532. 525

Theory of Laminar Flows in Rounded Pipe Entrance*

(1st Report, In Very Slow Flows When Mass Force Is Negligible)

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For a viscous fluid flowing very slowly near a pipe inlet and exit when the mass forces of Navier-Stokes equations are negligible, the author solved strictly the equations of motion and obtained the stream function, the velocity distribution, the coefficient of skin friction and the pressure gradient. The results are as follows.

(1) When the radius of a rounded pipe is large, the velocity distributions at a straight pipe entrance are near-parabolic. When it is small, they are close to the mean velocity profile. Also, when it is zero, i.e. the entrance is square-edge, the streamlines and the velocity distribution agree with Sampson's solution for a thin orifice.

(2) The pressure gradient on the rounded pipe surface is large near a straight pipe inlet, and is small at a large distance from it. Either for inflow or for outflow, separation can not occur as the pressure continues to decrease due to viscosity.

1. Introduction

There are no theoretical analyses of a viscous fluid flowing near a rounded pipe inlet or exit as far as the author knows. When the velocity of flow is large, it is well known that the velocity profile becomes uniform with an increasing radius of rounded entrance⁽¹⁾. In the preceding paper⁽²⁾, the author reported the theory of additional loss at a rounded pipe entrance. There, the velocity distribution was assumed to be the Poiseuille flow, i.e. parabolic profile because the fluid flows very slowly near the region of a straight pipe. However, it is not clear how well that assumption can hold. Then, the author tried to solve the Navier-Stokes equations directly. But it was too difficult to solve it strictly. Therefore, the author tried to solve the equations when a viscous fluid flows very slowly and mass forces are negligible. Assuming the stream function to be a descending power series, the velocity and the pressure distributions were obtained. From these results it is clear that the stream function assumed in its report approaches the function taken up to the second approximation of the series.

2. Fundamental equation

We may assume that the mass forces of Navier-Stokes equations are negligible for a slow flow of

viscous fluid. For steady, incompressible and axial symmetrical flows, putting the stream function ψ , from the γ component of the equation for the vorticity which used the general orthogonal coordinates (α, β, γ) , we have⁽³⁾

$$D^4\psi=0 \quad \dots\dots\dots(1)$$

where

$$D^2 \equiv \frac{h_3}{h_1 h_2} \left[\frac{\partial}{\partial \alpha} \left(\frac{h_2}{h_3 h_1} \frac{\partial}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{h_1}{h_2 h_3} \frac{\partial}{\partial \beta} \right) \right]$$

Solving Eq. (1) to satisfy the boundary conditions, we may obtain the stream function ψ .

The velocity components u and v in the directions of increasing α and β are

$$u = \frac{1}{h_3 h_2} \frac{\partial \psi}{\partial \beta}, \quad v = -\frac{1}{h_3 h_1} \frac{\partial \psi}{\partial \alpha} \quad \dots\dots\dots(2)$$

The α component of Navier-Stokes equations which omit the mass forces is⁽⁴⁾

$$\frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \alpha} (h_2 h_3 F_{\alpha\alpha}) + \frac{\partial}{\partial \beta} (h_3 h_1 F_{\beta\alpha}) \right] + \frac{F_{\alpha\beta}}{h_1 h_2} \frac{\partial h_1}{\partial \beta} - \frac{F_{\beta\beta}}{h_1 h_2} \frac{\partial h_2}{\partial \alpha} - \frac{F_{\gamma\gamma}}{h_1 h_3} \frac{\partial h_3}{\partial \alpha} = 0 \quad \dots\dots\dots(3)$$

where

$$\left. \begin{aligned} F_{\alpha\alpha} &= -p + \mu e_{\alpha\alpha}, & F_{\beta\beta} &= -p + \mu e_{\beta\beta} \\ F_{\gamma\gamma} &= -p + \mu e_{\gamma\gamma}, & F_{\alpha\beta} &= F_{\beta\alpha} = \mu e_{\alpha\beta} \\ \frac{1}{2} e_{\alpha\alpha} &= \frac{1}{h_1} \frac{\partial u}{\partial \alpha} + \frac{v}{h_1 h_2} \frac{\partial h_1}{\partial \beta} \\ \frac{1}{2} e_{\beta\beta} &= \frac{1}{h_2} \frac{\partial v}{\partial \beta} + \frac{u}{h_1 h_2} \frac{\partial h_2}{\partial \alpha} \\ \frac{1}{2} e_{\gamma\gamma} &= \frac{u}{h_3 h_1} \frac{\partial h_3}{\partial \alpha} + \frac{v}{h_2 h_3} \frac{\partial h_3}{\partial \beta} \\ e_{\alpha\beta} &= \frac{h_2}{h_1} \frac{\partial}{\partial \alpha} \left(\frac{v}{h_2} \right) + \frac{h_1}{h_2} \frac{\partial}{\partial \beta} \left(\frac{u}{h_1} \right) \end{aligned} \right\} \quad \dots\dots\dots(4)$$

