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Study of Optimal Steering*

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In the optimization problem of controlled system with steering mechanism, there exist some features proper to the problem. The typical points are listed below.

(1) The angular displacement of the rudder is not a control but one component of state vector.

(2) The control is the torque which is applied to the rudder mechanism.

(3) The angular displacement of the rudder (one component of the state vector) is limited physically, i.e. the problem belongs to the class of optimization problem with stop mechanism.

(4) Final value of the angular displacement of body (another component of state vector) is not specified to some fixed value but must be transferred to one of infinitely multiple values separated by 2π .

In this paper, minimum time control of course of ship is discussed as a typical concrete problem of optimal steering, and a synthesizing method of feedback system for optimal steering problem is presented.

1. Introduction

Although the ship course is controlled by the angular position of rudder, the rudder cannot be moved abruptly, so some dynamics must be taken account of for the motion of rudder. Then it turns out that the angular position of rudder is not a control and the rudder must be regarded as a part of controlled system, that is, the angular position of rudder must be one component of state vector. Question may be put then as what is the control in this problem, but it may be said that the torque u which the steering gear controlling the movement of rudder generates is exactly the control. The control should be considered to be subject to constraint such that its absolute value must be less than 1. This means that the absolute value of rudder speed is limited to within 1.

In this problem, there is another important factor that the rudder displacement is limited at the left and right ends. Assume that this limitation is ± 1 . This limitation is an inherent nature of the controlled system which consists of the ship body and the rudder. The system with such a nature is called a system with stop mechanism. Since the solution method for the system with stop mechanism is very similar to that for a bounded state variable

problem, the theory applied to the solution for the bounded state variable problem⁽¹⁾⁽²⁾ is applicable to this problem with slight modification.

2. Problem statement

Denote the ship course as x_1 , its angular velocity as x_2 and angular position of rudder as x_3 . $\{x_1, x_2, x_3\}$ forms the state vector. Control u is the driving torque of steering gear. Equations of motion of controlled system within admissible region i.e. within the region of $|x_3| \leq 1$ are given by

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \right\} \dots\dots\dots (1)$$

where $|u| \leq 1$. Minimum time control problem to the origin is considered, namely terminal time t_f is to be minimized with $x_i(t_f) = 0$, $i = 1, 2, 3$. In the angular control of a rotating body such as ship, the final value of x_1 is generally not confined to 0 but any of $2n\pi$ may be allowable. This is one of the features of ship course control problem. Leaving the subject till later on, the case of $n=0$ is firstly examined.

There are two boundaries $x_3 = 1$ and -1 in the state space in this problem. If the control is applied so as to move the state point towards the prohibited region when the state point lies on the boundary, the state point is restricted on the boundary by the appropriate reaction force caused by the stop mechanism. Such condition is expressed by the

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formula

$$\langle f[x, u], n(x) \rangle > 0, \dots\dots\dots(2)$$

where $n(x)$ is an outward normal to the boundary. When such control as satisfies formula (2) is applied, the equation of motion of controlled system changes from Eq. (1) to

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + v \end{aligned} \right\} \dots\dots\dots(3)$$

owing to the reaction caused by stop mechanism, where v is the reaction force with the passive nature of $v = -u$. Due to this nature, $\dot{x}_3$ becomes 0 and the motion of state point is restricted on $x_3 = 1$.

When $u = 0$ is applied, equation of motion of state point is still expressed by Eq. (1) and $\dot{x}_3 = 0$ holds, therefore the state point moves again along the boundary $x_3 = 1$. Combining the above two cases, $u \geq 0$ after the state point arrives at the boundary will surely move the state point along the boundary $x_3 = 1$. Similarly $u \leq 0$ after the state point arrives at the boundary will surely move the state point along the boundary $x_3 = -1$.

3. Tracking of optimal backward time trajectory in admissible region

State vector differential equation in admissible region is given by Eq. (1). Costate vector is defined by the following differential equations and boundary conditions,

$$\left. \begin{aligned} \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1 \\ \dot{p}_3 &= -p_2 \end{aligned} \right\} \dots\dots\dots(4)$$

$$p_1(t_f) = -\mu_1, p_2(t_f) = -\mu_2, p_3(t_f) = -\mu_3, \dots\dots(5)$$

Hamiltonian H is given by

$$H = p_1 x_2 + p_2 x_3 + p_3 u - 1. \dots\dots\dots(6)$$

It is clear that no singular solution exists, because if $p_3 = 0$ persists, $p_2 = 0$ and $p_1 = 0$ persist and $H = 0$ does not hold then. Optimal control condition is obtained as

$$u = \text{sgn } p_3. \dots\dots\dots(7)$$

Applying the condition $H(t_f) = 0$ results in

$$-\mu_3 = \pm 1 \dots\dots\dots(8)$$

Let us consider only the case of $-\mu_3 = 1$ in detail because a symmetric result is obtained for the case of $-\mu_3 = -1$.

Optimal backward time trajectories emanating from the origin are determined by the following equations

$$\left. \begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= -x_3 \\ \dot{x}_3 &= -u \\ u &= \text{sgn} \{ 1 + (-\mu_2)t + 1/2 \cdot (-\mu_1)t^2 \} \\ x_1(0) &= 0, x_2(0) = 0, x_3(0) = 0 \end{aligned} \right\} \dots\dots\dots(9)$$

where $-\mu_1$ and $-\mu_2$ are variables ranging from $-\infty$ to ∞ . $p_3(t)$ is given by

$$p_3(t) = 1 + (-\mu_2)t + 1/2 \cdot (-\mu_1)t^2, \dots\dots\dots(10)$$

which determines optimal backward time trajectories. Through the investigation of the evolution of sign of p_3 and optimal backward time trajectories, the pattern of switching of u obtained according to the various values of $\{-\mu_1, -\mu_2\}$ during the period of the state point being still within the limit of $x_3 = 1.0$ or -1.0 plane is made clear. This result is shown in Fig. 1 where switching times are 0, 1 and 2 for the region 0, I and II respectively. For each region optimal backward time trajectories and optimal control are given by:

