Thermal Stresses in a Circular Cylinder with
a Ring of Heat Sources*

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Linear steady-state thermo-elastic problems of a solid cylinder and a hollow one with a ring of uniformly distributed point heat sources of equal intensity $W$ are solved under plane strain theory using thermo-elastic displacement potential and Airy's stress function.

It is assumed that the surface of the cylinder is maintained at a constant temperature.

Numerical results for a particular case are given graphically and analyzed to determine the characteristics of the present problems.

1. Introduction

Stress problems on plates, discs and circular cylinders containing a ring of uniformly distributed circular holes of equal radii have been analyzed by several investigators through the use of such potential function as previously defined by Howland[10]. For example, torsion problem of a circular cylinder having a ring of axial holes was treated by Ling[11], and Saito[12] gave the solution for stresses in circular discs with symmetrically placed noncentral holes. With respect to the steady-state thermo-elastic problem, Udaguchi[13] treated the thermal stress on a circular disc with an eccentric hole, using bi-polar co-ordinates. Moreover Kurashige and Atsumi[14, 15] solved the thermo-elastic problems on a circular plate and cylinder with a ring of holes. Furthermore Takeuchi[16] analyzed the thermal stresses in a circular disc due to instantaneous line heat source, but thermal stress in a circular cylinder with a ring of point heat sources seems not yet to have been solved. The present paper deals with the steady-state thermo-elastic problems of a solid and a hollow cylinder with a ring of uniformly distributed point heat sources of equal intensity $W$ under plane strain theory using the potential of thermo-elastic displacement and Airy's stress function.

2. Relation of the co-ordinates

Consider a hollow cylinder defined by the exterior and the interior radii $a$ and $c$ respectively containing a ring of $m$-point heat sources, the centers of which are arranged symmetrically on a circle of radius $b$ (Fig. 1). Denoting the center of the cylinder $O$, the center of arbitrary one of point heat sources $O_a$, the center of the $k$-th point heat source counted from $O_a$, $O_b$ and the polar co-ordinate systems $(r, \phi)$, $(r_b, \theta_b)$, and $(r_b, \theta_b)$ which are taken referring to $O_a$, $O_b$ and $O_a$ as origins respectively as shown in Fig. 2, and then putting

$$z = r \exp (i \phi), \quad \zeta_b = r_b \exp (i \theta_b)$$

we find the following relations among these complex

![Fig. 1 Cross section of a hollow cylinder with a ring of point heat sources](image1)

![Fig. 2 Co-ordinate systems](image2)
planes:
\[ z = \beta (1 + \zeta_k) \exp(2\pi ki/m) \quad (k = 0, 1, \ldots, m - 1) \]  
\( \zeta_k = (1 + u_k \zeta_0)/(u_k - 1) \quad (k = 1, 2, \ldots, m - 1) \)  
\[ \]  
where
\[ u_k = i e^{-\pi ki/m}(2 \sin \pi k/m) \]  
\[ \alpha = c/a \]  
\[ \beta = b/a \]  
\[ \]  
For brevity, the subscript \( o \) referring to \( \zeta_0 \)-plane will be omitted when the meaning is clear. From Fig. 2 we obtain the following relations:
\[ \rho_k \cos \theta_k = (r/\beta_k) \cos(\phi - 2\pi k/m) - 1 \]  
\[ k = 0, 1, \ldots, m - 1 \]  
\[ \rho_k^2 = (2 \sin \pi k/m)^2 \]  
\[ -2(2 \sin \pi k/m)(\beta - \pi k/m) + \rho \]  
\[ (k = 1, 2, \ldots, m - 1) \]  
\[ \rho_k \cos \theta_k = \rho \cos(\beta - \pi k/m) - 2 \sin \pi k/m \]  
\[ (k = 1, 2, \ldots, m - 1) \]  
\[ \rho_k^2 = 1 + (r/\beta_k)^2 - 2(r/\beta_k) \cos(\beta - 2\pi k/m) \]  
\[ (k = 0, 1, \ldots, m - 1) \]  
\[ r^2 = \beta^2(1 + 20 \cos \theta + \rho^2) \]  
\[ \]  
where variables \( z, \beta \zeta_0, r \) and \( \beta \theta_0 \) denote dimensionless variables divided by the exterior radius \( a \). Series expansion of changing co-ordinate systems will be carried out by using above expressions.