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The pressure distribution in the direction of α may be obtained from Eq. (3).

In Fig. 1 the region of rounded pipe entrance is shown. Now, we may use the toroidal coordinates as the general orthogonal coordinates which satisfy this boundary. The relations between the Cartesian coordinates (x, y, z) and the toroidal coordinates (α, β, γ) are as follows.

$$\left. \begin{aligned} x &= \frac{k \sinh \beta \cos \gamma}{\cos \alpha + \cosh \beta} \\ y &= \frac{k \sinh \beta \sin \gamma}{\cos \alpha + \cosh \beta} \\ z &= \frac{k \sin \alpha}{\cos \alpha + \cosh \beta} \end{aligned} \right\} \dots\dots\dots (5)$$

From the above, Eliminating α or β respectively, we have

$$\left. \begin{aligned} (r - k \coth \beta)^2 + z^2 &= (k \operatorname{cosech} \beta)^2 \\ r^2 + (z + k \cot \alpha)^2 &= (k \operatorname{cosec} \alpha)^2 \end{aligned} \right\} \dots\dots\dots (6)$$

where

$$r^2 = x^2 + y^2$$

$\beta = \text{constant}$ shows the circle whose center is $(k \coth \beta, 0)$ and whose radius is $k \operatorname{cosech} \beta$, and also, $\alpha = \text{constant}$ shows the circle whose center is $(0, -k \cot \alpha)$ and whose radius is $k \operatorname{cosec} \alpha$. In Fig. 1, the broken lines show the former and the solid lines, the latter.

Now, let k and β_1 be chosen as follows.

$$k \coth \beta_1 = a + b, \quad k \operatorname{cosech} \beta_1 = b$$

where a is the radius of straight pipe and b is the radius of curvature of rounded pipe entrance. Therefore, the arc surface ABC is expressed with $\beta = \beta_1$, the pipe axis z with $\beta = 0$, and the cross-section at pipe entrance OC with $\alpha = 0$. If a and b are given, k and β_1 are

$$k = a\sqrt{1+2\varepsilon}, \quad \beta_1 = \cosh^{-1}(1+1/\varepsilon) \dots\dots\dots (7)$$

Fig. 1 Region of a pipe entrance and toroidal coordinates

where

$$\varepsilon = b/a$$

3. Stream function

Now we may assume that the stream function can be expanded with a descending power series as follows.

$$\psi = \sum_{n=0}^{\infty} \psi_n = \sum_{n=0}^{\infty} \frac{\Phi_n(s)}{(t+s)^n} \dots\dots\dots (8)$$

where

$$t = \cos \alpha \text{ and } s = \cosh \beta$$

The elements of length at (α, β, γ) in the direction of increasing α, β and γ are respectively $h_1 d\alpha, h_2 d\beta$ and $h_3 d\gamma$. Therefore, h_1, h_2 and h_3 are given as follows.

$$\left. \begin{aligned} h_1 &= \sqrt{\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2} \\ h_2 &= \sqrt{\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 + \left(\frac{\partial z}{\partial \beta}\right)^2} \\ h_3 &= \sqrt{\left(\frac{\partial x}{\partial \gamma}\right)^2 + \left(\frac{\partial y}{\partial \gamma}\right)^2 + \left(\frac{\partial z}{\partial \gamma}\right)^2} \end{aligned} \right\} \dots\dots\dots (9)$$