Region 0

$$\left. \begin{aligned} 0 &\leq t \leq 1 \\ x_1 &= -1/6 \cdot t^3, x_2 = 1/2 \cdot t^2, x_3 = -t, u = 1 \end{aligned} \right\} \dots\dots(11)$$

Region I

before switching

$$0 \leq t \leq t_1, \text{ Eq. (11)} \dots\dots\dots(12.1)$$

after switching

$$\left. \begin{aligned} 0 &\leq t \leq 1 + t_1, \\ x_1 &= -1/6 \cdot t_1^3 - 1/2 \cdot t_1^2 t - 1/2 \cdot t_1 t^2 + 1/6 \cdot t^3 \\ x_2 &= 1/2 \cdot t_1^2 + t_1 t - 1/2 \cdot t^2 \\ x_3 &= -t_1 + t, u = -1 \end{aligned} \right\} \dots\dots\dots(12.2)$$

where the origin of t is transferred to t_1 .

Region II

before the first switching

$$0 \leq t \leq t_1, \text{ Eq. (11) or Eq. (12.1)} \dots\dots\dots(13.1)$$

before the second switching after the first switch-

ing

$$0 \leq t \leq t_2 - t_1, \text{ Eq. (12.2)} \dots\dots\dots(13.2)$$

after the second switching

$$\left. \begin{aligned} 0 &\leq t \leq t_2 - 2t_1 + 1 \\ x_1 &= -1/6 \cdot t_1^3 - 1/2 \cdot t_1^2(t_2 - t_1) - 1/2 \cdot t_1(t_2 - t_1)^2 \\ &\quad + 1/6 \cdot (t_2 - t_1)^3 - \{ 1/2 \cdot t_1^2 + t_1(t_2 - t_1) \\ &\quad - 1/2 \cdot (t_2 - t_1)^2 \} t + 1/2 \cdot (t_2 - 2t_1)t^2 - 1/6 t^3 \\ x_2 &= 1/2 \cdot t_1^2 + t_1(t_2 - t_1) - 1/2 \cdot (t_2 - t_1)^2 \\ &\quad - (t_2 - 2t_1)t + 1/2 \cdot t^2 \\ x_3 &= t_2 - 2t_1 - t, u = 1 \end{aligned} \right\} \dots\dots\dots(13.3)$$

where the origin of t is transferred to $t_2 - t_1$. It should be noticed in the above that the ranges of t_1 and t_2 are given by

$$0 \leq t_1 \leq 1, t_1 \leq t_2 < 1 + 2t_1. \dots\dots\dots(14)$$

Fig. 1 Switching times of u determined by $-\mu_1$ and $-\mu_2$

Thus all possible optimal backward time trajectories have been found. Since they spread into 3-dimensional space and it is inconvenient to display optimal control law in 3-dimensional state space, we shall draw x_3 -cross sections in the following.

x_3 -cross section of trajectories described by Eq. (11) is given by

$$x_1 = 1/6 \cdot x_3^3, \quad x_2 = 1/2 \cdot x_3^2, \quad \dots \quad (15)$$

and the range of x_3 is $-1 \leq x_3 \leq 0$. Optimal control in this case is $u=1$.

x_3 -cross section of trajectories described by Eq. (12)

x_3 -cross section of trajectories described by Eq. (12.1) is the same as that of trajectories described by Eq. (11). x_3 -cross section of trajectories described by Eq. (12.2) is expressed by

$$\left. \begin{aligned} x_1 &= 1/6 \cdot x_3^3 - x_3 t_1 - t_1^3 \\ x_2 &= -1/2 \cdot x_3^2 + t_1^2 \end{aligned} \right\} \dots \quad (16)$$

Range of x_3 becomes $-t_1 \leq x_3 \leq 1$ from the relation of $0 \leq x_3 + t_1 < 1 + t_1$. Furthermore, $-1 \leq x_3 \leq 1$ results from Eq. (14), hence x_3 -cross section exists for all possible values of x_3 . Range of t_1 for any x_3 -cross section becomes

$$\left. \begin{aligned} 1 \geq t_1 > 0 & \text{ for } x_3\text{-cross section of } x_3 \geq 0 \\ 1 \geq t_1 > -x_3 & \text{ for } x_3\text{-cross section of } x_3 \leq 0 \end{aligned} \right\} \dots (17)$$

because of $t_1 > 0$. Optimal control in this case is $u=-1$.

x_3 -cross section of trajectories described by Eq. (13)

x_3 -cross section of trajectories described by Eq. (13.1) is the same as that of trajectories described by Eq. (11) and x_3 -cross section of trajectories described by Eq. (13.2) is the same as that of trajectories described by Eq. (12.2). x_3 -cross section of trajectories described by Eq. (13.3) is expressed by

Fig. 2 x_3 -cross section of admissible region for optimal backward time trajectories emanating from origin.

$$\begin{aligned}
 x_1 = & -1/6 \cdot t^3 - 1/2 \cdot t_1^2(t_2 - t_1) - 1/2 \cdot t_1(t_2 - t_1)^2 \\
 & + 1/6 \cdot (t_2 - t_1)^3 - \{1/2 \cdot t_1^2 + t_1(t_2 - t_1) \\
 & - 1/2 \cdot (t_2 - t_1)^2\}(t_2 - 2t_1 - x_3) \\
 & + 1/2 \cdot (t_2 - 2t_1)(t_2 - 2t_1 - x_3)^2 \\
 & - 1/6 \cdot (t_2 - 2t_1 - x_3)^3 \\
 x_2 = & 1/2 \cdot t_1^2 + t_1(t_2 - t_1) - 1/2 \cdot (t_2 - t_1)^2 \\
 & - (t_2 - 2t_1)(t_2 - 2t_1 - x_3) \\
 & + 1/2 \cdot (t_2 - 2t_1 - x_3)^2
 \end{aligned}
 \tag{18}$$

From the relation of $0 \leq t_2 - 2t_1 - x_3 \leq t_2 - 2t_1 + 1$ and Eq. (14), $-1 \leq x_3 \leq 1$ is obtained. Namely, x_3 -cross section for all possible values of x_3 exists. Range of t_1 , t_2 for any x_3 -cross section becomes

$$\text{range of } t_1; 0 \leq t_1 \leq 1 \tag{19}$$

range of t_2 ;

$$\left. \begin{aligned}
 2t_1 + x_3 \leq t_2 \leq 1 + 2t_1 \text{ for } t_1 + x_3 \geq 0 \\
 t_1 \leq t_2 \leq 1 + 2t_1 \text{ for } t_1 + x_3 \leq 0
 \end{aligned} \right\} \tag{20}$$

Optimal control $u=1$ is used on these cross sections.

