Buckling Strength of Metal Lining of
a Cylindrical Pressure Vessel*

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In this paper, the buckling strength of metal lining of a cylindrical pressure vessel is discussed from the viewpoint of the finite deflection theory. The lining generally forms a thin cylindrical shell reinforced circumferentially by ring stiffeners. Such a cylindrical shell is subjected to external pressure and restrained by the cylindrical rigid wall so that no radial outward displacements are permitted. Collapsing tests of such metal lining were conducted with cylindrical shells on an apparatus specially prepared for this purpose. The critical points of buckling are determined on the basis of the classical buckling theory, but they go up far beyond the experimental results. The results of experiments are in good accordance with the present finite deflection theory based on Tsien's concept.

1. Introduction

Anti-corrosive materials, such as titanium, nickel and stainless steel, are commonly used in chemical equipments as metal lining to protect structural members. In this paper the buckling strength of thin cylindrical shells used as metal lining for cylindrical vessels will be investigated. Such shells are usually reinforced circumferentially by rigid ring stiffeners.

Structures of vessels subjected to internal pressure are generally designed by disregarding metal lining. In case vacuum occurs in vessels, however, such metal lining may collapse by buckling. In general, the tightness of metal lining is inspected by inserting pressurized water of some 0.5 kg/cm² between lining and outer the vessels, and collapsing may occur in metal lining. Therefore, the scantling of metal lining and the spacing of its ring stiffeners should be determined so that it does not collapse under possible external pressure (at most 1 kg/cm²).

The buckling strength of a cylindrical shell subjected to external pressure has been investigated by many authors**. In the case of metal lining, however, it is restrained by the outer vessels which are rigid enough, and its buckling strength may increase beyond the buckling pressure determined by the ordinary theories. In this paper, a finite deflection theory of buckling of metal lining is developed on the basis of energy concepts. Moreover, Bucciarelli and Pian(1) recently investigated a similar problem, which corresponds to the buckling of metal lining due to thermal expansion.

2. Experiment

In order to observe buckling behaviors of metal lining due to external pressure, collapsing tests were conducted with cylindrical shells on an apparatus as shown in Fig. 1. The apparatus consists of a cylindrical thick shell ① and a water tank ②, which is connected to a hydraulic pump at ③, and to which a pressure gage is attached at ④. Experiments were conducted with duralumin or aluminum.

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![Fig. 1 Apparatus for experiments](image-url)
rectangular plates whose mechanical properties are shown in Table 1. Test plates bent in a cylindrical form were put in the apparatus. Four edges of test plates were fixed to the inner wall of the apparatus by rings and equipments prepared for this purpose. Water tightness at the four edges were kept by the help of a chemical bond EXVON. Scantlings of test plates are shown in Table 2.

A test plate was subjected to external pressure by pressurized water, and a deflection became visible first near the straight edges of the plate. At a certain value of water pressure, a buckling occurred in the central part of the bent plate, and several buckling patterns appeared one after another. For short plates of small \( L/R \) value, buckling occurred catastrophically, and the pressure went down suddenly. The maximum value of pressure was adopted as the critical buckling pressure \( p_c \) for experiments. The results of experiments are shown in Table 2.

3. Theory

3.1 Total potential energy

It should be noticed that a cylindrical shell of metal lining is restrained by the main vessel not to deflect externally. The rigidity of metal lining is generally very small in comparison with that of the main vessel. It can be assumed, therefore, that the main vessel is rigid enough and radial outward displacements of metal lining are prohibited by it. It is also assumed that to investigate the buckling phenomena, deflections in unbuckled states can be ignored in comparison with buckling deflection.

