Theory of Elastic Potential of Inhomogeneous Materials*

By Rōzō FURUHASHI** and Masaharu KATAOKA***

If the elastic coefficients are not constant throughout the whole body, such materials are called inhomogeneous. In this paper, using the concept of the parametrix of the fundamental equations in elasticity, the classical main results in the theory of elastic potentials developed by Fredholm, Weyl⁽¹⁾, Kupradze⁽²⁾ and many others have been extended to the case of inhomogeneous materials. Here, by the parametrix of the three dimensional elasticity, we mean the 3×3 square matrix constructed by the nine functions

$$H_s^{(k)}(x, y) = \frac{\lambda(y) + 3\mu(y)}{2\mu(y)[\lambda(y) + 2\mu(y)]} \frac{1}{|x - y|} \delta_s^{(k)} + \frac{\lambda(y) + \mu(y)}{2\mu(y)[\lambda(y) + 2\mu(y)]} \frac{(x_k - y_k)(x_s - y_s)}{|x - y|^3} \\ (k, s = 1, 2, 3)$$

where, λ and μ are Lamé's coefficients, x and y are the points in the elastic body, and δ_s^k is Kronecker's delta. Finally in this paper, the reduction of the fundamental boundary value problems in elasticity to Fredholm systems of singular integral equations is considered.

1. The basic integral formula

Let us assume that the three dimensional domain Ω is bounded by the finite closed Liapunov surface $\partial\Omega$, and $x = (x_1, x_2, x_3)$ will be denoted as the point of space. We assume that Lamé's coefficients $\lambda(x)$ and $\mu(x)$ are any twice differentiable

positive scalar functions of the position x . If the coefficients $\lambda(x)$ and $\mu(x)$ are constant identically throughout the whole domain Ω , this material will be called homogeneous; on the other hand, if $\lambda(x)$ and $\mu(x)$ are not constant, we call that material inhomogeneous.

Definition 1 In some bounded domain Ω , let us introduce the partial differential operators

$$L[u(x)] \equiv \mu(x) \Delta u(x) + [\lambda(x) + \mu(x)] \text{grad} \cdot \text{div} u(x) + \text{grad} \lambda(x) \cdot \text{div} u(x) \\ + \text{grad} \mu(x) \cdot \text{grad} u(x) + \text{grad} \mu(x) \cdot \frac{du(x)}{dx}$$

$$L^0[u(x)] \equiv \mu(x) \Delta u(x) + [\lambda(x) + \mu(x)] \text{grad} \cdot \text{div} u(x).$$

$$\mathfrak{D}[u(x)] \equiv 2\mu(x) \frac{\partial u(x)}{\partial n} + \lambda(x) n \cdot \text{div} u(x) + \mu(x) [n \times \text{curl} u(x)].$$

where, n is the external normal unit vector on the boundary $\partial\Omega$. Similarly, we shall introduce the operators which are frozen at any position $y = (y_1, y_2, y_3)$ in Ω .

$$\tilde{L}_x^0[u(x)] \equiv \mu(y) \Delta_x u(x) + [\lambda(y) + \mu(y)] \text{grad}_x \cdot \text{div}_x u(x).$$

$$\tilde{\mathfrak{D}}_x[u(x)] \equiv 2\mu(y) \frac{\partial u(x)}{\partial n_x} + \lambda(y) n_x \cdot \text{div}_x u(x) + \mu(y) [n_x \times \text{curl} u(x)].$$

where, n_x is the external normal unit vector at the point x on the boundary, and the notations Δ_x , grad_x , etc. mean the operations with respect to the variable x .

Definition 2 By the parametrix of the partial differential operator L_x , we mean the 3×3 matrix function constructed by nine elements

$$H_s^{(k)}(x, y) = \frac{\lambda(y) + 3\mu(y)}{2\mu(y)[\lambda(y) + 2\mu(y)]} \frac{1}{|x - y|} \delta_s^{(k)} + \frac{\lambda(y) + \mu(y)}{2\mu(y)[\lambda(y) + 2\mu(y)]} \frac{(x_k - y_k)(x_s - y_s)}{|x - y|^3} \quad (k, s = 1, 2, 3)$$

Lemma 1 If $x \neq y$, the matrix function $H(x, y)$ satisfies the partial differential equation.

$$\tilde{L}_x^0[H(x, y)] = 0.$$

Lemma 2 For certain constants, there hold the inequalities

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** Graduate Student, Graduate School Meiji University, Kawasaki.

*** Professor, Faculty of Engineering, Meiji University.

$$|H_s^{(k)}| < \frac{\text{const}}{|x-y|}, \quad \left| \frac{\partial H_s^{(k)}}{\partial x_i} \right| < \frac{\text{const}}{|x-y|^2}, \quad \left| \frac{\partial^2 H_s^{(k)}}{\partial x_i \partial x_j} \right| < \frac{\text{const}}{|x-y|^3}. \quad (k, s, i, j=1, 2, 3)$$

Lemma 3 In any bounded domain Ω , the formula

$$c(x)u(x) = \int_{\partial\Omega} [H(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[H(y, x)]] ds_y - \int_{\Omega} [H(y, x) \cdot L[u(y)] - u(y) \cdot L_y[H(y, x)]] dV_y$$

holds for any regular vector function $u(x)$, where $c(x)$ means

$$c(x) = \begin{cases} 4\pi, & \text{if } x \text{ is in the interior of } \Omega. \\ 2\pi, & \text{if } x \text{ is on the boundary of } \Omega. \end{cases}$$

First, we shall prove the next lemma.

Lemma 4 For any two regular vectors $u(x)$ and $v(x)$, the formula

$$e(u, v) = \int_{\partial\Omega} v \cdot \mathfrak{D}[u] ds - \int_{\Omega} v \cdot L[u] dV \quad e(u, v) = e(v, u), \quad e(u, u) \geq 0$$

is valid.