Based on the above preparative analysis, we will be able to draw an x_3 -cross section for any value of x_3 . The analysis so far was done for the value of $-\mu_3=1$. Since symmetric optimal backward time trajectories with respect to the origin will be obtained to the above in the case of $-\mu_3=-1$, the $-x_3$ -cross section in the case of $-\mu_3=-1$ will surely be symmetric with respect to the origin to x_3 -cross section in the case of $-\mu_3=1$. Thus x_3 -cross sections of all trajectories which emanate from the origin and exist in the admissible region, namely before reaching $x_3=1.0$ or -1.0 plane, can be drawn as shown in Fig. 2. The x_3 -cross section in which optimal control that has been found for any cross section is entered gives the very control law diagram. The number in the parentheses of Fig. 2 entered behind optimal control means the switching times needed during the period before the state point attains the origin from there.

4. Continuity condition at the point where the optimal backward time trajectory meets the boundary $x_3 = \pm 1$.

Now let us consider again the case of $-\mu_3=1$.

«In the case where $\{-\mu_1, -\mu_2\}$ lies in region I of Fig. 1.»

The state point reaches $x_3=-1$ at $t=1$ without any switching. At arrival point A the Hamiltonian is given by

$$H(A) = (-\mu_1)(1/2) + \{(-\mu_2) + (-\mu_1)\}(-1) + \{1 + (-\mu_1) + 1/2 \cdot (-\mu_2)\}u - 1, \dots \tag{21}$$

and u which maximizes $H(A)$ is 1. Then we obtain $\langle f[x(A), u], n(A) \rangle = -1$, hence the continuity condition is not satisfied.

«In the case where $\{-\mu_1, -\mu_2\}$ lies on the boundary of region 0, that is on the straight line

$$1/2 \cdot (-\mu_1) + (-\mu_2) + 1 = 0 \gg$$

The state point reaches $x_3=-1$ at $t=1$. At arrival point A, u which maximizes $H(A)$ is indefinite within the range of $[-1, 1]$. Now $\langle f[x(A), u], n(A) \rangle \geq 0$ will be obtained for $u \leq 0$. Therefore, the continuity condition is satisfied because $u \leq 0$ is surely included within the range of $[-1, 1]$. The continuity point is one point on $x_3=-1.0$ plane.

«In the case where $\{-\mu_1, -\mu_2\}$ lies in region I»

$u=1$ is used for $0 \leq t \leq t_1$ and is switched to $u=-1$ at t_1 . The state point reaches $x_3=1$ at $t=1+2t_1$, with the origin of time axis being unaltered. Hamiltonian at arrival point A is given by

$$\begin{aligned}
 H(A) = & (-\mu_1)(-1/2 + t_1^2) + \{(-\mu_2) \\
 & + (-\mu_1)(1 + 2t_1)\} \cdot 1 + 1 + (-\mu_2)(1 + 2t_1) \\
 & + 1/2 \cdot (-\mu_1)(1 + 2t_1)^2 u - 1 \dots \tag{22}
 \end{aligned}$$

and $p_3(A)$ is negative, hence u which maximizes $H(A)$ is -1 . Now, $\langle f[x(A), u], n(A) \rangle = -1$ will be obtained. Therefore, the continuity condition is not satisfied in this case.

«In the case where $\{-\mu_1, -\mu_2\}$ lies on the boundary $8(-\mu_2)^2 + 2(-\mu_1)(-\mu_2) - (-\mu_1)^2 - 18(-\mu_1) = 0$ of region I»

$p_3(A)$ becomes just 0 in this case and u which maximizes $H(A)$ written in Eq. (22) is indefinite within the range of $[-1, 1]$. $\langle f[x(A), u], n(A) \rangle \geq 0$ will be obtained for $u \geq 0$, hence the continuity is satisfied, because $u \geq 0$ is included in the range of $[-1, 1]$. The continuity point is any point on the curve on $x_3=1.0$ plane (see $x_3=1.0$ plane in Fig. 2).

«In the case where $\{-\mu_1, -\mu_2\}$ lies in region II»

$u=1$ for $0 \leq t \leq t_1$, $u=-1$ for $t_1 \leq t < t_2$ and $u=1$ after t_2 are used. The state point reaches $x_3=-1$ at $t=1+2t_2-2t_1$. u which maximizes $H(A)$ at arrival point A is 1. Then $\langle f[x(A), u], n(A) \rangle = -1$ will be obtained, hence the continuity condition is not satisfied.

After all it may be concluded that the optimal backward time trajectory can connect to the one on the boundary when $\{-\mu_1, -\mu_2\}$ takes the value of the boundary (straight line) of region 0 and the

Fig. 3 Continuity points between optimal backward time trajectory in admissible region and the one on the boundary

value of the curved part of boundary of region I.

In the case of $-\mu_3 = -1$, a symmetric result to the above can be obtained. Figure 3 shows all continuity points for the case of $-\mu_3 = 1$ as well as $-\mu_3 = -1$.

5. Tracking of optimal backward time trajectory on the boundary

State vector differential equation on the boundary is given by

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 1 \text{ or } -1 \end{aligned} \right\}, \dots\dots\dots(23)$$

while x_3 is disqualified for state variable. Costate vector becomes two-dimensional too on the boundary and is described by $\phi = \{\phi_1, \phi_2\}$. Its differential equation is obtained as

$$\phi_1 = 0, \quad \phi_2 = -\phi_1 \dots\dots\dots(24)$$

from Eq. (23).