Consider a thin cylindrical shell of length \( L \), radius \( R \), and thickness \( t \). The length \( L \) is the free span between two successive ring frames which support the shell rigidly. The shell is restrained by a cylindrical rigid wall of the main vessel not to deflect outwards and is subjected to external pressure. We can introduce such orthogonal curvilinear coordinates \( x, y, z \) that the \( x \)- and \( y \)-coordinate curves are in the axial and circumferential directions, respectively, and the \( z \)-coordinate curves are normal to the midsurf ace \( S_m \) which is expressed by \( z=0 \). It is assumed that \( t/R \) is small enough in comparison with unity. The circular ends of the cylindrical shell are expressed by \( x= \pm L/2 \). The displacements \( u, v, w \) of the midsurface are given as functions of \( x \) and \( y \). The stress resultants \( N_x, N_y \), and \( N_{xy} \) per a unit breadth should satisfy the following condition of equilibrium in the \( x \)- and \( y \)-directions:

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_y}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \text{ in } S_m
\]

Then stress resultants in equilibrium can be expressed in terms of an appropriate stress function \( F \) as

\[
N_x = \frac{\partial^2 F}{\partial y^2}, \quad N_y = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = \frac{\partial^2 F}{\partial x \partial y}
\]

The strains in the midsurface can be expressed in two alternative forms as

![Fig. 2](image-url)
\[
\epsilon_{xx} = \frac{1}{E} \left( \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right), \quad \epsilon_{yy} = \frac{1}{E} \left( \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) , \quad \epsilon_{xy} = \frac{1}{E} \left( \frac{\partial^2 w}{\partial x \partial y} \right)
\]

where \( E \) is Young’s modulus and \( \nu \) is Poisson’s ratio.

Elimination of \( u \) and \( v \) from Eqs. (3) yields the condition of compatibility in the midsurface;

\[
\frac{1}{E} \left( \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) = \frac{1}{E} \left( \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right)
\]

The deflection \( w \) should satisfy the boundary conditions at both ends (\( x = \pm L/2 \)) and should be in conformity to the restraint of the cylindrical rigid wall. Corresponding to such a deflection \( w \), a stress function \( F \) can be determined by integrating Eq. (4) so as to satisfy appropriate boundary conditions. The total potential energy due to the occurrence of \( w \) is expressed in terms of \( w \) and \( F \). The deflection \( w \) in an equilibrium state can be determined by the principle of minimum potential energy.

Omitting the terms of higher order in the expression of displacement, the strain energy \( U(\epsilon) \) per unit area of midsurface can be expressed as follows;

\[
U(\epsilon) = U'(F) + U''(w)
\]

where

\[
U'(F) = \frac{1}{2Et} \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right)^2 + 2(1+\nu) \left( \frac{\partial^2 F}{\partial x \partial y} \right)^2
\]

\[
U''(w) = \frac{D}{2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 + 2(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{w^2}{R^2} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)
\]

where \( D = \frac{Et}{12(1-\nu^2)} \).

The cylindrical shell under consideration is subjected to uniform external pressure \( p \) on the cylindrical outer surface, but is not subjected to axial thrusts. Before buckling occurs, the stress resultants are given approximately by

\[
N_{x} = 0, \quad N_{y} = -pR, \quad N_{z} = 0
\]

and the stress couples may be considered to be zero. Such an equilibrium state is taken as the reference state to determine the increase of the total potential energy, \( \Pi \):

\[
\Pi = \Pi_{s} + \Pi_{p}
\]

where \( \Pi_{s} \) is the increase of strain energy due to the additional displacements \( u, v, \) and \( w \), and \( \Pi_{p} \) is the work done by the external pressure \( p \). The \( \Pi_{s} \) is obtained in the form

\[
\Pi_{s} = \int \int \left[ -pR \left( \frac{\partial u}{\partial y} + \frac{w}{R} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right) \right) \right] dy dx
\]

\[
+ \frac{2(1-\nu^2)}{E} \int \int \left[ \left( \epsilon_{xx} + \epsilon_{yy} \right)^2 - 2(1-\nu) \left( \epsilon_{xx} \epsilon_{yy} + \nu \epsilon_{yx} \right) \right] dy dx
\]

\[
+ D \int \int \left[ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + \frac{w^2}{R^2} \left( \frac{\partial^2 w}{\partial y^2} - \nu \frac{\partial^2 w}{\partial x^2} \right) \right] dy dx
\]