Proof: Let us introduce the notation

$$\tau_{jk}(u) = \lambda(x) \operatorname{div} u(x) \delta_{jk} + \mu(x) \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right) \quad (j, k=1, 2, 3)$$

where, δ_{jk} means Kronecker's delta. By using Gauss' formula, we have

$$\begin{aligned} \int_{\Omega} \sum_{j=1}^3 \sum_{k=1}^3 v_j \frac{\partial \tau_{jk}(u)}{\partial x_k} dV &= \int_{\partial\Omega} \sum_{j=1}^3 \sum_{k=1}^3 v_j \tau_{jk}(u) n_k dS \\ &- \int_{\Omega} \left\{ \lambda(x) \left(\sum_{k=1}^3 \frac{\partial v_k}{\partial x_k} \right) \left(\sum_{k=1}^3 \frac{\partial u_k}{\partial x_k} \right) + \frac{1}{2} \mu(x) \sum_{j=1}^3 \sum_{k=1}^3 \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right) \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) \right\} dV. \end{aligned}$$

From the definition 1, we see

$$v \cdot L[u] = \sum_{j=1}^3 \sum_{k=1}^3 v_j \frac{\partial \tau_{jk}(u)}{\partial x_k}, \quad v \cdot \mathfrak{D}[u] = \sum_{j=1}^3 \sum_{k=1}^3 v_j \tau_{jk}(u) n_k$$

Here we shall define the notation

$$e(u, v) = \int_{\Omega} \left\{ \lambda(x) \left(\sum_{k=1}^3 \frac{\partial v_k}{\partial x_k} \right) \left(\sum_{k=1}^3 \frac{\partial u_k}{\partial x_k} \right) + \frac{1}{2} \mu(x) \sum_{j=1}^3 \sum_{k=1}^3 \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right) \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) \right\} dV$$

From these relations, we can easily get the lemma 4. **Proof of the lemma 3:** First of all, we shall consider the case of $x \in \Omega$. Let us introduce the sphere $\sigma(x; \varepsilon)$ whose centre coincides with the point x and whose radius equals to ε . Obviously in the finite domain $\Omega_\varepsilon = \Omega - \sigma(x; \varepsilon)$ the parametrix $H(x, y)$ is always regular. Applying the lemma 4, we have

$$\int_{\Omega_\varepsilon} [H(y, x) \cdot L[u(y)] - u(y) \cdot L_y[H(y, x)]] dV_y = \int_{\partial\Omega_\varepsilon} [H(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[H(y, x)]] dS_y \dots\dots (1)$$

Let us rewrite as

$$H(y, x) \cdot L[u(y)] - u(y) \cdot L_y[H(y, x)] = -u(y) \cdot L_y^0[H(y, x)] + \xi(y, x)$$

where,

$$\xi(y, x) = H(y, x) \cdot L[u(y)] - u(y) \cdot \left\{ \operatorname{grad} \lambda(y) \cdot \operatorname{div}_y H(y, x) + \operatorname{grad} \mu(y) \cdot \operatorname{grad}_y H(y, x) + \operatorname{grad} \mu(y) \cdot \frac{dH(y, x)}{dy} \right\}$$

Now remembering the definition of the parametrix $H(x, y)$, the inequalities

$$|\xi^{(k)}(x, y)| < \frac{\text{const}}{|x-y|^2}, \quad (k=1, 2, 3)$$

hold. On the other hand from the lemma 1, if $y \in \Omega_\varepsilon$, we have

$$\begin{aligned} L_y^0[H(y, x)] &= L_y^0[H(y, x)] - \bar{L}_y^0[H(y, x)] \\ &\equiv [\mu(y) - \mu(x)] \Delta_y H(y, x) + \{[\lambda(y) + \mu(y)] - [\lambda(x) + \mu(x)]\} \operatorname{grad}_y \cdot \operatorname{div}_y H(y, x) \end{aligned}$$

However using Taylor expansion, when x and y are mutually at the near points, then we can easily see

$$\mu(y) - \mu(x) = \text{const} \cdot |x-y| + o(|x-y|^2).$$

Therefore from the lemma 2, there hold the inequalities

$$|L_y^0{}^{(j)}[H^{(k)}(y, x)]| < \frac{\text{const}}{|x-y|}, \quad (j, k=1, 2, 3). \dots\dots\dots (2)$$

Then the inequalities

$$|H_k^{(j)}(y, x) \cdot L^{(k)}[u(y)] - u_k(y) \cdot L_y^{(k)}[H^{(j)}(y, x)]| < \frac{\text{const}}{|x-y|^2} \quad (j, k=1, 2, 3) \dots\dots\dots (3)$$

are valid.

According to the inequalities (3) when $\varepsilon \rightarrow 0$, we have

$$\left| \int_{\sigma(x;\varepsilon)} (H_k^{(j)}(y, x) \cdot L^{(k)}[u(y)] - u_k(y) \cdot L_y^{(k)}[H^{(j)}(y, x)]) dV_y \right| < \text{const} \cdot \varepsilon \rightarrow 0.$$

Hence the volume integral of left side of Eq. (1) converges to a certain limit when $\varepsilon \rightarrow 0$. Similarly we see that the integral of the finite term of right side of Eq. (1) converges to a certain limit. Because

$$\left| \int_{\sigma(x;\varepsilon)} H(y, x) \cdot \mathfrak{D}[u(y)] dS_y \right| < \text{const} \int_{\sigma(x;\varepsilon)} \frac{1}{|x-y|} dS_y \xrightarrow{\varepsilon \rightarrow 0} 0$$

holds.

Next we shall study the integral of the second term of right side of Eq. (1).

By keeping in mind the definition 1, let us rewrite as

$$\mathfrak{D}_y[H^{(k)}(y, x)] = \tilde{\mathfrak{D}}_y[H^{(k)}(y, x)] + \mathfrak{D}_y^*[H^{(k)}(y, x)], \quad (k=1, 2, 3)$$

Here we can easily see about the above second term of right side

$$|\mathfrak{D}_y^*[H^{(k)}(y, x)]| < \frac{\text{const}}{|x-y|}, \quad (j, k=1, 2, 3)$$

from the definition of the operators.

Because of the definition of $H(x, y)$

$$\begin{aligned} \tilde{\mathfrak{D}}_y^{(j)}[H^{(k)}(y, x)] = & \left\{ \left(\frac{\lambda(x)+3\mu(x)}{\lambda(x)+2\mu(x)} - 1 \right) \delta_{jk} + \frac{6[\lambda(x)+\mu(x)]}{\lambda(x)+2\mu(x)} \frac{(x_k-y_k)(x_j-y_j)}{|x-y|^2} \right\} \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \\ & + \left(1 - \frac{2[\lambda(x)+\mu(x)]}{\lambda(x)+2\mu(x)} \right) \left(\cos(n_y, x_k) \frac{\partial}{\partial y_j} \frac{1}{|x-y|} - \cos(n_y, x_j) \frac{\partial}{\partial y_k} \frac{1}{|x-y|} \right) \end{aligned}$$

However when $\varepsilon \rightarrow 0$, the second term of the above equality becomes

$$\left| \int_{\sigma(x;\varepsilon)} \mathfrak{D}_y^*[H^{(k)}(y, x)] dS_y \right| < \text{const} \cdot \varepsilon \rightarrow 0, \quad (j, k=1, 2, 3).$$

Accordingly we can easily get the relations

$$\lim_{\varepsilon \rightarrow 0} \int_{\sigma(x;\varepsilon)} u(y) \cdot \mathfrak{D}_y[H^{(k)}(y, x)] dS_y = 4\pi u_k(x).$$

Then there holds the equality

$$\lim_{\varepsilon \rightarrow 0} \int_{\sigma(x;\varepsilon)} [H(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[H(y, x)]] dS_y = -4\pi u(x). \quad (4)$$

Because, if we take $\varepsilon \rightarrow 0$ in Eq. (1), the formula

$$4\pi u(x) = \int_{\partial\Omega} [H(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[H(y, x)]] dS_y - \int_{\Omega} [H(y, x) \cdot L[u(y)] - u(y) \cdot L_y[H(y, x)]] dV_y$$

holds. If $x \in \Omega$, the lemma 3 has been proved.