Control u must be $u \geq 0$ on $x_3 = 1.0$ plane and $u \leq 0$ on $x_3 = -1.0$ plane from the condition that the state point moves along the boundary, but since a state vector differential equation does not contain u , the optimal problem degenerates on the boundary.

Now let us consider the case of $-\mu_3 = 1$ only. The only continuity points are A_{-1} in $x_3 = -1.0$ plane and curve $A_1 B_1$ (excluding point A_1) in $x_3 = 1.0$ plane. Let the backward time origin be transferred to the continuity point hereafter.

«Trajectory emanating from the point A_1 »

From Eq. (23) the optimal backward time trajectory is given by

Fig. 4 Optimal backward time trajectories on the boundary

Fig. 5 Portion, where branch-off is realizable, of trajectory emanating from A_{-1}

$$\left. \begin{aligned} x_1 &= -1/6 - 1/2 \cdot t - 1/2 \cdot t^2 \\ x_2 &= 1/2 + t \end{aligned} \right\}, \dots\dots\dots(25)$$

and costate vector becomes

$$\left. \begin{aligned} \phi_1 &= -\mu_1 \\ \phi_2 &= -1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t \end{aligned} \right\}, \dots\dots\dots(26)$$

«Trajectories emanating from any point of $A_1 B_1$ »

Optimal backward time trajectories which are given by

$$\left. \begin{aligned} x_1 &= 1/6 - t_1^2 - t_1^3 - (-1/2 + t_1^2)t + 1/2 \cdot t^2 \\ x_2 &= -1/2 + t_1^2 - t \end{aligned} \right\}, \dots\dots\dots(27)$$

form a region on $x_3 = 1.0$ plane. Costate vector is written as

$$\left. \begin{aligned} \phi_1 &= 2/t_1(1+2t_1) \\ \phi_2 &= (1+t_1)/t_1(1+2t_1) + 2t/t_1(1+2t_1) \end{aligned} \right\}, \dots\dots\dots(28)$$

Figure 4 shows optimal backward time trajectory on the boundary including the case of $-\mu_3 = -1$.

6. Branch-off of optimal backward time trajectory from the boundary into admissible region

«Branch-off from any point on $A_{-1} A'_{-1}$ »

Let us investigate the branch-off condition of optimal backward time trajectory into admissible region at point B, the location of the state point at time t_B since continuation is realized. Assuming a branch-off, the Hamiltonian $H(\rho, u)$ becomes

$$H(\rho, u) = (-\mu_1)(1/2 + t_B) + \{-1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B\}(-1) + \rho u - 1. \dots\dots\dots(29)$$

The condition that $\langle -f[x(B), u], n(B) \rangle$ is negative is satisfied by $u < 0$. u and ρ which maximize and make zero $H(\rho, u)$ described by Eq. (29) under the condition $u < 0$ are given by

$$u = [-1, 0), \quad \rho = 0. \dots\dots\dots(30)$$

If the origin of backward time is transferred to this point, costate vector will be described by

$$\left. \begin{aligned} p_1 &= -\mu_1 \\ p_2 &= -1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B + (-\mu_1)t \\ p_3 &= \{-1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B\}t + 1/2 \cdot (-\mu_1)t^2 \end{aligned} \right\} \dots\dots\dots(31)$$

The branch-off condition becomes

$$p_2(B) = -1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B < 0. \dots\dots\dots(32)$$

Equation (32) is necessarily satisfied in the case of $-\mu_1 \leq 0$ and the branch-off becomes realizable at any point on $A_{-1} A'_{-1}$, but in the case of $-\mu_1 > 0$,

$$t_B < \{1 - (1/2)(-\mu_1)\}/(-\mu_1) \dots\dots\dots(33)$$

must be satisfied in order for Eq. (32) to hold. Figure 5 shows the region where branch-off is realizable and the one where branch-off is unrealizable.

«Branch-off from the trajectory emanating from $A_1 B_1$ »

Equation for the family of optimal backward time trajectories on the boundary is shown in Eq. (27). Assuming branch-off is realizable at point B, the location of the state point at time t_B since continuation, the Hamiltonian becomes

$$H(\rho, u) = 2/t_1(1+2t_1) \cdot (-1/2 + t_1^2 - t_B) + \{(1+t_1)/t_1(1+2t_1) + 2t_B/t_1(1+2t_1)\} \cdot 1 + \rho u - 1. \dots (34)$$

The condition that $\langle -f[x(B), u], n(B) \rangle$ is negative is satisfied by $u < 0$. u and ρ which maximize and make zero $H(\rho, u)$ described by Eq. (34) are given by

$$u = (0, 1], \quad \rho = 0. \dots (35)$$

Then, if time origin is transferred to this point, costate vector will be described

$$\left. \begin{aligned} p_1 &= 2/t_1(1+2t_1) \\ p_2 &= (1+t_1)/t_1(1+2t_1) + 2t_B/t_1(1+2t_1) + 2/t_1(1+2t_1) \cdot t \\ p_3 &= \{(1+t_1)/t_1(1+2t_1) + 2t_B/t_1(1+2t_1)\}t + t^2/t_1(1+2t_1) \end{aligned} \right\} \dots (36)$$

Therefore, surely $p_3 > 0$ and $u=1$, and hence the state point goes into admissible region, that is, branch-off is realizable.

7. The second tracking of optimal backward time trajectory in admissible region

«Optimal backward time trajectories which branch off from A_{-1} , A'_{-1} »

As stated in the last section, branch-off is realizable from any point in the case of $-\mu_1 < 0$, while in the case of $-\mu_1 > 0$ branch-off is realizable only from the region where inequality (33) is satisfied. Figure 6 shows such a relation between t_B and $-\mu_1$. Difference of region B from C in Fig. 6 will be explained later.