3. Temperature function

3.1 Solid cylinder

First we consider the two-dimensional, steady-state temperature field in a solid cylinder containing a ring of uniformly distributed point heat sources of equal intensity \( W \), the exterior surface of which has temperature \( T_\infty \). If we assume that there is no axial heat flow in the cylinder, the present analysis will be reduced to a solution of the heat conduction equation.

\[ p^2 T = 0 \]  
\[ (13) \]  

with the boundary conditions:

\[ T = T_\infty \quad r = 1 \]  
\[ (14) \]  

\[ W = -\int_0^{\pi} \frac{\partial T}{\partial \rho} \rho d\theta \]  
\[ (15) \]  

where \( \lambda \) is the thermal conductivity and \( p^2 \) is the Laplace operator and takes the following forms referring to the \( (r, \phi) \) and \( (\beta \theta, \theta) \) co-ordinate systems respectively:

\[ p^2 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) \]  
\[ (16) \]  

\[ p^2 = \frac{1}{\beta^2} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \right) \]  
\[ (17) \]  

A general solution of the Laplace equation (13) is taken in the form

\[ T = \frac{W}{2\pi \lambda} \sum_{s=0}^{\infty} \left( a_{s} r^{m_s} \cos m_s \phi + c_{s} J_{m_s}(u_k - 1) \right) \]  
\[ (18) \]  

where the terms containing the unknown coefficients \( a_{s}, c_{s} \) are the homogeneous solutions of Eq. (13) and the term containing the unknown coefficient \( c_{0} \) is the potential function defined by Howland which has singularities at the \( m \)-point heat sources of required periodicity and is expressible in the form

\[ c_{0} = -R_{e} \left[ \log \frac{R_{e}}{u_{k} - 1} \right] \]  
\[ (19) \]  

In order to satisfy the set of boundary conditions, it is convenient to expand the temperature function \( T \) into \( (r, \phi) \) polar co-ordinates or \( (\beta \rho, \theta) \) co-ordinates:

\[ T(\rho, \theta) = \frac{W}{2\pi \lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( a_{m, n} r^{m} \cos m_n \phi \right) \cos(m_n \theta) \]  
\[ (18.a) \]  

\[ -c_{0} \frac{R_{e}}{u_{k} - 1} m \log \left( \frac{R_{e}}{u_{k} - 1} \right) \]  
\[ (\rho < 1) \]  

\[ T(r, \phi) = \frac{W}{2\pi \lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( a_{m, n} r^{m} \cos m_n \phi \right) \cos m_n \theta \]  
\[ (18.b) \]  

where

\[ \tau_{n} = \sum_{r=1}^{n-1} \log(2(1 - \cos 2\pi k/m)) \]  
\[ (x=1) \]  

\[ \alpha_{n, s} = \frac{1}{s} \sum_{s=1}^{n-1} x_{s} \]  
\[ (s) \]  

and \( x_{s} \) is a root of the equation

\[ (x=1) \]  

\[ (20) \]  

\[ (21) \]  

\[ (22) \]  

3.2 Hollow cylinder

Next we consider the two-dimensional, steady-state temperature field in a hollow cylinder containing a ring of uniformly distributed point heat sources of equal intensity \( W \), the exterior and the interior surfaces of which have temperature \( T_\infty \) and \( T_0 \) respectively. Thus the present analysis may be reduced to a solution of heat conduction equation being subjected to the boundary conditions

\[ T = T_\infty \quad r = 1 \]  
\[ (14) \]  

\[ T = T_{0} \quad r = \alpha \]  
\[ (14) \]  