Next we shall consider the case of $x \in \partial\Omega$. Similar to the case of $x \in \Omega$, we shall draw the sphere of radius ε whose centre coincides with the point x on the boundary $\partial\Omega$. Let us denote the part of spherical surface described above which is in the exterior of Ω as the notation $S(\varepsilon)$, and we shall denote the remaining part of $\partial\Omega$ which is cut out by $S(\varepsilon)$ as the notation $\partial\Omega(\varepsilon)$. In the finite domain $\Omega(\varepsilon)$ which is bounded by the surface $\partial\Omega(\varepsilon) + S(\varepsilon)$, the parametrix $H(x, y)$ is always regular; then applying the lemma 3, we have

$$\int_{\partial\Omega(\varepsilon)} [H(y, x) \cdot L[u(y)] - u(y) \cdot L_y[H(y, x)]] dV_y = \int_{\partial\Omega(\varepsilon) + S(\varepsilon)} [H(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[H(y, x)]] dS_y$$

Here using the previous result, we get

$$\int_{\partial\Omega(\varepsilon) + S(\varepsilon)} [H(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[H(y, x)]] dS_y = -4\pi u(x)$$

However repeating the similar considerations as above, we can easily arrive at the equality

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial\Omega(\varepsilon)} [H(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[H(y, x)]] dS_y = -2\pi u(x).$$

Then if $x \in \partial\Omega$, the formula

$$2\pi u(x) = \int_{\partial\Omega} [H(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[H(y, x)]] dS_y - \int_{\Omega} [H(y, x) \cdot L[u(y)] - u(y) \cdot L_y[H(y, x)]] dV_y$$

is valid. Then the lemma 3 has been proved.

Definition 3 By the notation $\phi_s^{(h)}(x, y)$, we mean the nine scalar functions which are twice differentiable when $x \neq y$, and in any closed part of finite domain Ω the inequalities

$$|\phi_s^{(h)}| < \frac{\text{const}}{|x-y|^{1-h}}, \quad \left| \frac{\partial \phi_s^{(h)}}{\partial x_i} \right| < \frac{\text{const}}{|x-y|^{2-h}}, \quad \left| \frac{\partial^2 \phi_s^{(h)}}{\partial x_i \partial x_j} \right| < \frac{\text{const}}{|x-y|^{3-h}} \quad (k, s, i, j=1, 2, 3), h > 0$$

hold. The matrix function which is constructed by the elements $\phi_s^{(h)}$ shall be assumed as $\phi(x, y)$.

By the fundamental solution of the partial differential operator L_x , we mean the 3×3 matrix function $\mathfrak{E}(x, y)$,

which is represented in the domain Ω as the sum $\mathfrak{E}(x, y) = H(x, y) + \phi(x, y)$ and if $x \neq y$, it satisfies the partial differential equation $L_x[\mathfrak{E}(x, y)] = 0$.

Then, $|\mathfrak{E}_s^{(k)}| < \frac{\text{const}}{|x-y|}, \left| \frac{\partial \mathfrak{E}_s^{(k)}}{\partial x_i} \right| < \frac{\text{const}}{|x-y|^2}, \left| \frac{\partial^2 \mathfrak{E}_s^{(k)}}{\partial x_i \partial x_j} \right| < \frac{\text{const}}{|x-y|^3} \quad (k, s, i, j = 1, 2, 3).$

From the lemma 3 we can easily deduce the next theorem.

Theorem 1 For the fundamental solution $\mathfrak{E}(x, y)$ and any regular vector function $u(x)$ which satisfies the differential equation $L[u(x)] = 0$ in the finite domain Ω , the formula

$$c \cdot u(x) = \int_{\partial\Omega} [\mathfrak{E}(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[\mathfrak{E}(y, x)]] dS_y$$

is valid.

Corollary 1 For any constant vector $c = (c_1, c_2, c_3)$, we have

$$\int_{\partial\Omega} c \cdot \mathfrak{D}_y[\mathfrak{E}(y, x)] dS_y = \int_{\partial\Omega} \mathfrak{D}_y[\mathfrak{E}(y, x)] c dS_y = -c(x) \cdot c.$$

2. Introduction of the quasi-potential and the method of construction of the fundamental solution $\mathfrak{E}(y, x)$ for the bounded domain

Definition 4 We denote $\rho(x)$ as the vector function defined in the finite domain Ω . This time the integral

$$U(x) = \int_{\Omega} H(x, y) \cdot \rho(y) dS_y$$

is called the quasi-potential with density ρ . The next lemma can be easily proved.

Lemma 5 For any continuous vector function ρ , there exist the first derivatives of quasi-potential

$$\frac{\partial U(x)}{\partial x_i} = \int_{\Omega} \frac{\partial H(x, y)}{\partial x_i} \rho(y) dV_y, \quad (i = 1, 2, 3)$$

Lemma 6 We shall assume that the density ρ is once differentiable. Then in any sphere of radius ε with its centre at x , the relations

$$\frac{\partial U}{\partial x_i} = \int_{\sigma(x; \varepsilon)} H(x, y) \frac{\partial \rho(y)}{\partial y_i} dV_y - \int_{\sigma(x; \varepsilon)} H(x, y) \rho(y) \cos(n, y_i) dS_y + o(\varepsilon), \quad (i = 1, 2, 3)$$

hold, where $\sigma(x; \varepsilon)$ means the surface of the sphere $\sigma(x; \varepsilon)$.

Proof: From the obvious relations

$$\frac{\partial}{\partial y_i} [H(x, y) \cdot \rho(y)] = -\frac{\partial H(x, y)}{\partial x_i} \rho(y) + H(x, y) \frac{\partial \rho(y)}{\partial y_i} + \xi_{(i)}^{(k)}(x, y)$$

where

$$\xi_{(i)}^{(k)}(x, y) = \frac{\partial}{\partial y_i} \left(\frac{\lambda(y) + 3\mu(y)}{2[\lambda(y) + 2\mu(y)]} \right) \cdot \frac{1}{|x-y|} \rho_k(y) + \sum_{s=1}^3 \frac{\partial}{\partial y_i} \left(\frac{\lambda(y) + \mu(y)}{2[\lambda(y) + 2\mu(y)]} \right) \frac{(x_k - y_k)(x_s - y_s)}{|x-y|^2} \rho_k(y) \quad (i, k = 1, 2, 3)$$

we can easily obtain the inequalities

$$\left| \xi_{(i)}^{(k)}(x, y) \right| < \frac{\text{const}}{|x-y|}, \quad (i, k = 1, 2, 3)$$

According to the result of lemma 5, we'll see that the equalities

$$\frac{\partial U(x)}{\partial x_i} = \int_{\sigma(x; \varepsilon)} H(x, y) \frac{\partial \rho(y)}{\partial y_i} dV_y - \int_{\sigma(x; \varepsilon)} \frac{\partial}{\partial y_i} [H(x, y) \rho(y)] dV_y + o(\varepsilon)$$

hold. Applying Gauss' formula on the second term of the above equalities, we can easily conclude lemma 6.