(i) Region A

After branch-off p_3 becomes negative and $u=-1$ is applied, then the following is obtained:

$$\left. \begin{aligned} x &= \begin{bmatrix} -1/6 - 1/2 \cdot t_B - 1/2 \cdot t_B^2 - 1/2 \cdot t - t_B t \\ -1/2 \cdot t^2 + 1/6 \cdot t^3 \\ 1/2 + t_B + t - 1/2 \cdot t^2 \\ -1 + t \end{bmatrix} \\ p &= \begin{bmatrix} -\mu_1 \\ -1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B + (-\mu_1)t \\ \{-1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B\}t \\ + 1/2 \cdot (-\mu_1)t^2 \end{bmatrix} \end{aligned} \right\} \dots (37)$$

The state point described by Eq. (37) reaches $x_3=1.0$ plane at $t=2$. Hamiltonian at arrival point A is given by

$$H(A) = (-\mu_1)(1/2 + t_B) + \{-1 + 5/2 \cdot (-\mu_1) + (-\mu_1)t_B\} \cdot 1 + \{-2 + 3(-\mu_1) + 2(-\mu_1)t_B\}u - 1 \dots (38)$$

and u which maximizes $H(A)$ is -1 because

$$-2 + 3(\mu_1) + 2(-\mu_1)t_B < 0.$$

$$\text{Then } \langle f[x(A), u], n(A) \rangle = -1$$

results, and hence the continuity condition is not satisfied.

(ii) Region B i.e. $0 \leq t_B < \{1 - 3/2 \cdot (-\mu_1)\} / (-\mu_1)$

Some time after branch-off, the state point reaches $x_3=1.0$ plane while p_3 is still negative, and the continuity condition is not satisfied again just the same as in region A.

«Boundary between regions B and C i.e.

$$t_B = \{1 - 3/2 \cdot (-\mu_1)\} / (-\mu_1) \gg$$

The state point reaches $x_3=1$ at $t=2$ but $p_3(A)$ at arrival point becomes just 0 this time. u which maximizes $H(A)$ at arrival point is indefinite within the range of $[-1, 1]$. Since u which makes $\langle f[x(A), u], n(A) \rangle \geq 0$ i.e. $u \geq 0$ is included within the range of $[-1, 1]$, the continuity condition is surely satisfied at arrival point.

(iii) Region C

After branch-off $u=-1$ is first applied, and u is switched to $u=1$ at

$$t = \{-1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B\} / \{-1/2 \cdot (-\mu_1)\},$$

then the state point comes back to $x_3=-1.0$ plane again at

$$t = \{-2 + (-\mu_1) + 2(-\mu_1)t_B\} / \{-1/2 \cdot (-\mu_1)\}.$$

Since $p_3(A)$ at arrival point is positive, u which maximizes $H(A)$ is 1. Continuity condition is not satisfied then because $\langle f[x(A), u], n(A) \rangle = -1$. Optimal backward time trajectory in region C is given by

$$\left. \begin{aligned} 0 \leq t \leq \frac{-1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B}{-1/2 \cdot (-\mu_1)} \end{aligned} \right\} \dots (39)$$

$$\text{Eq. (37)}$$

$$\left. \begin{aligned} \frac{-1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B}{-1/2 \cdot (-\mu_1)} \leq t \\ \leq \frac{2\{-1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B\}}{-1/2 \cdot (-\mu_1)} \end{aligned} \right\}$$

$$x = \begin{bmatrix} -1/6 - 1/2 \cdot t_B - 1/2 \cdot t_B^2 + 1/3 \cdot t_s^3 - 1/2 \cdot t \\ -t_B t - t_s^2 t - 1/2 \cdot t^2 + t_s t^2 - 1/6 \cdot t^3 \\ 1/2 + t_B + t_s^2 + t - 2t_s t + 1/2 \cdot t^2 \\ -1 + 2t_s - t \end{bmatrix} \dots (40)$$

where

$$t_s = \{-1 + 1/2 \cdot (-\mu_1) + (-\mu_1)t_B\} / \{-1/2 \cdot (-\mu_1)\}.$$

Fig. 6 Division of t_B , $(-\mu_1)$ plane

«Optimal backward time trajectory in admissible region which branches off from the trajectory emanating from A_1B_1 »

Branch-off is always possible in this case. After branch-off, since $p_3 > 0$ as shown in Eq. (36), optimal control is fixed to 1. Then the state point is described by

$$x = \begin{bmatrix} 1/6 - t_1^2 - t_1^3 - (-1/2 + t_1^2)t_B + 1/2 \cdot t_B^2 \\ + 1/2 \cdot t - t_1^2 t + t_B t + 1/2 \cdot t^2 - 1/6 \cdot t^3 \\ - 1/2 + t_1^2 - t_B - t + 1/2 \cdot t^2 \\ 1 - t \end{bmatrix} \quad \dots\dots\dots(41)$$

The state point reaches $x_3 = -1$ at $t = 2$ and u which maximizes $H(A)$ at arrival point becomes 1 because $p_3 > 0$. Then $\langle f[x(A), u], n(A) \rangle = -1$ results and clearly the continuity condition is not satisfied.

After all, only the optimal backward time trajectory which branches off from $A_{-1}A'_{-1}$ under the condition of $t_B = \{1 - 3/2 \cdot (-\mu_1)\} / (-\mu_1)$ can connect to the optimal one on the boundary $x_3 = 1.0$. $x(A)$ forms a curve on $x_3 = 1.0$ plane which may be described by

$$x_1 = 31/24 - 1/2 \cdot (x_2 + 2)^2. \quad \dots\dots\dots(42)$$

8. The second tracking of optimal backward time trajectory on the boundary

If $u \geq 0$ is applied after the state point reaches the curve given by Eq. (42), the state point moves along $x_3 = 1.0$ plane. If the origin of backward time is transferred to the arrival point, backward time trajectories are given by

$$\left. \begin{aligned} x_1 &= -11/6 - 5/2 \cdot t_B - 1/2 \cdot t_B^2 - 1/2 \cdot t \\ &\quad - t_B t + 1/2 \cdot t^2 \\ x_2 &= 1/2 + t_B - t \end{aligned} \right\} \quad \dots\dots\dots(43)$$

which forms a region adjacent to the region formed by Eq. (27). Costate vector becomes

$$\phi_1 = -\mu_1, \quad \phi_2 = -\mu_1 + (-\mu_1)t. \quad \dots\dots\dots(44)$$