The \( \Pi_{p} \) is obtained in the form

\[
\Pi_{p} = -p \int \int \left[ wR \frac{\partial w}{\partial y} + \frac{w}{R} \right] dy dx
\]

To evaluate \( \Pi_{p} \), the term \( u(\partial w/\partial z) \) can be ignored in comparison with \( w^2/(2R^2) \). Since \( \epsilon_{yy} \) given by (3) remains small even if \( w/R \) and \( \partial w/\partial y \) take large values in buckled states, the approximate relation

\[
\frac{\partial w}{\partial y} \approx \frac{w}{R}
\]

can be used for the evaluation of \( \Pi_{p} \). It then follows, from Eqs. (11) and (12), that

\[
\Pi = \Pi_{s} + \Pi_{p} = \int \int \left[ U'(F) + U''(w) \right] dy dx - p \int \int \left[ \frac{R}{2} \left( \frac{\partial w}{\partial y} \right)^2 - \frac{w^2}{2R^2} \right] dy dx
\]

Equilibrium states after buckling can be determined for each step of pressure \( p \) by the condition that the variation of \( \Pi \) vanishes. The \( \Pi \) in such an equilibrium state for each step of \( p \) can be evaluated by using Eq. (14). If the \( \Pi \) vanishes for a certain value of \( p \), such a pressure \( p \) is the buckling pressure in Tsien’s concept.
3.2 A cylindrical metal lining fixed at both ends

For a cylindrical shell fixed at both ends, the buckling pattern will be assumed in the form

$$w = -\frac{\delta}{4} \left(1 + \cos \frac{\pi y}{R\beta} \right) \left(1 + \cos \frac{2\pi x}{L}\right)$$

$$= -f(1 + \cos ax)(1 + \cos by),$$

$$|x| < \frac{L}{2}, \quad |y| < \frac{R\beta}{2} \quad \text{.........(15)}$$

where \(f = \delta / 4, \quad a = 2\pi / L, \quad b = \pi / (R\beta),\) and where \(2\beta\) is the angular range of a buckling pattern in circumferential direction, and \(\delta\) is the maximum deflection. It can be seen that the buckling pattern (15) is symmetric with respect to the axis \(x = 0\). It is obvious that Eq. (15) satisfies the conditions

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{for any } \alpha \quad \text{at } x = \pm \frac{L}{2}$$

$$w = -\delta, \quad \frac{\partial w}{\partial y} = 0 \quad \text{for } \alpha = 0 \quad \text{at } x = 0$$

$$w = \frac{\partial w}{\partial y} = 0 \quad \text{for } \alpha = \pm \beta \quad \text{at } x = 0 \quad \text{.........(16)}$$

where \(\alpha = y / R\) is the angular coordinates in the circumferential direction. Substituting Eq. (15) into Eq. (4), the stress function \(F\) can be determined approximately by

$$F = C_x x^2 + C_y y^2 + \frac{Et\alpha^2}{R} \left[\frac{1}{\alpha^2} + \frac{1}{(\alpha^2 + b^2)^2} \cos by\right] \cos ax - \frac{Et\alpha^2 b^2}{2} \left[\frac{1}{b^2} \cos by + \frac{1}{16b^4} \cos 2by\right]$$

$$+ \left[\frac{1}{16b^4} + \frac{1}{(\alpha^2 + b^2)^2} \cos by\right] \cos 2ax + \left[\frac{1}{\alpha} + \frac{2}{(\alpha^2 + b^2)^2} \cos by + \frac{1}{(\alpha^2 + b^2)^2} \cos 2by\right] \cos ax \quad \text{.........(17)}$$

