrt{n^2(2ax-x^2)+a^2}}, \quad \nu: \text{Poisson's ratio.}$$

When $d=2a \gg \rho$, then $n \rightarrow \infty$ and $\eta \rightarrow 1/\sqrt{1+(2x/\rho)}$.

Therefore,

$$\sigma_1(x) = \frac{1}{2\sqrt{1+\frac{2x}{\rho}}} \left(1 + \frac{1}{1+\frac{2x}{\rho}} \right) \dots\dots\dots (4)'$$

This expression coincides with Eq. (2)'.

From Eqs. (2), (2)', (4) and (4)', we can calculate the stress distribution of a round bar having a

Fig. 23 Stress distribution near the root of a notch (corresponds to shallow notches)

Fig. 24 Stress distribution near the root of a notch (corresponds to deep notches)

shallow or a deep circumferential groove, subjected to bending moment.

Fig. 23 shows that the stress distribution near the root of a shallow notch having a constant notch radius does not change in the wide range of notch depth (for example, $t/\rho: 1 \sim \infty$).

Fig. 24 shows that the stress distribution near the root of a deep notch having a constant notch radius does not change in the wide range of the diameter of the minimum section (for example, $a/\rho: 9 \sim \infty$).

Comparing Fig. 23 with Fig. 24, we find that the stress distribution near the root of a shallow notch ($1 < t/\rho < \infty$) is nearly equal to the one near the root of a deep notch ($9 < a/\rho < \infty$). This fact suggests that the root radius of a notch is the main factor which controls the stress distribution near the root of a notch.

5. Comparisons with previous studies

From those four points mentioned in chapter 3, in the repetitions of bending stress, the fatigue limit σ_{w1} and the crack strength σ_{w2} of a round bar having arbitrary material properties and an arbitrary notch (the diameter of the minimum section $d=d_1$; the notch depth $t=t_1$; the radius of the notch $\rho=\rho_1$; the stress concentration factor $\alpha=\alpha_1$; and the stress gradient $\chi=\chi_1$) are obtained in the following way. From the rotary bending test of the specimens of the same material as the notched specimen, we obtain the root radius of a notch ρ_0 at the branch point and the relations between $\alpha\sigma_{w1}$ and χ , varying ρ alone, while keeping d and t constant (for example, $d=d_2$, $t=t_2$). Then we can calculate the stress concentration factor $\alpha|_{\rho=\rho_0}$ for the case when $d=d_1$, $t=t_1$, and $\rho=\rho_0$.

Thus, (i) in order to obtain σ_{w1} , obtain the maximum stress at the crack initiation limit, in the first place, from the $\alpha\sigma_{w1}-\chi$ curve for the case of $\chi=\chi_1$, then divide it by α_1 .

(ii) The necessity of the crack strength σ_{w2} arises only when $\rho_1 < \rho_0$, for if $\rho_1 > \rho_0$, there exists no non-propagating crack, and therefore, if the crack is initiated, it is destined to the fracture. σ_{w2} is obtained from the maximum stress at the crack initiation limit σ_0 for the case when $d=d_2$, $t=t_2$, $\rho=\rho_0$, and the stress concentration factor at the branch point $\alpha|_{\rho=\rho_0}$. That is, $\sigma_{w2}=\sigma_0/\alpha|_{\rho=\rho_0}$. ρ_0 and σ_0 are the material constants to be determined only by the material properties as mentioned in chapter 3.

According to the results of our study, if the relationship between $\alpha\sigma_{w1}$ and χ , and the root radius of a notch ρ_0 at the branch point are obtained from the rotary bending test with a set of specimens

having $d=d_2$, $t=t_2$, the stresses at the crack initiation limit and at the fracture limit of round bars, having arbitrary diameter, notch depth, and notch radius, are obtained.

5.1 Isibasi's study⁽¹⁰⁾

According to Isibasi σ_{w1} is obtained from the following equation.

$$\alpha\sigma_{w1} = \frac{1}{1 - \chi\epsilon_0 + a_2\left(\frac{\epsilon_0}{\rho}\right)^2 + \dots} \sigma_{w0} \quad (\epsilon_0, a_2, \sigma_{w0}: \text{constants}) \dots\dots\dots (5)$$

This equation was obtained on the supposition that when the crack is initiated at the root of a notch, the stress at the point which is apart ϵ_0 in the direction of the maximum stress gradient from the root of the notch, is equal to the fatigue limit of a plain specimen.

The standpoint that σ_{w1} is considered to be determined only by the stress gradient χ , is almost the same as that stated above. For instance, when ρ is large, Eq. (5) becomes as follows.

$$\alpha\sigma_{w1} = \frac{1}{1 - \chi\epsilon_0} \sigma_{w0} \doteq (1 + \chi\epsilon_0) \sigma_{w0} \dots\dots\dots (6)$$

Equation (6) means that the relation between $\alpha\sigma_{w1}$ and χ is expressed by a straight line for the part where χ is near 0 (that is, ρ is large) (see Figs. 15 and 16). ϵ_0 is a parameter deciding the inclination of the straight line. Ohuchida, making a wide range of experiments on large-sized specimens, shows how well the size effect of the large-sized specimen is explained by Eq. (6)⁽¹¹⁾⁽¹²⁾.