Substituting Eqs. (5) into Eqs. (9) respectively, we have

$$h_1 = h_2 = k/(t+s), \quad h_3 = k \sinh \beta / (t+s) \dots\dots\dots (10)$$

Substituting only the n -th term ψ_n into Eq. (1) and arranging it, we have

$$D^4 \psi_n = (1/k^4) [A_{4n} \Phi_n'''' + A_{3n} \Phi_n''' + A_{2n} \Phi_n'' + A_{1n} \Phi_n' + A_{0n} \Phi_n] \dots\dots\dots (11)$$

where the primes now denote differentiation with respect to s , and

$$\left. \begin{aligned} A_{4n} &= (s^2 - 1)^2 / (t+s)^{n-4} \\ A_{3n} &= (s^2 - 1) [4s / (t+s)^{n-4} \\ &\quad - 2(2n-3)(s^2 - 1) / (t+s)^{n-3}] \\ A_{2n} &= (s^2 - 1) [-2(n-1)(n-2) / (t+s)^{n-4} \\ &\quad + 2(n-4)(2n-3)s / (t+s)^{n-3} \\ &\quad + (2n-1)(2n-3)(s^2 - 1) / (t+s)^{n-2}] \\ A_{1n} &= (s^2 - 1) [2n(n-1)(2n-3) / (t+s)^{n-3} \\ &\quad - (2n-1)(2n-3)s^2 / (t+s)^{n-2}] \\ A_{0n} &= n[(n-1)(n-2)(n-3) / (t+s)^{n-4} \\ &\quad - 2(n-1)(n-2)(2n-3)s / (t+s)^{n-3} \\ &\quad + (2n-1)(2n-3)\{(n-2)s^2 + 1\} \\ &\quad / (t+s)^{n-2}] \end{aligned} \right\} \dots\dots\dots (12)$$

Then, using all the terms and solving $(t+s)^{-4} D^4 \psi = 0$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} [(t+s)^{-n} \{ (s^2 - 1)^2 \Phi_n'''' + 4s(s^2 - 1) \Phi_n''' \\ - 2(n-1)(n-2)(s^2 - 1) \Phi_n'' \\ + n(n-1)(n-2)(n-3) \Phi_n' \\ - 2(2n-3)(t+s)^{-(n+1)} \{ (s^2 - 1)^2 \Phi_n''' \\ - (n-4)s(s^2 - 1) \Phi_n'' - n(n-1)(s^2 - 1) \Phi_n' \\ + n(n-1)(n-2)s \Phi_n \} \\ + (2n-1)(2n-3)(t+s)^{-(n+2)} \{ (s^2 - 1)^2 \Phi_n'' \\ - (2n-3)s(s^2 - 1) \Phi_n' + n[(n-2)s^2 + 1] \Phi_n \}] = 0 \end{aligned} \dots\dots\dots (13)$$

Collecting terms of the same powers of $(t+s)^{-(n+2)}$, we have

$$\sum_{n=-2}^{\infty} (t+s)^{-(n+2)} [(2n-1)(2n-3)\{(s^2-1)^2\Phi_n'' - (2n-3)s(s^2-1)\Phi_n' + n[(n-2)s^2+1]\Phi_n\} - 2(2n-1)\{(s^2-1)^2\Phi_{n+1}''' - (n-3)s(s^2-1)\Phi_{n+1}'' - n(n+1)(s^2-1)\Phi_{n+1}' + n(n+1)(n-1)s\Phi_{n+1}\} + \{(s^2-1)^2\Phi_{n+2}'''' + 4s(s^2-1)\Phi_{n+2}''' - 2n(n+1)(s^2-1)\Phi_{n+2}'' + n(n+1)(n+2)(n-1)\Phi_{n+2}\}] = 0 \dots (14)$$

In order that Eq. (14) may consist independently of all the values of t , each coefficient of $(t+s)^{-(n+2)}$ must be zero. In case of $n=-2$

$$(s^2-1)\Phi_0'''' + 4s\Phi_0''' - 4\Phi_0'' = 0 \dots (15)$$

with the boundary conditions

$$s=s_1: u=v=0, \Phi_0=0; s=1: u=\text{finite}, v=0$$

Solving this equation, we have

$$\Phi_0 = c_0(s-s_1)^2(s+2s_1) \dots (16)$$

Constant c_0 depends upon the flowrate and is evaluated later.