First two terms in the right-hand side of this expression are solutions of the homogeneous equation \((\partial^2 / \partial x^2 + \partial^2 / \partial y^2)F = 0\) and correspond to a uniform stress distribution. In view of Eq. (3), \(u\) and \(v\) are related to \(w\) and \(F\) by

$$\frac{\partial u}{\partial x} = \frac{1}{E\alpha} \left(\frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2}\right) - \frac{1}{2} \frac{(\partial w)^2}{(\partial x)^2},$$

$$\frac{\partial v}{\partial y} = \frac{1}{E\beta} \left(\frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{\partial y^2}\right) - \frac{1}{2} \frac{(\partial w)^2}{(\partial y)^2} - w \quad \text{.........(18)}$$

Substituting Eqs. (15) and (17) into Eqs. (18) and integrating the resulting equations, \(u\) and \(v\) are obtained as

$$u = \frac{2Et}{R} \left(C_v - \nu C_x\right) x + \frac{f a^2 b^2}{(\alpha^2 + b^2)^2} \cos by \left(\frac{1}{\alpha^2} + \frac{1}{(\alpha^2 + b^2)^2} \cos by\right) \sin ax + \frac{f a^2 b^2}{2(\alpha^2 + b^2)^2} \left(\frac{1}{b^2} \cos by + \frac{1}{16b^4} \cos 2by\right) \cos 2ax$$

$$+ \frac{f a^2 b^2}{2} \left[\frac{1}{b^2} \cos by + \frac{1}{(\alpha^2 + b^2)^2} \cos 2by\right] x \left(1 + \cos ax\right) \left(1 - \frac{1}{2a^2} \sin 2ax\right) \quad \text{.........(19)}$$

$$v = \frac{2Et}{R} \left(C_v - \nu C_y\right) y + \frac{f a^2 b^2}{(\alpha^2 + b^2)^2} \sin by \left(\frac{1}{\alpha^2} + \frac{1}{(\alpha^2 + b^2)^2} \sin by\right) \cos ax + \frac{f a^2 b^2}{2(\alpha^2 + b^2)^2} \left(\frac{1}{b^2} \sin by + \frac{1}{16b^4} \sin 2by\right) \cos 2ax$$

$$+ \frac{f a^2 b^2}{2} \left[\frac{1}{b^2} \sin by + \frac{1}{(\alpha^2 + b^2)^2} \sin 2by\right] y \left(1 + \cos ax\right) \left(1 + \frac{1}{2a^2} \sin 2ax\right) \quad \text{.........(20)}$$

The unknown constants \(C_x\) and \(C_y\) in Eq. (17) are determined by the boundary conditions for \(u\) and \(v\) and the conditions for the stress resultants. The condition that \(v\) vanishes at the straight lines \(y = \pm R\beta\), or \(v = 0\) at \(y = \pm R\beta\) \quad \text{.........(21)}

and the condition that the primary part of \(N_z\) vanishes can be used for this purpose. It then follows that

$$C_x = 0, \quad C_y = -\frac{Et}{R} \left(\frac{3b^2 f^2}{8} - f\right) \quad \text{.........(22)}$$
Substituting Eqs. (15), (17), and (22) into Eq. (14), the expression for $\Pi$ is given by

$$\frac{\Pi}{EtR^2} = A_1(\beta)\eta^4 + B_1(\beta)\eta^3 + \left[C_1(\beta) - \frac{pR}{Et}D_1(\beta)\right]\eta^2$$

where

$$\frac{f}{R} = \eta, \quad \frac{l}{R} = \tau, \quad \frac{L}{2R} = \lambda$$

$$A_1(\beta) = \pi^\frac{4}{(\beta\lambda)^3} \left[\frac{17}{64} + \frac{1}{2(\theta + \phi)^2} + \frac{1}{8(\theta + 4\phi)^2} + \frac{1}{8(4\theta + \phi)^2}\right]$$

$$B_1(\beta) = -\pi\frac{4}{\beta} \left[\frac{5}{2\lambda^3} + \frac{1}{\lambda^3(\theta + \phi)^2}\right]$$

$$C_1(\beta) = \pi^\frac{3}{2\lambda^3} \left[\frac{3}{\beta} - \beta^3\right]$$

The equilibrium state is determined by the principle of minimum potential energy:

$$\frac{\partial \Pi}{\partial \eta} = 0, \quad \frac{\partial \Pi}{\partial \beta} = 0$$

or

$$A_1(\beta)\eta^4 + B_1(\beta)\eta^3 + \left[C_1(\beta) - \frac{pR}{Et}D_1(\beta)\right]\eta^2 = 0$$

$$A_2(\beta)\eta^4 + B_2(\beta)\eta^3 + \left[C_2(\beta) - \frac{pR}{Et}D_2(\beta)\right]\eta^2 = 0$$

where

$$A_2(\beta) = 4A_1(\beta), \quad B_2(\beta) = 3B_1(\beta), \quad C_2(\beta) = 2C_1(\beta), \quad D_2(\beta) = 2D_1(\beta)$$

$$A_3(\beta) = \pi\frac{4}{\beta^4} \left[\frac{17}{64} - \frac{105}{104}\left(\frac{1}{2(\theta + \phi)^2} + \frac{1}{8(\theta + 4\phi)^2} + \frac{1}{8(4\theta + \phi)^2}\right)\right]$$

$$B_3(\beta) = -\pi\frac{4}{\beta^4} \left[\frac{5}{2\lambda^3} + \frac{1}{\lambda^3(\theta + \phi)^2}\right]$$

$$C_3(\beta) = \left[\frac{3}{\beta} - \beta^3\right]$$

$$D_3(\beta) = -\frac{3}{2}\pi\frac{4}{\beta^4} \left[\frac{1}{\lambda^3(\theta + 3\phi)^2}\right]$$

In the case of the classical theory of buckling, non-linear terms in Eq. (4) are disregarded. Correspondingly, terms of higher order in Eqs. (26) can be ignored, and the critical pressure $p_c$ is determined by

$$C_1(\beta) - \frac{pR}{Et}D_1(\beta) = 0$$

$$C_2(\beta) - \frac{pR}{Et}D_2(\beta) = 0$$

By determining $\eta$ and $p$ which satisfy Eq. (26) simultaneously for each $\beta$, the pressure-deflection curves for deflected equilibrium states are obtained as shown in Fig. 6. In the same figure, the $\Pi$ in equilibrium states is also shown, and the buckling pressure is determined by the condition $\Pi = 0$.

3.3 A cylindrical metal lining simply supported at both ends

For a cylindrical shell supported simply at both ends, the buckling pattern will be assumed in the form

![Fig. 4 Assumed buckling pattern for supported ends](image-url)
\[
\omega = -\frac{\partial}{\partial z} \left( \frac{1}{2} (1 + \cos \pi y) \cos \frac{\pi x}{L} \right)
\]

\[
= -f(1 + \cos by) \cos ax, \quad |x| < \frac{L}{2}, \quad |y| < R \beta
\]  

where \( f = \beta / 2, \quad a = \pi / L, \quad b = \pi / (R \beta) \), and where \( 2 \beta \) and \( \beta \) are similar to those defined under 3.2. It is obvious that Eq. (29) satisfies the conditions

\[
\begin{align*}
\omega &= \frac{\partial^4 \omega}{\partial z^4} = 0 \quad \text{for any } \alpha \text{ at } x = \pm \frac{L}{2} \\
\omega &= -\delta, \quad \frac{\partial \omega}{\partial y} = 0 \quad \text{for } \alpha = 0 \text{ at } x = 0 \\
\omega &= \frac{\partial \omega}{\partial y} = 0 \quad \text{for } \alpha = \pm \beta \text{ at } x = 0
\end{align*}
\]

Substituting Eq. (29) into Eq. (4), the stress function \( F \) can be determined approximately as

\[
\begin{align*}
F &= C_s \eta^3 + C_v \eta^3 + \frac{E f a^2}{R} \left[ \frac{1}{a^2 + (a^2 + b^2)^2} \cos by \right] \cos ax \frac{E f a^2 b^2}{2 R} \left[ \frac{1}{b^2 + (a^2 + b^2)^2} \cos by \right] \\
&\quad \left\{ \frac{1}{16a^2 + (4a^2 + b^2)^2} \cos by \right\} 2ax
\end{align*}
\]