Lemma 7 If $x \in \Omega$, the formula

$$L^0[U(x)] = -4\pi\rho(x) + \int_{\Omega} L_x^0[H(x, y)] \rho(y) dV_y$$

is valid for any once differentiable vector function ρ .

Proof: Similar to the previous considerations, we denote as $\Omega = \Omega_\varepsilon + \sigma(x; \varepsilon)$, and we shall assume the spherical surface of σ as $\sigma(x; \varepsilon)$. From the lemma 6, the second derivatives of quasi-potential $U(x)$ are represented formally by

$$\frac{\partial^2 U}{\partial x_j \partial x_i} = \int_{\Omega} \frac{\partial^2 H(x, y)}{\partial x_j \partial x_i} \rho(y) dV_y + \int_{\sigma(x; \varepsilon)} \frac{\partial H(x, y)}{\partial x_j} \frac{\partial \rho(y)}{\partial y_i} dV_y - \int_{\sigma(x; \varepsilon)} \frac{\partial H(x, y)}{\partial y_i} \rho(y) \cos(n, y_j) dS_y + o(\varepsilon), \quad (i, j = 1, 2, 3)$$

Here, let us use the elementary relations

$$\int_{\bar{\sigma}(x;\varepsilon)} \frac{\partial H_j^{(p)}}{\partial y_s} \cos(n, y_i) dS_y = \int_{\sigma(x;\varepsilon)} \frac{\partial^2 H_j^{(p)}}{\partial y_i \partial y_s} dV_y = \int_{\bar{\sigma}(x;\varepsilon)} \frac{\partial H_j^{(p)}}{\partial y_i} \cos(n, y_s) dS_y$$

Further we see that according to the definition 1, the i -element of the operator $L_x^0[u]$ takes the form

$$L_x^0[u(x)] = \mu(x) \sum_{j=1}^3 \frac{\partial^2 u_j(x)}{\partial x_j^2} + [\lambda(x) + \mu(x)] \sum_{j=1}^3 \frac{\partial u_j(x)}{\partial x_j}$$

By using the above variable relations, we can represent formally

$$L^0[U(x)] = \int_{\Omega_\varepsilon} L_x^0[H(x, y)] \rho(y) dV_y - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{s=1}^3 e_j \int_{\bar{\sigma}(x;\varepsilon)} \left\{ \lambda(x) \sum_{p=1}^3 \frac{\partial H_j^{(p)}(x, y)}{\partial y_p} \delta_{si} + \mu(x) \left(\frac{\partial H_j^{(i)}(x, y)}{\partial y_s} - \frac{\partial H_j^{(s)}(x, y)}{\partial y_i} \right) \right\} \cos(n, y_s) \rho_i(y) dS_y + o(\varepsilon).$$

where e_j means the unit vector in the direction of x_j -coordinate axis. At the time we can show that

$$\sum_{i=1}^3 \sum_{j=1}^3 \sum_{s=1}^3 e_j \left\{ \lambda(x) \sum_{p=1}^3 \frac{\partial H_j^{(p)}}{\partial y_p} \delta_{si} + \mu(x) \left(\frac{\partial H_j^{(i)}}{\partial y_s} - \frac{\partial H_j^{(s)}}{\partial y_i} \right) \right\} \cos(n, y_s) \rho_i(y) = \hat{\mathfrak{D}}_y[H(x, y)] \rho(y).$$

Then we get

$$L^0[U(x)] = \int_{\Omega_\varepsilon} L_x^0[H(x, y)] \rho(y) dV_y - \int_{\bar{\sigma}(x;\varepsilon)} \hat{\mathfrak{D}}_y[H(x, y)] \rho(y) dV_y + o(\varepsilon) \dots \dots \dots (5)$$

However if $x \in \Omega$, we know from the previous discussion (cf. Proof of the lemma 3) that the next equality holds.

$$\lim_{\varepsilon \rightarrow 0} \int_{\bar{\sigma}(x;\varepsilon)} \hat{\mathfrak{D}}_y[H(x, y)] \rho(y) dS_y = 4\pi \rho(x).$$

Let us rewrite as

$$L_x^0[H(x, y)] = \tilde{L}_x^0[H(x, y)] + \xi(x, y)$$

where, remembering the definition 1, we see that $\xi(x, y)$ satisfies the inequalities

$$|\xi^{(j)}(x, y)| < \frac{\text{const}}{|x-y|^2}, \quad (j=1, 2, 3)$$

On the other hand from the lemma 1, if $x \in \Omega$, the parametrix $H(x, y)$ satisfies the differential equation

$$\tilde{L}_x^0[H(x, y)] = 0.$$

Then the inequalities

$$|L_x^0[H^{(k)}(x, y)]| < \frac{\text{const}}{|x-y|^2}, \quad (j, k=1, 2, 3) \dots \dots \dots (6)$$

are valid. Namely, the integrand of right side of Eq. (5) has only quasi-singularity. Accordingly there holds

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} L_x^0[H(x, y)] \rho(y) dV_y = \int_{\Omega} L_x^0[H(x, y)] \rho(y) dV_y.$$

Since at the limit of $\varepsilon \rightarrow 0$, the formula

$$L^0[U(x)] = -4\pi \rho(x) + \int_{\Omega} L_x^0[H(x, y)] \rho(y) dV_y$$

holds. q.e.d.

Because of this lemma, the next theorem is obvious.

Theorem 2 If $x \in \Omega$, the formula

$$L[U(x)] = -4\pi \rho(x) + \int_{\Omega} L_x[H(x, y)] \rho(y) dV_y$$

is valid for any once differentiable vector function ρ . Applying the above theorem, we can construct the fundamental solution $\mathfrak{E}(x, y)$ defined in the finite domain Ω as follows.