9. Branch-off into admissible region

Let us investigate the branch-off condition about trajectories described by Eq. (43) at point B' , the location of the state point at $t_{B'}$ after continuation to the trajectory on the $x_3 = 1.0$ plane. Hamiltonian $H(\rho, u)$ at point B' is given by

$$H(\rho, u) = (-\mu_1)(1/2 + t_B - t_{B'}) + \{-\mu_1 + (-\mu_1)t_B\} \cdot 1 + \rho u - 1 = \rho u. \quad \dots\dots\dots(45)$$

u and ρ which make $\langle f[x(B'), u], n(B') \rangle$ negative and at the same time maximize and make zero $H(\rho, u)$ described by Eq. (45) are

$$u = (0, 1], \quad \rho = 0. \quad \dots\dots\dots(46)$$

Then costate vector p becomes

$$p = \begin{bmatrix} -\mu_1 \\ -\mu_1 + (-\mu_1)t_{B'} + (-\mu_1)t \\ \{(-\mu_1) + (-\mu_1)t_{B'}\}t + 1/2 \cdot (-\mu_1)t^2 \end{bmatrix} \quad \dots\dots\dots(47)$$

p_3 is surely positive and $u = 1$ is fixed. Therefore, optimal backward time trajectory can branch off into admissible region at any point on the trajectories.

10. The third tracking of optimal backward time trajectory in admissible region

Since p_3 is necessarily positive as indicated by Eq. (47), the state point that moves under the control of $u = 1$ reaches $x_3 = -1.0$ in time, and clearly, the continuity condition is not satisfied any more. The state point is given by

$$x = \begin{bmatrix} -11/6 - 5/2 \cdot t_B - 1/2 \cdot t_B^2 - 1/2 \cdot t_{B'} - t_B t_{B'} \\ + 1/2 \cdot t_{B'}^2 - 1/2 \cdot t - t_B t + t_{B'} t + 1/2 \cdot t^2 \\ - 1/6 \cdot t^3 \\ 1/2 + t_B - t_{B'} - t + 1/2 \cdot t^2 \\ 1 - t \end{bmatrix} \quad \dots\dots\dots(48)$$

11. x_3 -cross section and optimal control law

As above, all possible optimal backward time trajectories have been tracked so far as continuity and branch-off conditions are satisfied for the case of $-\mu_3 = 1$. All trajectories that emanate from the origin and lead to the boundary $x_3 = 1.0$ or -1.0 plane were investigated in section 3 and its x_3 -cross section was obtained as shown in Fig. 2. The trajectories which connect to the ones on the boundary and then branch off into admissible region were examined in section 4 and subsequent sections. The

Fig. 7 Three dimensional figure of optimal backward time trajectories in the case of $-\mu_3 = 1$ (parts of admissible region for trajectories emanating from origin are excluded)

three-dimensional figure of all trajectories is shown in Fig. 7.

The trajectories described by Eq. (37) form a curve

$$\left. \begin{aligned} x_1 &= -1/6 - 1/2 \cdot t_B - 1/2 \cdot t_B^2 - (1/2 + t_B)(x_3 + 1) \\ &\quad - 1/2 \cdot (x_3 + 1)^2 + 1/6 \cdot (x_3 + 1)^3 \\ x_2 &= 1/2 + t_B + (x_3 + 1) - 1/2 \cdot (x_3 + 1)^2 \end{aligned} \right\} \dots\dots\dots (49)$$

on x_3 -cross section. x_3 -cross section of Eq. (40) is given by

$$\left. \begin{aligned} x_1 &= -1/6 - 1/2 \cdot t_B - 1/2 \cdot t_B^2 + 1/3 \cdot t_s^3 + (-1/2 \\ &\quad - t_B - t_s^2)(2t_s - 1 - x_3) + (-1/2 + t_s) \\ &\quad \times (2t_s - 1 - x_3)^2 - 1/6 \cdot (2t_s - 1 - x_3)^3 \\ x_2 &= 1/2 + t_B + t_s^2 + (1 - 2t_s)(2t_s - 1 - x_3) \\ &\quad + 1/2 \cdot (2t_s - 1 - x_3)^2 \end{aligned} \right\} \dots\dots\dots (50)$$

which forms a region in the cross section. x_3 -cross section of Eq. (48) is given by

$$\left. \begin{aligned} x_1 &= -11/6 - 5/2 \cdot t_B - 1/2 \cdot t_B^2 - 1/2 \cdot t_{B'} - t_B t_{B'} \\ &\quad + 1/2 \cdot t_{B'}^2 + (-1/2 - t_B + t_{B'})(1 - x_3) \\ &\quad + 1/2 \cdot (1 - x_3)^2 - 1/6 \cdot (1 - x_3)^3 \\ x_2 &= 1/2 + t_B - t_{B'} - (1 - x_3) + 1/2 \cdot (1 - x_3)^2 \end{aligned} \right\} \dots\dots\dots (51)$$

which forms a region in the cross section. x_3 -cross section of Eq. (41) is given by

$$\left. \begin{aligned} x_1 &= 1/6 - t_1^2 - t_1^3 - (-1/2 + t_1^2)t_B + 1/2 \cdot t_B^2 \\ &\quad + (1/2 - t_1^2 + t_B)(1 - x_3) + 1/2 \cdot (1 - x_3)^2 \\ &\quad - 1/6 \cdot (1 - x_3)^3 \\ x_2 &= -1/2 + t_1^2 - t_B - (1 - x_3) + 1/2 \cdot (1 - x_3)^2 \end{aligned} \right\} \dots\dots\dots (52)$$

which forms a region in the cross section.