Substituting \( \omega \) and \( F \) from Eqs. (29) and (31) into Eqs. (18), and integrating the resulting equations, \( u \) and \( v \) are obtained as

\[
\begin{align*}
u &= \frac{2}{E f} \left( C_v - C_s \right) x + \frac{fa}{R} \left( \frac{\nu + \frac{\nu a^2 - b^2}{4a^2 + b^2} \cos by}{\cos ax} \right) + \frac{f^2 a b^2}{4 R} \left[ \frac{1}{a^2 + b^2} \cos by \right] \sin 2ax
\end{align*}
\]

Substituting the constants \( C_v \) and \( C_s \) in the expression Eq. (31) are determined by the boundary conditions for \( u \) and \( v \) and the conditions for the stress resultants as before. It then follows that

\[
C_v = 0, \quad C_s = \frac{E f a^2}{16b^2}
\]

Introducing Eqs. (29), (31), and (34) into Eq. (14), the expression for \( \Pi \) is given by a similar form

\[
\frac{\Pi}{E I R^2} = A_v(\beta) \eta^3 + B_v(\beta) \eta^3 + \left\{ C_v(\beta) - \frac{E f}{E I} D_v(\beta) \right\} \eta^3
\]

where

\[
\begin{align*}
\frac{f}{R} &= \bar{\eta}, \quad \frac{t}{a} = \tau, \quad \frac{L}{R} = \lambda
\end{align*}
\]

\[
\begin{align*}
A_v(\beta) &= \frac{\pi^2}{4} \left( \frac{1}{\lambda} + \frac{1}{\beta^2} \right) \\
B_v(\beta) &= \frac{\pi^4}{4} \left( \frac{1}{\beta^2} + \frac{1}{\lambda^2} \right) \\
C_v(\beta) &= \frac{\pi^2}{4} \left( \frac{1}{\lambda} + \frac{1}{\beta^2} \right)
\end{align*}
\]

The equilibrium condition is given by a similar form:

\[
A_1(\beta) \eta + B_1(\beta) \eta + \left\{ C_1(\beta) - \frac{E f}{E I} D_1(\beta) \right\} \eta = 0, \quad A_2(\beta) \eta + B_2(\beta) \eta^3 + \left\{ C_2(\beta) - \frac{E f}{E I} D_2(\beta) \right\} \eta^3 = 0
\]

where

\[
\begin{align*}
\frac{f}{R} &= \bar{\eta}, \quad \frac{t}{a} = \tau, \quad \frac{L}{R} = \lambda
\end{align*}
\]

\[
\begin{align*}
A_v(\beta) &= \frac{\pi^2}{4} \left( \frac{1}{\lambda} + \frac{1}{\beta^2} \right) \\
B_v(\beta) &= \frac{\pi^4}{4} \left( \frac{1}{\beta^2} + \frac{1}{\lambda^2} \right) \\
C_v(\beta) &= \frac{\pi^2}{4} \left( \frac{1}{\lambda} + \frac{1}{\beta^2} \right)
\end{align*}
\]
\[ A_1(\beta) = 4A_0(\beta), \quad B_1(\beta) = 3B_0(\beta), \quad C_1(\beta) = 2C_0(\beta), \quad D_1(\beta) = 2D_0(\beta) \]
\[ A_2(\beta) = \frac{\pi^2\theta^2}{\lambda^2} \left[ \frac{17\beta^2 - 9\beta^4}{64} + \frac{3\theta + \phi}{8(\theta + 4\phi)^3} \right] \]
\[ B_2(\beta) = -\pi \left[ \frac{2}{3} \beta \lambda + \frac{\theta(3\beta - 2\phi)}{(\theta + 4\phi)^3} \left( \frac{3\beta + \phi}{(\theta + 4\phi)^3} \right) \right] \]
\[ C_2(\beta) = \frac{\lambda}{24(1 + \nu)} \left[ \frac{5\theta + \phi}{(\theta + 4\phi)^3} \right] \]
\[ D_2(\beta) = \pi \frac{\lambda^2}{2} \left( \frac{\theta + 3\phi}{(\theta + 4\phi)^3} \right) \]