Theorem 3 Let us rewrite the fundamental solution $\mathfrak{E}(x, y)$ of the differential operator L_x in the form

$$\mathfrak{E}(x, y) = H(x, y) + \int_{\Omega} H(x, x') \Phi(x', y) dV_{x'},$$

where 3×3 matrix function $\Phi(x, y)$ satisfies the integral equation

$$4\pi \Phi(x, y) - \int_{\Omega} L_x[H(x, x')] \Phi(x', y) dV_{x'} = L_x[H(x, y)].$$

Proof: After the operation L_x on the above expression of $\mathfrak{E}(x, y)$, by using the theorem 2, we get

$$L_x[\mathfrak{E}(x, y)] = L_x[H(x, y)] - 4\pi \Phi(x, y) + \int_{\Omega} L_x[H(x, x')] \Phi(x', y) dV_{x'}.$$

Because of the fact that the matrix $\mathfrak{E}(x, y)$ must be the fundamental solution of L_x , if $x \neq y$, it satisfies the differential equation

$$L_x[\mathfrak{E}(x, y)] = 0$$

Then we can get the above integral equation with respect to $\Phi(x, y)$.

3. The generalized Poisson's formula for the bounded domain

Definition 5 By the generalized volume potential with the density vector ρ in the bounded domain Ω , we mean the integral of the form

$$\dot{U}(x) = \int_{\Omega} \mathfrak{E}(x, y) \rho(y) dV_y$$

Lemma 8 If $x \in \Omega$, the generalized volume potential satisfies the partial differential equation

$$L[\dot{U}(x)] = -4\pi\rho(x)$$

for any once differentiable vector function ρ .

Proof: Using the theorem 3, we can easily arrive at the equality

$$\begin{aligned} L[\dot{U}(x)] &= -4\pi\rho(x) \\ &+ \int_{\Omega} \left\{ L_x[H(x, y)] - 4\pi\Phi(x, y) \right. \\ &\left. + \int_{\Omega} L_x[H(x, x')] \Phi(x', y) dV_{x'} \right\} \rho(y) dV_y \end{aligned}$$

However the inside of $\{\dots\}$, according to the lemma 2, equals to zero. Then the above lemma is proved.

$$L[W(x)] = \frac{1}{2\pi} \int_{\partial\Omega} L_x \{ \mathfrak{D}_y [\mathfrak{E}(x, y)^t] \}^t \cdot \varphi(y) dS_y$$

Here, we shall introduce the well-known expression of matrix differential operator of L and $\mathfrak{D}$. This time we can arrive immediately at the equality

$$\begin{aligned} L_x \{ \mathfrak{D}_y [\mathfrak{E}(x, y)^t] \}^t &= \left(L_x^{(ij)}, i \downarrow 1, 2, 3, j \rightarrow 1, 2, 3 \right) \left(\sum_{k=1}^3 \mathfrak{D}_y^{(ik)} \mathfrak{E}_j^{(k)}(x, y), i \downarrow 1, 2, 3, j \rightarrow 1, 2, 3 \right) \\ &= \left(\sum_{k=1}^3 \sum_{p=1}^3 \mathfrak{D}_y^{(ik)} \cdot L_x^{(ip)} \mathfrak{E}_p^{(k)}(x, y), i \downarrow 1, 2, 3, j \rightarrow 1, 2, 3 \right) \end{aligned}$$

However when $x \neq y$ (this condition always holds at this place), according to the definition of the fundamental solution $\mathfrak{E}(x, y)$, we have

$$\sum_{p=1}^3 L_x^{(ip)} [\mathfrak{E}_p^{(k)}(x, y)] = 0, \quad (i, k = 1, 2, 3)$$

Then we can see that if $x \in \Omega$, the vector function $W(x)$ satisfies the equation

$$L[W(x)] = 0.$$

Theorem 6 For the generalized potential of double layers with any Hölder continuous density φ , if we are allowed to approach the point x from the interior of Ω to the point x_0 on the boundary $\partial\Omega$, then there holds the next formula,

$$W(x_0)_{(i)} = -\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \varphi(y) \cdot \mathfrak{D}_y [\mathfrak{E}(x_0, y)^t] dS_y$$

where the notation $W(x_0)_{(i)}$ means the interior limit of $W(x)$ on the boundary.

Proof: First let us rewrite as follows.

$$\begin{aligned} W(x) &= \frac{1}{2\pi} \int_{\partial\Omega} \varphi(y) \cdot \mathfrak{D}_y [\mathfrak{E}(x, y)^t] dS_y = \frac{1}{2\pi} \int_{\partial\Omega} \varphi(x_0) \cdot \mathfrak{D}_y [\mathfrak{E}(y, x)^t] dS_y \\ &+ \frac{1}{2\pi} \int_{\partial\Omega} \varphi(x_0) \{ \mathfrak{D}_y [\mathfrak{E}(x, y)^t - \mathfrak{E}(y, x)^t] \} dS_y + \frac{1}{2\pi} \int_{\partial\Omega} [\varphi(y) - \varphi(x_0)] \mathfrak{D}_y [\mathfrak{E}(x, y)^t] dS_y \dots \dots \dots (7) \end{aligned}$$

Here by using the definition of the fundamental solution $\mathfrak{E}(x, y)$, we can easily get the inequalities

4. The generalized potentials of simple and double layers for the bounded domain

Definition 6 By the generalized potential of simple layer with the density vector φ , we mean the integral of the form

$$V(x) = \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{E}(x, y) \cdot \varphi(y) dS_y$$

The next results are obvious.

Lemma 9 The generalized potential of simple layer with the continuous density is continuous everywhere including the boundary.

Theorem 4 If $x \in \Omega$, the vector function $V(x)$ satisfies the partial differential equation

$$L[V(x)] = 0.$$

Definition 7 By the generalized potential of double layers with density vector φ , we mean the integral of the form

$$W(x) = \frac{1}{2\pi} \int_{\partial\Omega} \varphi(y) \cdot \mathfrak{D}_y [\mathfrak{E}(x, y)^t] dS_y$$

where, $\mathfrak{E}(x, y)^t$ means the transposed matrix of $\mathfrak{E}(x, y)$.

Theorem 5 If $x \in \Omega$, the vector function $W(x)$ satisfies the partial differential equation

$$L[W(x)] = 0.$$

Proof: After the operation L on the above expression of $W(x)$, we have

$$|\mathfrak{D}_y^{(j)}[\mathfrak{E}^{(k)}(x, y)^t - \mathfrak{E}^{(k)}(y, x)^t]| < \frac{\text{const}}{|x-y|}, \quad (j, k=1, 2, 3)$$

On the other hand, for the reason that the vector function $\varphi(x)$ is Hölder continuous, the third term of right side in Eq. (7) has only quasi-singularity. Then we see that the vector function

$$\alpha(x) = W(x) - \frac{1}{2\pi} \int_{\partial\Omega} \varphi(x_0) \cdot \mathfrak{D}_y[\mathfrak{E}(x, y)^t] dS_y$$

is continuous everywhere including the boundary. Here using the lemma 8, we get

$$W(x_0)_{(i)} + 2\varphi(x_0) = W(x_0) + \varphi(x_0).$$

From this relation we can obviously arrive at the object.