As above, x_3 -cross sections of all possible backward time trajectories have been obtained as figures described by Eqs. (49) ~ (52) besides Fig. 2. Since the figure described by Eq. (49) makes a part of contour of the region described by Eq. (50), the figures with the region are Eqs. (50) ~ (52) alone. Optimal backward time trajectories symmetric to the above with respect to the origin are obtained in the case of $-\mu_3 = -1$. Thus the following are obtained:

$$\left. \begin{aligned} x_1 &= 1/6 + 1/2 \cdot t_B + 1/2 \cdot t_B^2 - 1/3 \cdot t_s^3 - (-1/2 - t_B \\ &\quad - t_s^2)(2t_s - 1 + x_3) - (-1/2 + t_s)(2t_s - 1 + x_3)^2 \\ &\quad + 1/6 \cdot (2t_s - 1 + x_3)^3 \\ x_2 &= -1/2 - t_B - t_s^2 - (1 - 2t_s)(2t_s - 1 + x_3) \\ &\quad - 1/2 \cdot (2t_s - 1 + x_3)^2 \end{aligned} \right\} \dots\dots\dots (50)'$$

$$\left. \begin{aligned} x_1 &= 11/6 + 5/2 \cdot t_B + 1/2 \cdot t_B^2 + 1/2 \cdot t_{B'} + t_B t_{B'} \\ &\quad - 1/2 \cdot t_{B'}^2 - (-1/2 - t_B + t_{B'})(1 + x_3) \\ &\quad - 1/2 \cdot (1 + x_3)^2 + 1/6 \cdot (1 + x_3)^3 \\ x_2 &= -1/2 - t_B + t_{B'} + (1 + x_3) - 1/2 \cdot (1 + x_3)^2 \end{aligned} \right\} \dots\dots\dots (51)'$$

$$\left. \begin{aligned} x_1 &= -1/6 + t_1^2 + t_1^3 + (-1/2 + t_1^2)t_B \\ &\quad - 1/2 \cdot t_B^2 - (1/2 - t_1^2 + t_B)(1 + x_3) \\ &\quad - 1/2 \cdot (1 + x_3)^2 + 1/6 \cdot (1 + x_3)^3 \\ x_2 &= 1/2 - t_1^2 + t_B + (1 + x_3) - 1/2 \cdot (1 + x_3)^2 \end{aligned} \right\} \dots\dots\dots (52)'$$

x_3 -cross section of all possible optimal backward time trajectories emanating from the origin is shown by Fig. 2 superimposed over the figures specified by Eqs. (50) ~ (52) and those specified by Eqs. (50)' ~ (52)'. The x_3 -cross section thus obtained is shown in Fig. 8. The x_3 -cross section is divided into eight domains in each of which the number of equation that specifies the domain and the optimal control to be applied are inserted. Figure 8, therefore, can be used as the optimal control law diagram. Also, the x_3 -cross section is defined by twelve curves which are given by

- ① $x_1 = 1/6 \cdot x_3^3 - \rho^2 x_3 + \rho^3$, $x_2 = 1/2 \cdot x_3^2 - \rho^2$
 $x_3 \leq \rho \leq 1$ if $x_3 \geq 0$
 $0 \leq \rho \leq 1$ if $x_3 \leq 0$
- ② $x_1 = 1/6 \cdot x_3^3 - \rho^2 x_3 - \rho^3$, $x_2 = -1/2 \cdot x_3^3 + \rho^2$
 $0 \leq \rho \leq 1$ if $x_3 \geq 0$
 $-x_3 \leq \rho \leq 1$ if $x_3 \leq 0$
- ③ $x_1 = 1/6 \cdot x_3^3 + x_3(-\rho^2 + 4\rho - 3) - \rho^3 + 6\rho^2$
 $-10\rho + 5$
 $x_2 = -1/2 \cdot x_3^2 + \rho^2 - 4\rho + 3$, $2 - x_3 \leq \rho \leq 3$
- ④ $x_1 = 1/6 \cdot x_3^3 + x_3(-\rho^2 + 4\rho - 3) + \rho^3 - 6\rho^2$
 $+10\rho - 5$
 $x_2 = 1/2 \cdot x_3^2 - \rho^2 + 4\rho - 3$, $2 + x_3 \leq \rho \leq 3$
- ⑤ $x_1 = 1/6 \cdot x_3^3 + x_3(\rho^2 - 1) + \rho^3 + 2\rho^2 - 1$
 $x_2 = -1/2 \cdot x_3^2 - \rho^2 + 1$, $0 \leq \rho \leq 1$
- ⑥ $x_1 = 1/6 \cdot x_3^3 + x_3(\rho^2 - 1) - \rho^3 - 2\rho^2 + 1$
 $x_2 = 1/2 \cdot x_3^2 + \rho^2 - 1$, $0 \leq \rho \leq 1$
- ⑦ $x_1 = 1/6 \cdot x_3^3 - \rho x_3 - 1/2 \cdot \rho^2 - 1/2 \cdot \rho + 2$
 $x_2 = -1/2 \cdot x_3^2 + \rho$, $\rho \geq 0$
- ⑧ $x_1 = 1/6 \cdot x_3^3 - \rho x_3 + 1/2 \cdot \rho^2 + 1/2 \cdot \rho - 2$
 $x_2 = 1/2 \cdot x_3^2 - \rho$, $\rho \geq 0$
- ⑨ $x_1 = 1/6 \cdot x_3^3 + \rho x_3 + 1/2 \cdot \rho^2 + 7/2 \cdot \rho + 2$
 $x_2 = -1/2 \cdot x_3^2 - \rho$, $\rho \geq 0$
- ⑩ $x_1 = 1/6 \cdot x_3^3 + \rho x_3 - 1/2 \cdot \rho^2 - 7/2 \cdot \rho - 2$
 $x_2 = 1/2 \cdot x_3^2 + \rho$, $\rho \geq 0$
- ⑪ $x_1 = 1/6 \cdot x_3^3 + x_3(-\rho - 1) + 1/2 \cdot \rho^2 + 3/2 \cdot \rho + 1$
 $x_2 = 1/2 \cdot x_3^2 - \rho - 1$, $\rho \geq 0$
- ⑫ $x_1 = 1/6 \cdot x_3^3 + x_3(-\rho - 1) - 1/2 \cdot \rho^2 - 3/2 \cdot \rho - 1$
 $x_2 = -1/2 \cdot x_3^2 + \rho + 1$, $\rho \geq 0$