Here Eqs. (13), (15), (16) and (17) described in the case of a solid cylinder can be used too. Considering the geometry, the general solution of Eq. (13) for the case of a hollow cylinder is taken in the following form:
\begin{equation}
T = \frac{W}{2 \pi \alpha} \left[ a_n^r + b_n^r \log r + \sum_{n=1}^{\infty} \left( a_n^{r=\alpha} + b_n^{r=\alpha} \right) \cos mn \phi + \sum_{n=1}^{\infty} a_n^r \left( \frac{mn}{s} \right) \beta^{mn} \rho^s \cos (s \phi) \right] \tag{18'}
\end{equation}

where \( a_n^r, b_n^r \) (n=0, 1, ...) and \( c_n^r \) are unknown coefficients to be determined by the boundary conditions. As in the previous case, the temperature function (18') will be expanded in the \((r, \phi)\) polar co-ordinates and \((\beta, \theta)\) polar co-ordinates:

\begin{align*}
T &= \frac{W}{2 \pi \alpha} \left[ a_n^r + b_n^r \log r + \sum_{n=1}^{\infty} \left( a_n^{r=\alpha} + b_n^{r=\alpha} \right) \cos mn \phi + \sum_{n=1}^{\infty} a_n^r \left( \frac{mn}{s} \right) \beta^{mn} \rho^s \cos (s \phi) \right] \\
&\quad + \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} \left( \frac{mn}{s} \right) \cos (s \phi) \\
&\quad - \sum_{n=1}^{\infty} a_n^r \left( \frac{mn}{s} \right) \beta^{mn} \rho^s \cos (s \phi) \right] \left( \rho < 1 \right) \tag{18-a'}
\end{align*}

\begin{align*}
T &= \frac{W}{2 \pi \alpha} \left[ a_n^r + b_n^r \log r + \sum_{n=1}^{\infty} \left( a_n^{r=\alpha} + b_n^{r=\alpha} \right) \cos mn \phi - \sum_{n=1}^{\infty} a_n^r \left( \frac{mn}{s} \right) \beta^{mn} \rho^s \cos (s \phi) \right] \left( r > \beta \right) \tag{18-b'}
\end{align*}

\begin{align*}
T &= \frac{W}{2 \pi \alpha} \left[ a_n^r + b_n^r \log r + \sum_{n=1}^{\infty} \left( a_n^{r=\alpha} + b_n^{r=\alpha} \right) \cos mn \phi - \sum_{n=1}^{\infty} a_n^r \left( \frac{mn}{s} \right) \beta^{mn} \rho^s \cos (s \phi) \right] \left( r < \beta \right) \tag{18-c'}
\end{align*}

Substituting the temperature functions (18-a'), (18-b') and (18-c') into Eqs. (14)' and (15), we obtain

\begin{align*}
t_a^r &= \frac{2 \pi \alpha T_0}{W} \tag{23} \\
t_b^r &= \left( m \log r + 2 \pi \alpha T_0 / W \right) \tag{24} \\
t_a^r &= \frac{\alpha^{mn} \beta^{mn}}{\left( n \alpha^{mn} - (n-1) \right) \left( n \alpha^{mn} - (n+1) \right)} \tag{25} \quad \text{(n=1, 2, ...)} \\
t_b^r &= \frac{\alpha^{mn} \beta^{mn} \left( n \alpha^{mn} - (n+1) \right)}{\left( n \alpha^{mn} - (n-1) \right)} \tag{26} \quad \text{(n=1, 2, ...)} \\
t_c^r &= 1 \tag{27}
\end{align*}

4. Potential of thermo-elastic displacement

To find the state of stress, we shall use the potential of thermo-elastic displacement \( \phi \), which is determined as a particular solution of the Poisson equation

\begin{equation}
\phi' = \frac{1 + \nu}{1 - \nu} \alpha T \tag{28}
\end{equation}

where \( \nu \) is the Poisson ratio and \( \alpha \) is a coefficient of thermal expansion. The Poisson equation may be reduced to a bi-harmonic equation since \( \phi' = 0 \).