Since Eq. (5) generally prescribes the relationship between $\alpha\sigma_{w1}$ and χ through ϵ_0 alone, it is difficult to apply Eq. (5) to the wide range of χ . Isibasi, considering the possibility of ϵ_0 to vary a little with ρ , however, expressed the approximate value of ϵ_0 by a linear equation of ρ and furthermore neglecting the higher terms than ϵ_0^3 , he rewrote Eq. (5) as follows.

$$\alpha\sigma_{w1} = \frac{1}{A_0 - A_1 \frac{1}{\rho} + A_2 \frac{1}{\rho^2}} \dots\dots\dots (7)$$

As for A_0 , A_1 and A_2 , he obtained them directly from the experiment. Since $1/\rho$ is substantially in proportion to χ , Eq. (7) expresses approximately the relationship between $\alpha\sigma_{w1}$ and χ by the reciprocal of the quadratic equation of χ . And therefore, Eq. (7) substantially corresponds to the standpoint to relate $\alpha\sigma_{w1}$ directly to χ , and can fully express the size effect of σ_{w1} . Conversely Eq. (7) seems to be justified, because $\alpha\sigma_{w1}$ is nearly determined by χ alone.

Isibasi extended also his study to the non-propagating crack, and found the relation $\alpha_0\sigma_w \cong \sigma_{sl}$ at the branch point of steels (σ_w : the fatigue limit,

α_0 : α at the branch point, and σ_{sl} : the lower yield point). Except for the case when the notch is shallow enough, however, the relation $\alpha\sigma_{w1} \cong \sigma_{sl}$ holds approximately in the wide range of α , and therefore, it is generally difficult to estimate α at the branch point, on the basis of this relation.

According to the author's study, at the branch point, the root radius of a notch becomes a constant ρ_0 to be determined by the material properties, and ρ_0 has no relation to the diameter of the minimum section or the notch depth. That is, in the specimen of which the notch radius is less than ρ_0 , the non-propagating crack is observed, while it is not observed in the specimen of which the notch radius is larger than ρ_0 . As mentioned in chapter 3, the size effect of σ_{w2} is explained only by this fact.

5.2 Frost's study⁽¹³⁾

According to Frost, when the reversed cycles of stress under tension and compression are added to a strip with a crack as in Fig. 25 (a) or (b), the following relation is found to hold experimentally between the limit stress for crack propagation σ and the effective crack length l .

$$\sigma^3 l = C \quad (C: \text{constant}) \dots\dots\dots (8)$$

Here l is the sum of the notch depth and the crack length. σ is the nominal stress at the minimum section when l is small in comparison with the plate width, but when l becomes large it needs some correction. He utilized the equation to obtain the crack strength σ_{w2} of notched specimens under tension and compression. That is, according to Eq. (8).

$$\sigma_{w2} = \frac{K}{t^{1/3}} \quad (K: \text{constant}) \dots\dots\dots (9)$$

t is the notch depth (the crack length is considered to be short in comparison with t).

When the crack strength σ_{w2} is obtained by the author's method in the case of tension and compression for the strip with a comparatively small notch

Fig. 25

(the length is $2t$) in the middle of the plate width, it is expressed from the result in chapter 3 as follows.

$$\sigma_{w2} = \frac{\sigma_0}{\alpha|_{\rho=\rho_0}} = \frac{\sigma_0}{1 + 2\sqrt{\frac{t}{\rho_0}}}, \quad (\sigma_0, \rho_0: \text{constants}) \dots\dots\dots (10)$$

When K and σ_0 are chosen so that the crack strength calculated by Eqs. (9) and (10) (putting $\rho_0 = 0.5 \text{ mm}$) may become 1 in the case of $t = 1 \text{ mm}$, the dependence of σ_{w2} calculated by both equations on the notch depth t is shown in Table 3, which shows that the crack strengths calculated by Frost's equation agree well with the one by the author's equation. This means that $\alpha|_{\rho=\rho_0}/t^{1/3}$ becomes almost constant over a fairly wide range of t , when $\rho_0 \cong 0.5 \text{ mm}$. Conversely Eq. (9) seems to be obtained from the fact that physically the root radius of a notch, namely the stress distribution near the root of a notch, is constant at the branch point. When ρ_0 is much less than 0.5 mm , therefore, there is the possibility that Frost's equation will not give the crack strength properly.