12. Optimal control law from the view point of angular position control of rotating body

We have sought the optimal control law for the

Fig. 8 x_3 -cross section for any x_3 which is also valid as optimal control law

Fig. 9 Optimal control law from the view point of rotating body

terminal point $\{0, 0, 0\}$ so far, but as mentioned in section 2 the terminal value of x_1 may be $2n\pi$, $n=0, \pm 1, \pm 2, \dots$ from the point of view of angular position control of rotating body. It should be noticed that the real terminal point must be such that the required time for the minimum time control to the terminal point is minimum among all conceivable terminal points $\{2n\pi, 0, 0\}$, where $n=0, \pm 1, \pm 2, \dots$.

Optimal control law diagrams for the terminal points $\{2n\pi, 0, 0\}$ are easily obtained from Fig. 8. Superimpose infinitely many control law diagrams obtained for $n=0, \pm 1, \pm 2, \dots$, then choose a real terminal point and decide the optimal control for the chosen terminal point by comparing required times. Figure 9 shows the real optimal control law on $x_3=1.0, 0.5, 0, -0.5$, and -1.0 cross sections decided by comparing the times required.

13. Conclusion

There are some points to be noted in this problem. Firstly the position of rudder is not a control but one component of state vector, and hence the dimensions of state become three. Control is considered to be the driving torque of steering gear. The rudder has necessarily right and left maximum strokes on account of system structure. It turns out, therefore, that this problem must be dealt with as an optimal problem with stop mechanism. Finally, angular position control problem of a rotating body such as ship has some peculiarity. These various situations are treated precisely and a perfect synthesis of optimal control law is accomplished in this paper.

References

- (1) K. Ichikawa et al.: *Trans. SICE*, Vol. 4, No. 1 (1968), p. 1.
- (2) K. Ichikawa and K. Tamura: *IFAC Symposium, Düsseldorf*, (1968).

Discussion

H. HATANAKA (1) Although the optimal feedback control system may be synthesized by utilizing the optimal control law for each cross section shown in the paper, the practical hardware is thought to be pretty complicated or difficult to realize. How do authors think about this point?

S. KANOH (2) By what simplification is Eq. (1) obtained which denotes the equation of motion of steering system?

(3) There are optimal trajectories starting from any initial point which terminate at $2n\pi$, $(2n-1)\pi$ or $(2n+1)\pi$, and hence some boundaries may be obtained which separate the trajectories which terminate at $2n\pi$ from the trajectories which terminate at $(2n-1)\pi$ or those which terminate at $(2n+1)\pi$, by comparing the values of criterion function for each trajectory. Then the boundary will play a very important role in the practice of control. What is the concrete shape of boundary?

(4) How do authors think to realize the optimal feedback control law?

R. MIURA (5) The optimal control system as dealt with in this paper seems to have the characteristic such that at most two switchings will suffice unless x_3 is restricted and when x_3 is restricted, trajectory can proceed along the boundary $x_3=1$ or $x_3=-1$ only once respectively because there exists no trajectory which is tangent to the boundary. The problem is thought to be solved more simply by utilizing this characteristic.

(6) The relation between costate vector p and φ together with the relation between results in this paper and jump condition will clarify the peculiarity of this problem in the sense of author's assertion.

Authors' closure

(1) The main aim of this paper is in neither the mechanization of the optimal feedback system nor practical working test but in finding out the optimal control law theoretically. The authors consider, however, that it may be realized by some proper three input-one output function generator because the dimensions of state vector and control vector are respectively three and one. The mechanization of the function generator is generally not easy but the following two methods are conceivable; the first is to obtain the optimal control as the output of digital computer, in which the optimal control

law function has been memorized, by applying to it the three dimensional state vector, and the second is to make arrangement of many cross sections for each value of x_3 (scores or so may be enough) and to decide whether u is 1 or -1 according to the set $\{x_1, x_2\}$ through some optical means on the particular cross section which has been selected automatically according to the value of x_3 .

(2) The equation of the motion of steering system may be described by

$$I\ddot{x}_3 + D\dot{x}_3 = u,$$

where I is total equivalent inertia of rudder and masses moving together with the rudder about the hinge of rudder and D is drag coefficient. In the case that inertia term $I\ddot{x}_3$ is far less than drag term $D\dot{x}_3$, the equation of motion can be approximately described by

$$D\dot{x}_3 = u,$$

and D is taken as 1 is in this paper.

(3) As shown in this paper, the shape of boundary is exactly obtained although some numerical calculation is needed.

(4) Refer to the authors' closure (1).

(5) Instinctively it may be considered that the optimal trajectory can proceed at most once along boundary $x_3=1.0$ $x_3=-1.0$ respectively, but it is authors' assertion that the optimal solution must be obtained only through the correct theory without any guess. The characteristic pointed out by the discussor is obtained as the important theoretical result in this paper. The authors think that the method mentioned in this paper is general and standard for this kind of problem and the solution method is not troublesome but clear-cut. They also think that any simplification beyond this will have something doubtful left even if it leads to an exact solution by chance.

(6) Although the problem with stop mechanism is essentially different from the bounded state variable problem, the solution method itself is very similar and has little difference. Therefore the paper about bounded variable problem will give a clear-cut answer to the question. Refer to Ref. (a) and for more elaborate papers to Ref. (b) and (c).

- (a) K. Ichikawa et al.: *Trans. SICE*, Vol. 4, No. 1 (1968), p. 1.
- (b) K. Ichikawa and K. Tamura: *Jour. Japan Ass. of ACE*, Vol. 12, No. 5 (1968), p. 281.
- (c) K. Ichikawa and K. Tamura: *Preprint of IFAC Symposium, Düsseldorf*, (1968-10).