In case of $n=-1$, substituting Eq. (16), we have

$$(s^2-1)\Phi_1'''' + 4s\Phi_1''' = -36c_0(5s^2-1) \dots (17)$$

Solving Eq. (17) with the same boundary conditions, we have

$$\Phi_1 = -(3/2)c_0(s-s_1)^2(s-1)(s+1+2s_1) \dots (18)$$

Similarly in case of $n=0$

$$\Phi_2 = (3/8)c_0(s-s_1)^2(s-1)^2(s+2+2s_1) \dots (19)$$

By the same method, Φ_n may be obtained. According to Eq. (16), Eq. (18) and Eq. (19), Φ_n will be supposed as follows.

$$\Phi_n = B_n c_0 (s-s_1)^2 (s-1)^n (s+n+2s_1) \dots (20)$$

Substituting this equation into Eq. (14), we have

$$B_n = \left(\frac{2n-5}{n+1} \right) \left(B_{n-1} - \frac{(2n-7)}{4n} B_{n-2} \right) \quad (n \geq 2) \dots (21)$$

where $B_0=1, B_1=-(3/2)$

Fig. 2 Streamlines at $\varepsilon=5$
— from Eq. (22)

Fig. 3 Streamlines at $\varepsilon=1$
— from Eq. (22)
--- from Eq. (24)

Then, all the values of B_n may be given from the above equation. Therefore, the stream function ψ is

$$\psi = \sum_{n=0}^{\infty} B_n c_0 (s-s_1)^2 (s-1)^n (s+n+2s_1) / (t+s)^n \dots (22)$$

This equation may be an exact solution when the fluid flows very slowly, because it satisfies Eq. (1).

The flowrate Q is

$$Q = 2\pi[\psi_{s=s_1} - \psi_{s=1}] = -2\pi c_0 (s_1-1)^2 (1+2s_1)$$

Therefore, constant c_0 is given as follows.

$$c_0 = -Q/2\pi (s_1-1)^2 (1+2s_1) \dots (23)$$

The stream function that was assumed in the preceding paper is

$$\psi = \frac{Q}{2\pi} \left(\frac{1+\cos \alpha}{1+\cos \alpha_1} \right)^2 \left(\frac{1+\cos \alpha_1 \cosh \beta}{\cos \alpha + \cosh \beta} \right)^2 \dots (24)$$

Expressing Eq. (24) in the same form as this paper, we have

$$\psi = \psi_0(s) + \psi_1(s)/(t+s) + \psi_2(s)/(t+s)^2 \dots (25)$$

where $\psi_0(s) = c_0(s-s_1)^2(1+2s_1)$

$$\psi_1(s) = -2c_0(s-s_1)^2(s-1)(1+2s_1)$$

$$\psi_2(s) = c_0(s-s_1)^2(s-1)^2(1+2s_1)$$

These are very similar to the expression of Eq. (22). Figures 2, 3 and 4 show the streamlines at $\varepsilon=5, 1$ and 0 respectively. Solid lines are the streamlines drawing from Eq. (22) and broken lines from Eq. (24). Numbers in those figures are the

Fig. 4 Streamlines and velocity distributions at $\varepsilon=0$
— from Eq. (22)
--- from Eq. (24)
--- from Miyazu's solution on a convergent flow

values of ψ/ψ_0 . Here, ψ_0 is the value of ψ on the pipe axis. In Fig. 2, the streamlines which were taken up to the first approximation of Eq. (22) are shown, because the value of ψ_2 is that of 1/1000 order of ψ_0 . These lines coincide with those of Eq. (24). In Fig. 3, the streamlines are shown up to the second approximation, because ψ_3 is 1/300 order of ψ_0 . In this case, the difference between the

streamlines based on Eq. (22) and Eq. (24) is not so large. In Fig. 4, the streamlines are shown up to the seventh approximation, because ψ_8 is 1/1000 order of ψ_0 . Both differences are large near the pipe inlet. These lines coincide with Sampson's solution⁽⁵⁾ for a slow flow through a thin orifice and, also, with Miyazu's solution⁽⁶⁾ for a convergent flow at a large distance from the entrance.