Theorem 7 If the vector function φ is Hölder continuous, for the generalized potential of simple layer $V(x)$ with the density φ , there exists the formula

$$\mathfrak{D}[V(x_0)]_{(i)} = \varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}_{x_0}[\mathfrak{E}(x_0, y)] \varphi(y) dS_y$$

Proof: From the definition of the fundamental solution $\mathfrak{E}(x, y)$, we see

$$\begin{aligned} \mathfrak{D}[V(x)] &= \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}_x[\mathfrak{E}(x, y)] \varphi(y) dS_y = \frac{1}{2\pi} \int_{\partial\Omega} (\mathfrak{D}_x[H(x, y)] + \mathfrak{D}_y[H(x, y)]) \varphi(y) dS_y \\ &\quad - \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}[H(x, y)] [\varphi(y) - \varphi(x_0)] dS_y - \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}_y[H(x, y)] \varphi(x_0) dS_y + \frac{1}{2\pi} \int_{\partial\Omega} \xi(x, y) \varphi(y) dS_y \end{aligned}$$

.....(8)

where $\xi(x, y)$ satisfies the inequalities

$$|\xi_j^{(k)}(x, y)| < \frac{\text{const}}{|x-y|^{2-h}} \quad (j, k=1, 2, 3)$$

From the definition of the parametrix $H(x, y)$, we see that

$$\begin{aligned} &(\mathfrak{D}_x^{(s)}[H^{(k)}(x, y)] + \mathfrak{D}_y^{(s)}[H^{(k)}(x, y)]) \\ &= \frac{1}{\lambda(y) + 2\mu(y)} \left\{ \left[\mu(y) \delta_{sk} + 3[\lambda(y) + \mu(y)] \frac{(x_s - y_s)(x_k - y_k)}{|x-y|^2} \cdot \frac{\cos(r, n_x) - \cos(r, n_y)}{|x-y|^2} \right] \right. \\ &\quad \left. + \mu(y) \left[(\cos(n_x, x_k) - \cos(n_y, x_k)) \frac{\partial}{\partial x_s} \frac{1}{|x-y|} - (\cos(n_x, x_s) - \cos(n_y, x_s)) \frac{\partial}{\partial x_k} \frac{1}{|x-y|} \right] \right\} \\ &+ \eta_k^{(s)}(x, y), \quad |\eta_k^{(s)}(x, y)| < \frac{\text{const}}{|x-y|^{1-h}}, \quad (s, k=1, 2, 3) \end{aligned}$$

where, the notation r denotes the unit vector directed from the point y to x . Here as is well-known, if the boundary $\partial\Omega$ consists of Liapunov surface of index h , the fundamental relation

$$|\cos(n_x, x_k) - \cos(n_y, x_k)| < \text{const} \cdot |x-y|^h$$

holds⁽³⁾. Then the inequalities

$$|\mathfrak{D}_x^{(s)}[H^{(k)}(x, y)] + \mathfrak{D}_y^{(s)}[H^{(k)}(x, y)]| < \frac{\text{const}}{|x-y|^{2-h}} \quad (s, k=1, 2, 3)$$

are valid. Accordingly, we see that the integrand of right side of Eq. (8) has only quasi-singularity. Similarly, because of the fact that the vector function φ is Hölder continuous, the integrand of the second term in Eq. (8) has no more than quasi-singularity. Then we know that the vector function

$$\alpha(x) = \mathfrak{D}[V(x)] + \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}_y[H(x, y)] \varphi(x_0) dS_y$$

is continuous everywhere. Therefore the vector function

$$\beta(x) = \mathfrak{D}[V(x)] + \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}_y[\mathfrak{E}(y, x)] \varphi(x_0) dS_y$$

is continuous in every possible point of x . Here if the point x is allowed to approach the point x_0 on the boundary, and if we use the result of lemma 8 on the second term of the above equality, we have immediately

$$\mathfrak{D}[V(x_0)]_{(i)} - 2\varphi(x_0) = \mathfrak{D}[V(x_0)] - \varphi(x_0). \quad \text{q.e.d.}$$

5. The elastic potentials in the infinite domain

We shall now assume that in the exterior of any large sphere Σ including the bounded domain Ω in it, the elastic body turns into homogeneous media. The boundary value problems of this type were considered by W. Sternberg⁽⁴⁾ about the problems of the elliptic differential equation of the second order with variable coefficients.

We shall show that Lamé's coefficients $\lambda(x)$ and $\mu(x)$ can be extended to the outer region of the sphere Σ discussed above with sufficiently smoothness, and in the exterior of Σ , we assume that λ and μ equal to certain constants. The fundamental solution which is extended to the infinite domain in the sense of the method described above shall be expressed by the notation $\Gamma(x, y)$. This time, we can easily prove that the matrix function $\Gamma(x, y)$ has the symmetric properties

$$\Gamma_j^{(k)}(x, y) = \Gamma_k^{(j)}(y, x), \quad (j, k=1, 2, 3)$$

Theorem 8 For any regular vector function $u(x)$ which satisfies the differential equation $L[u]=0$ in the interior of the bounded domain Ω , the basic formula

$$\eta(x)u(x) = \int_{\partial\Omega} (\Gamma(y, x) \cdot \mathfrak{D}[u(y)] - u(y) \cdot \mathfrak{D}_y[\Gamma(y, x)]) dS_y$$

is valid. Where,

$$\eta(x) = \begin{cases} 4\pi, & \text{when } x \in \Omega_i \text{ (the interior of } \Omega) \\ 2\pi, & \text{when } x \in \partial\Omega \\ 0, & \text{when } x \in \Omega_e \text{ (the exterior of } \Omega) \end{cases}$$

Proof: In the case of $x \in \Omega_i$ and $x \in \partial\Omega$, the above formula can be deduced easily from the theorem 1. When $x \in \Omega_e$, the relation $x \neq y$ holds always. Then from the definition of the fundamental solution $\Gamma(x, y)$, this formula is obvious.

Definition 8 By the generalized volume potential with the density ρ , we mean the integral of the form

$$\tilde{U}(x) = \int \Gamma(x, y) \rho(y) dV_y$$

Here the next lemma holds.

Lemma 10 For any once differentiable vector function ρ with the support on some bounded domain Ω , the generalized volume potential satisfies the partial differential equations

$$L[\tilde{U}(x)] = -4\pi\rho(x), \quad \text{when } x \text{ is in the interior of } \Omega.$$

$$L[U(x)] = 0, \quad \text{when } x \text{ is in the exterior of } \Omega.$$

Definition 9 By the generalized potential of simple layer with the density vector φ , we mean the integral of the form

$$V(x) = \frac{1}{2\pi} \int_{\partial\Omega} \Gamma(x, y) \varphi(y) dS_y$$

Definition 10 By the generalized potential of double layers with the density vector φ , we mean the integral of the form

$$W(x) = \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x, y) \varphi(y) dS_y$$

where, $\Gamma_I(x, y)$ means the transposed matrix of $\mathfrak{D}_y[\Gamma(x, y)^t]$.