5.3 Study by Siebel and Stieler⁽¹⁴⁾

Siebel and Stieler obtained the relationship between α/β and χ (β : the notch factor) for a wide range of materials under various load-conditions and showed that α/β , namely the ratio of the maximum stress at the root of a notch at the fatigue limit based on fracture of notched specimens to the fatigue limit of plain specimens under tension and compression, could be expressed by a single curve for each material.

If the fatigue limit based on fracture is determined by the crack initiation limit, this consideration is the same as that of Isibasi or the author. As Isibasi points out⁽¹⁾, however, study by Siebel and Stieler is incomplete because they do not distinguish σ_{w1} and σ_{w2} .

5.4 Some other theories

There are some other theories such as Moore's theory⁽¹⁵⁾, Peterson's theory⁽¹⁶⁾, Heywood's theory⁽¹⁷⁾, Harris' theory⁽¹⁸⁾, etc. concerning the size effect or

Table 3 Comparison of calculated crack strengths under reversed axial load

Notch depth mm	Author, Eq. (10) ($\rho_0 = 0.5 \text{ mm}$)	Frost, Eq. (9)
0.1	2.02	2.15
0.2	1.69	1.71
0.5	1.28	1.26
1.0	1.00	1.00
2.0	0.77	0.79
3.0	0.65	0.69
4.0	0.58	0.63
6.0	0.48	0.55

the notch effect. These theories seem to be as incomplete as the theory by Siebel and Stieler because they do not distinguish σ_{w1} and σ_{w2} .

If the fatigue limit to be obtained from those theories is σ_{w1} , those theories seem to be useful for obtaining σ_{w1} , but actually they have some inconvenience as follows. If they are applied to σ_{w1} , the following equations are obtained.

$$\alpha\sigma_{w1} = \frac{1 + \sqrt{\frac{c}{\rho}}}{1 + \frac{1}{\alpha} \sqrt{\frac{c}{\rho}}} \sigma_{w0}, \quad (\text{Moore}) \dots \dots \dots (11)$$

$$\alpha\sigma_{w1} = \frac{1 + \frac{c}{\rho}}{1 + \frac{1}{\alpha} \frac{c}{\rho}} \sigma_{w0}, \quad (\text{Peterson}) \dots \dots \dots (12)$$

$$\alpha\sigma_{w1} = \left\{ 1 + \left(1 - \frac{1}{\alpha} \right) \frac{c}{\rho} \right\} \sigma_{w0}, \quad (\text{Heywood}) \dots \dots \dots (13)$$

$$\alpha\sigma_{w1} = \frac{1}{1 - \left(1 - \frac{1}{\alpha} \right) e^{-\rho/c}} \sigma_{w0}, \quad (\text{Harris}) \dots \dots \dots (14)$$

Here σ_{w0} is the fatigue limit of plain specimens and c is a constant. $1/\rho$ is nearly proportional to λ and therefore, so far as the stress concentration factors α are equal as in the case of similar specimens, those expressions correspond to the standpoint of relating $\alpha\sigma_{w1}$ directly to λ by a single parameter c . Those equations, however, have contradictions in the case when ρ is constant and α varies, as in the case of a deep hyperboloidal notch. That is, in the case of a deep hyperboloidal notch, if ρ is the same, even though a/ρ varies in the range of $9 \sim \infty$ (then α varies from 2.6 to ∞), the stress distribution near the root of a notch does not change practically (see Fig. 24). Therefore, if two specimens are made, whose ρ is same but d is different, $\alpha\sigma_{w1}$ is supposed to be equal for both specimens, even though α varies in the range of $2.6 \sim \infty$. But since each equation from (11) to (14) includes α , $\alpha\sigma_{w1}$ calculated by these Eqs. varies with α to a certain degree.

6. Conclusions

(1) In the case of rotary bending tests, the root radius of a notch ρ_0 at the branch point is a constant to be determined by the material, and has no relation to the diameter d , of the minimum section and to the notch depth t , or to the existence of work hardened layers. This is due to the fact that the stress distribution near the root of a notch is

almost completely determined by the root radius of the notch ρ alone, except for the special case, irrespective of d or t .

(2) The cause of size effect of the fatigue limit σ_{w1} is different from that of the crack strength σ_{w2} .

a) The size effect of σ_{w1} (the fatigue limit based on crack initiation) appears from the cause that the maximum stress at the root of a notch at the crack initiation limit becomes constant, irrespective of d , t or α if the stress gradient is constant.

b) The size effect of σ_{w2} (the fracture limit within the range of the non-propagating crack existing) appears from the causes that the root radius of a notch becomes constant at the branch point and the crack strength has no relations with ρ .

(3) If the relationship between $\alpha\sigma_{w1}$ and λ , and the root radius of a notch ρ_0 at the branch point are obtained experimentally by varying ρ for a set of d , t and a material, σ_{w1} and σ_{w2} of the notched specimen of an arbitrary size made of the same material are obtained from the above facts (1) and (2).

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