4. Velocity distribution

As the stream function was obtained in the preceding chapter, the velocity components are given from Eq. (2), that is

$$u = \frac{1}{k^2} \sum_{n=0}^{\infty} \left[\frac{\Phi_n'}{(t+s)^{n-2}} - \frac{n\Phi_n}{(t+s)^{n-1}} \right], \quad v = -\frac{1}{k^2} \sqrt{\frac{1-t^2}{s^2-1}} \sum_{n=0}^{\infty} \frac{n\Phi_n}{(t+s)^{n-1}} \quad (26)$$

Taking $\bar{u}$ as the mean velocity in a pipe and using Eq. (23), we have

$$\bar{u} = -2c_0(s_1-1)^2(1+2s_1)/a^2 \quad (27)$$

Arranging Eq. (26) with the aid of Eq. (7) and Eq. (27), the velocity components are expressed as follows.

$$\left. \begin{aligned} \frac{u}{\bar{u}} &= -\frac{1}{2c_0(s_1^2-1)(1+2s_1)} \left[\Phi_0'(t+s)^2 + \Phi_1'(t+s) + \sum_{m=0}^{\infty} \frac{\Phi_{m+2}' - (m+1)\Phi_{m+1}}{(t+s)^m} \right] \\ \frac{v}{\bar{u}} &= \frac{1}{2c_0(s_1^2-1)(1+2s_1)} \sqrt{\frac{1-t^2}{s^2-1}} \sum_{n=1}^{\infty} \frac{n\Phi_n}{(t+s)^{n-1}} \end{aligned} \right\} \quad (28)$$

In Figs. 5 and 6, the velocity distributions are shown, where they are drawn along the surface $\alpha = \text{constant}$ and $\bar{u}$ is set at unity. $v/\bar{u}$ is negligible because it is of the order 0.03 comparing with $u/\bar{u}$. Though the magnitude and the direction of the velocity at the positions $\alpha = \pi/3, \pi/6$ and 0 are expressed with a length and an arrow, they are not ones on the same streamlines. Figure 5 shows that the velocity distributions are almost parabolic at a straight pipe entrance. But, in Fig. 6 they are not. The difference between them, however, is not

so large. On the contrary, in Fig. 4, when the pipe entrance has a square-edge the velocity distribution becomes uniform rather than parabolic.

It is well known that the velocity distribution becomes uniform in the flow of a large Reynolds number when the radius of roundness is large. But it is contrary to the flow of a small Reynolds number.

Of course, the maximum velocity at the pipe entrance occurs on the pipe axis. Making α and β zero in Eqs. (28), the maximum velocity is

$$u_{\max}/\bar{u} = 3(s_1+1)(2s_1+1) \quad (29)$$

From the above, at $s_1 \rightarrow 1$, $u_{\max}/\bar{u} \rightarrow 2$, the velocity distribution becomes parabolic, and also at $s_1 \rightarrow \infty$, $u_{\max}/\bar{u} \rightarrow 1.5$, it becomes increasingly uniform.

Fig. 5 Velocity distributions at $\varepsilon=5$

Fig. 6 Velocity distributions at $\varepsilon=1$

$$I[m] = \int \frac{d\alpha}{(t+s)^m}$$

Figure 10 shows an example of pressure distribution on a rounded surface calculated from Eq. (38). When θ is larger than 50° , p is almost constant and

Fig. 9 Pressure gradients on rounded pipe entrance for various values of θ

Fig. 10 Pressure distribution on rounded pipe entrance at $\epsilon = 1$

pressure drop is found to be between $\theta = 0^\circ \sim 30^\circ$. Both for inflow and for outflow, pressure continues to decrease due to viscosity and separation can not occur.

7. Conclusions

For a viscous fluid flowing very slowly near a rounded pipe entrance and exit, when the mass forces of Navier-Stokes equations are negligible, the author solved the equations of motion and obtained the results as follows.

(1) The stream function is expressed in the form of a descending power series. When the radius of rounded entrance is larger than that of pipe, the stream lines of approximation reported in the preceding paper resemble each other in the form taken up to the second approximation of this series. When the radius of it is small, i.e. the entrance is a square-edge, they agree with Sampson's solution for a slow flow through a thin orifice.

(2) The velocity distribution near the region of a pipe inlet becomes parabolic when the radius of rounded entrance is large, and it approaches uniform velocity when the radius of it is small. This is contrary to the flows of a high Reynolds number.

(3) The coefficient of skin friction is large near the pipe entrance and small at a large distance from it. The smaller the radius of rounded entrance becomes, the larger the coefficient of skin friction becomes.

(4) The pressure gradient is large near the pipe entrance and small far from it. Both for inflow and for outflow, pressure drops due to viscosity and separation can not occur in such a slow flow.

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