Repeating the similar considerations in the case of the finite domain, we can easily get the next results.

Lemma 11 The generalized potential of simple layer with the continuous density continuous everywhere.

Theorem 9 When $x \notin \partial\Omega$, $V(x)$ satisfies the partial differential equation

$$L[V(x)] = 0$$

Similarly, the vector function $W(x)$ satisfies the equation if $x \notin \partial\Omega$.

$$L[W(x)] = 0.$$

Theorem 10 For the generalized potential of

simple layer with any Hölder continuous density φ , there exist the formulas

$$\begin{aligned} \mathfrak{D}[V(x_0)]_{(i)} &= \varphi(x_0) \\ &+ \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}_{x_0}[\Gamma(x_0, y)] \varphi(y) dS_y \\ \mathfrak{D}[V(x_0)]_{(e)} &= -\varphi(x_0) \\ &+ \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}_{x_0}[\Gamma(x_0, y)] \varphi(y) dS_y, \quad x_0 \in \partial\Omega. \end{aligned}$$

where, the notation $\mathfrak{D}[V(x_0)]_{(i)}$ and $\mathfrak{D}[V(x_0)]_{(e)}$ will be taken respectively for the interior and exterior limit of $\mathfrak{D}[V(x_0)]$ on the boundary $\partial\Omega$.

Theorem 11 For the generalized potential of double layer with any Hölder continuous density φ , there exist the formulas

$$\begin{aligned} W(x_0)_{(i)} &= -\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x_0, y) \varphi(y) dS_y \\ W(x_0)_{(e)} &= \varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x_0, y) \varphi(y) dS_y \\ &x_0 \in \partial\Omega. \end{aligned}$$

6. Application to the boundary value problems in elasticity

In this chapter, we shall point out that by using the results explained in the previous chapters, the basic boundary value problems in elasticity can be reduced to systems of singular integral equations of Fredholm type.

By the first to the third boundary value problems in the theory of elasticity, we mean the problems to find the regular vector function $u(x)$ which satisfies the next conditions. If the point x does not lie on the boundary $\partial\Omega$, the vector function $u(x)$ satisfies the strongly elliptic partial differential equation

$$L[u(x)] = 0.$$

and if the point x lies on the boundary, the function $u(x)$ has to satisfy one of the next conditions in the form

$$\begin{aligned} (a) \quad u(x_0) &= f(x_0) \\ (b) \quad \mathfrak{D}[u(x_0)] &= f(x_0) \\ (c) \quad \mathfrak{D}[u(x_0)] + \Pi(x_0)u(x_0) &= f(x_0) \end{aligned} \quad x_0 \in \partial\Omega.$$

where, $f(x_0)$ is the known Hölder continuous vector function defined on $\partial\Omega$, and $\Pi(x_0)$ means any Hölder continuous, positive definite, and symmetric matrix function.

This time the problem (a) represents the elastic problems which describe the displacements on the boundary, similarly the problem (b) represents the elastic problems which describe the external forces on the boundary.

If the domain is infinite, $u(x)$ must satisfy the so-called radiation conditions at infinity,

$$\text{when } |x| \rightarrow \infty;$$

$$|u_j(x)| = o(|x|^{-1}), \quad \left| \frac{\partial u_j(x)}{\partial x_i} \right| = o(|x|^{-2})$$

$$(i, j=1, 2, 3)$$

a) First boundary value problem

On the basis of the theorem 9, we seek the solution of the first problem in the form of the generalized potential of double layers

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x, y) \varphi(y) dS_y$$

Then using the theorem 11, we get the system of singular integral equations of Fredholm type about the vector function φ

$$-\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x_0, y) \varphi(y) dS_y = f(x_0)$$

for the interior first problem, and the integral equation

$$\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x_0, y) \varphi(y) dS_y = f(x_0)$$

for the exterior first problem

b) Second boundary value problem

On the basis of the theorem 9, we seek the solution of the second problem in the form of the generalized potential of simple layer

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \Gamma(x, y) \varphi(y) dS_y$$

Then, using the theorem 10, we get the system of singular integral equations of Fredholm type about the vector function φ

$$\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}_{x_0}[\Gamma(x_0, y)] \varphi(y) dS_y = f(x_0)$$

for the interior second problem, and the integral equation

$$-\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \mathfrak{D}_{x_0}[\Gamma(x_0, y)] \varphi(y) dS_y = f(x_0)$$

for the exterior second problem.

c) Third boundary value problem

Similar to the second problem, we seek the solution in the form of the generalized potential of simple layer. Then, we obtain the integral equation

$$\begin{aligned} \varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \{ \mathfrak{D}_{x_0}[\Gamma(x_0, y)] \\ + \Pi(x_0) \Gamma(x_0, y) \} \varphi(y) dS_y = f(x_0) \end{aligned}$$

for the interior third problem, and the integral equation

$$\begin{aligned} -\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \{ \mathfrak{D}_{x_0}[\Gamma(x_0, y)] \\ + \Pi(x_0) \Gamma(x_0, y) \} \varphi(y) dS_y = f(x_0) \end{aligned}$$

for the exterior problem.

At the end of this paper, the authors extend sincere thanks to Miss Aki Adachi of Meiji University for her many advices.

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Discussion

I. NAKAHARA: Judging from the conclusions of your paper, I guess your point as follows.

The following is an example of the first boundary value problem whose surface force is given by $f(x_0)$. Suppose now that

$$L[u(x)] = 0, \quad x \in \Omega \quad \dots\dots\dots(i)$$

is the differential equation of displacement $u(x)$ in an elastic interior domain Ω . Then obtain the solution of that Eq. (i) in the form

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x, y) \varphi(y) dS_y \quad \dots\dots\dots(ii)$$

where $\partial\Omega$ is the surface of elastic body and $\varphi(y)$ is the solution of the integral equation

$$\begin{aligned} -\varphi(x_0) + \frac{1}{2\pi} \int_{\partial\Omega} \Gamma_I(x_0, y) \cdot \varphi(y) dS_y = f(x_0). \\ \dots\dots\dots(iii) \end{aligned}$$

$\mathfrak{D}[u]$ is a stress vector generated from the displacement u and $\Gamma_I(x, y)$ is a transposed matrix of $\mathfrak{D}_y[\mathfrak{E}(x, y)^t]$. If we put

$$\mathfrak{E}(x, y) = H(x, y) + \phi(x, y)$$

then $H(x, y)$ is the solution of the equation

$$L_x^0[H(x, y)] = 0.$$

(1) I wonder you intend to come to the same conclusion as in your previous paper⁽⁵⁾ in the method of parametrix.