Table 3: Comparison of the values of \( \left( \frac{\rho R}{E} \right) \) in experiments and theories (for \( t/R = 0.004, \nu = 0.3 \))

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<th>( L/R )</th>
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Fig. 5

Fig. 6 Equilibrium states and the increase of the total potential energy (shells fixed at both ends)

Fig. 7 Equilibrium states and the increase of the total potential energy (shells supported at both ends)
In the case of the classical theory of buckling, non-linear terms in Eq. (4) are disregarded as before, and the classical critical pressure $p_c$ can then be determined by

$$C_1(\beta) - \frac{2R}{EI} D_1(\beta) = 0,$$

$$C_2(\beta) - \frac{2R}{EI} D_2(\beta) = 0$$

By determining $\eta$ and $p$ by Eq. (37) for each $\beta$ numerically, the pressure-deflection curves for buckled states are obtained as shown in Fig. (7). In the same figure, the corresponding $\Pi$ is also shown. The buckling pressure in Tsien's concept can be determined by the condition $\Pi = 0$.

4. Consideration on the theory

In reality, uniform radial deflections $\delta_0$ due to external pressure occur in the first stage of loading, and are the same as those in the case of cylindrical shells free in the radial direction. Buckling may occur first when the pressure reaches the critical value for unrestrained cylindrical shell. Since the distance between the rigid vessel and the outer surface of the deflected cylindrical shell is very short, the deflection of the cylindrical shell comes into contact with the outer vessel as soon as it appears, and its further growth is prevented. Such deformations affect slightly the actual buckling in Tsien's concept. It was observed by experiments that the deflection $\delta$ and the angular range $2\beta$ of a buckling pattern increase gradually under buckling pressure until they arrive at certain values, which may correspond to the equilibrium state determined by the present theory.

As can be seen in Fig. 8, the critical pressure determined by the classical theory of buckling goes up far beyond experiments. Experimental buckling pressure is two times as large as the theoretical buckling pressure for cylindrical shells unrestrained radially.

5. Conclusion

In this paper, the buckling phenomena of metal lining of a cylindrical pressure vessel were observed experimentally and discussed theoretically from the viewpoint of the finite deflection theory based on Tsien's concept. Such metal lining is of cylindrical shells restrained by a cylindrical rigid vessel, and its radial outward displacements are not permitted. The cylindrical shells are subjected to external pressure, and may buckle. A theory of buckling was developed on the basis of Tsien's concept for the boundary condition that both circular ends are fixed or freely supported. The results of experiments are in good accordance with the present finite deflection theory. The experimental buckling pressure goes up beyond the theoretical buckling pressure of unrestrained cylindrical shells because of the influence of restraint.

On the basis of the present theory, the space of ring stiffeners or the thickness of metal lining can be determined rationally for design purposes.

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References

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Discussion

S. Nakagiri (1) I should like to know the procedure for determination of the solutions of Eqs. (26) and (37).

I. Hayashi (2) The authors retain $v/R$ in the term of rotation appearing in Eq. (10) though $v/R$ is omitted in Eq. (3). Is there any special reason?

M. Sunakawa (3) I should like to know the author's opinion about Tsien's concept.

Authors' closure

(1) The authors solved Eqs. (26) and (37) according to the flowchart as shown in Appendix.-Fig. 1.

(2) Since $v/R$ is very small in comparison with $\partial u/\partial y$, it can be omitted. By retaining $v/R$ in Eq. (9), the subsequent manipulation can be done naturally.

(3) Tsien's concept is applicable for practical problems. The authors believe that Tsien's concept plays an essential role, only when buckling pattern is assumed in a simple form, and the buckling pressure or load determined by Tsien's concept must be in good accordance with experiments.