(2) In which formula can we get $\phi(x, y)$?

(3) Is $H(x, y)$ the solution of the elasticity of homogeneous body?

(4) You said that you could transform the solution of Eq. (i) to formula (ii) by Theorem 9. Has anybody said this before? Are these theorems and lemmas new results? If not, please divide these ones into knowns and unknowns.

(5) Please show me your aim and conclusion.

(6) Probably we can not analyse the boundary value problems concretely in your conclusion. How do we use it?

(7) Please show me how to use the method in those examples.

"Example 1" Find the stress in a homogeneous bar of circular section with the radius a under the uniform pressure along its axis.

"Example 2" Find the stress of an infinite body under the uniform pressure in the direction of y whose Young's modulus varies in the direction of x .

A. YOSHIDA: (8) Has the word "regular", appears in 'regular vector u ' of Lemma 4, the same meaning as that used in ' $u(x)$ is regular' of reference (5)? The latter is defined only when u is the solution vector.

(9) You obtain $\mathcal{E}$ from Φ which is constructed by the integral equation of Theorem 3. In this case, if we suppose $x=y$, has $\mathcal{E}$ the singularity in Definition 3?

(10) Please show me how to solve the singular integral equation. What is your opinion about the regularity of the solution and mixed boundary value problem?

Authors' closure

The first boundary value problem is not a problem in which are given the surface forces on the boundary, but a problem in which are given the displacements of the surface.

(1) Yes.

(2) The method of construction of the fundamental solution is explained in Theorem 3, where the integral $\int_{\Omega} H(x, x') \Phi(x', y) dV_{x'}$, corresponds to $\phi(x, y)$.

(3) The parametrix $H(x, y)$ is not a solution of the homogeneous materials. If $\lambda(x)$ and $\mu(x)$ are infinite times differentiable it satisfies the partial differential equation

$$\tilde{L}_x^0[H(x, y)] = -4\pi\delta(x, y)I$$

in the sense of distribution. (Where $\delta(x, y)$ is the delta function, I is 3×3 unit matrix). However the condition of infinite times differentiability is not necessary in this paper, it is sufficient only if $\lambda(x)$ and $\mu(x)$ are twice differentiable.

(4) All lemmas and theorems in this paper are new results, and they are the extensions of the basic classical results in the case of homogeneous materials to the inhomogeneous case. Particularly, on the concept of parametrix of elasticity explained in this paper, we do not know any other examples which have been used to study the theory of elasticity of inhomogeneous materials.

(5) The authors' main aim is to consider the boundary value problems of elasticity which has no theoretical obscurities, and particularly, the main object of this paper is to prepare for them.

(6) and (7) In the case of homogeneous materials the method of integral equations is thought to be useful for 'calculation' of the concrete problems using the computer. Particularly if the shape of the boundary is complex, it is very difficult to con-

sider the three dimensional problems by any other method.

"Example 1" We assume that the uniform pressure $-p$ is distributed on the plane $x_1 = \pm h$. Further we shall denote the part of boundary under the external force by the notation $\partial\Omega_1$ and the remaining part by $\partial\Omega_2$. By the Green tensor of the second problem $N(x, y)$ the solution of this problem is expressed as

$$u(x) = -p \int_{\partial\Omega_1} \cos(n, x_1) N^{(1)}(x, y) dS_y$$

If the materials are homogeneous the stresses τ_{jk} become $\tau_{jk} = -p\delta_{1j}\delta_{1k}$. However in the case of inhomogeneous materials the stresses in elastic body are not uniform.

"Example 2" If the domain is infinite, it is essential to establish the radiation conditions at infinity. Accordingly when 'calculating' formally the elastic problems which do not satisfy the radiation conditions, it is not adequate to use the method of elastic potentials. For the solution of the problems of infinite body under the uniform pressure can not satisfy the radiation conditions. It is known that a regular solution of the equation $L[u] = 0$ everywhere in the infinite body does not exist except the identical zero. However in the case when the body force $-4\pi K(x)$ is distributed in some finite domain Ω of an infinite body, the solution of this problem is expressed by

$$u(x) = \int_{\Omega} \Gamma(x, y) K(y) dV_y$$

(8) It has the same meaning.

(9) The fundamental solution $\mathcal{E}(x, y)$ is given by

$$\mathcal{E}(x, y) = H(x, y) + \int_{\Omega} H(x, x') \Phi(x', y) dV_{x'}$$

where $\Phi(x, y)$ satisfies the integral equation

$$4\pi\Phi(x, y) = L_x[H(x, y)] + \int_{\Omega} L_x[H(x, x')] \Phi(x', y) dV_{x'}$$

However we can see easily that

$$|L_x[H(x, y)]| < \frac{\text{const}}{|x-y|^{3-h}}$$

From this we get

$$|\Phi(x, y)| < \frac{\text{const}}{|x-y|^{3-h}}$$

Accordingly, using Lemma 2, the inequality

$$\left| \int_{\Omega} L_x[H(x, x')] \Phi(x', y) dV_{x'} \right| < \frac{\text{const}}{|x-y|^{1-h}}$$

is valid. This relation shows that the above matrix function $\mathcal{E}(x, y)$ satisfies the conditions of Definition 3.

(10) If the elastic body is homogeneous the approximate method will be useful, but in the inhomogeneous case, we do not intend to calculate the

concrete problems by using the integral equations obtained in this paper. We regard them as the preparation to introducing the other analytical method.

Next we explain simply concerning the regularity of the solutions. For brevity we assume that the boundary $\partial\Omega$ is finite, and the solution φ of the integral equations is integrable over $\partial\Omega$, that is

$$\left| \int_{\partial\Omega} \varphi(x) dS_x \right| < \infty$$

Then the potential $V(x)$ takes the value $|V(x)| \rightarrow 0$ when $|x| \rightarrow \infty$, accordingly the regularity conditions at infinity are satisfied. The other cases are also similar.

The regularity conditions in elasticity are necessary to study exactly the problems. But we know that there exist many 'practically useful' solutions which do not satisfy the regularity conditions. Therefore we think that some generalizations of the

concept of solutions in elasticity to cover the case which can be included such as 'theoretically not orthodox, but practically useful solutions' will be necessary.

For the three dimensional mixed boundary value problems of any shape domain, the next formal consideration will be possible. We assume $N(x, y)$ as the Green tensor for the second problem, then the solution of the second problem is given by

$$u(x) = \int_{\partial\Omega} f(y) N(x, y) dS_y.$$

Now, in the subregion $\partial\Omega - \partial\Omega^*$ of $\partial\Omega$ we suppose that $f=0$. then

$$u(x) = \int_{\partial\Omega^*} f(y) N(x, y) dS_y.$$

Accordingly the stresses under the stamp, which is generated by the displacement of $\partial\Omega^*$, will be obtained from the above Fredholm integral equation of first kind.