4.1 Solid cylinder

Considering the temperature functions (18-a)', (18-b') and (18-c') the bi-harmonic function \( \phi \) will be expressed in the form

\begin{equation}
\phi = \frac{\alpha W}{8 \pi \alpha} \left[ \sum_{n=0}^{\infty} a_n^{r=\alpha} \cos mn \phi + \sum_{n=1}^{\infty} a_n^r \left( \frac{mn}{s} \right) \beta^{mn} \rho^s \log \rho \right] \tag{29}
\end{equation}

and it takes the following form, apart from trivial terms, owing to singularities on the circle \( r = \beta \),

\begin{align*}
\phi &= \frac{\alpha W}{8 \pi \alpha} \left[ \sum_{n=0}^{\infty} a_n^{r=\alpha} \cos mn \phi + \sum_{n=1}^{\infty} a_n^r \left( \frac{mn}{s} \right) \beta^{mn} \rho^s \cos (s \phi) \right] \\
&\quad + \sum_{n=1}^{\infty} c_n^r \beta^{mn} \rho^s \log \rho \tag{29a}
\end{align*}

where \( \alpha, \beta, \phi \) are determined by the same method as used in the case of a solid cylinder:

\begin{align*}
a_n^r &= a_n^{r=\alpha} - b_n^{r=\alpha} + \frac{m}{n} - \log (n-1) \tag{30} \\
a_n^r &= - \frac{m}{n} \log (n+1) \tag{31} \\
c_n^r &= \frac{1}{\sum_{n=0}^{\infty} \beta^{mn} \rho^s \log \rho} \tag{32}
\end{align*}

4.2 Hollow cylinder

Considering the temperature functions (18-a)', (18-b') and (18-c') we find a potential of thermo-elastic displacement for a case of hollow cylinder expressible in the form

\begin{equation}
\phi = \frac{\alpha W}{8 \pi \alpha} \left[ \sum_{n=0}^{\infty} a_n^{r=\alpha} \cos mn \phi + \sum_{n=1}^{\infty} a_n^r \left( \frac{mn}{s} \right) \beta^{mn} \rho^s \cos (s \phi) \right] \\
+ b_n^r \beta^{mn} \rho^s \log \rho \tag{33}
\end{equation}

and we introduce Airy's stress function satisfying the bi-harmonic equation \( \phi''' = 0 \).

5. Stress function

As the stresses obtained by the potential of thermo-elastic displacement do not satisfy the boundary conditions on the surface of the cylinder, we will introduce Airy's stress function satisfying the bi-harmonic equation \( \phi''' = 0 \).
5-1 Solid cylinder

The stress function is obtained by using relations of co-ordinate systems (3) and (4) with the result:

$$\phi = \frac{E}{1-\nu} \frac{\alpha W}{8\pi \lambda} \left( A r^2 + \sum_{n=1}^{\infty} \left( A_n r^2 + B_n \right) r^{2n} \cos m \theta \right)$$

where $E$ is the Young’s modulus and $A_n$, $A_n$ and $B_n$ ($n=1, 2, \cdots$) are unknown coefficients to be determined by the following boundary conditions:

$$\sigma_r = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \phi}{\partial r} \right) \left( \frac{E}{1+\nu} \right) = 0; \quad r=1$$

$$\tau_{r\phi} = -\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \phi}{\partial \phi} \right) \left( \frac{E}{1+\nu} \right) = 0; \quad r=1$$

Substituting expressions (33) and (39) into expressions (40) and (41) we find

$$A_n = \frac{m^2}{2} (1-\beta^2)$$

$$A_n = \frac{m^2 \beta^n (1-\beta^2)}{(n+1)(n-1)}$$

$$B_n = \frac{\beta^n (\beta^2 + mn(1-\beta^2))}{n(n-1)}$$

5-2 Hollow cylinder

Considering the difference between these two cylinders, we know that the stress function of hollow cylinder is given by the form

$$\phi = \frac{E}{1-\nu} \frac{\alpha W}{8\pi \lambda} \left( A r^2 + D r \log r + \sum_{n=1}^{\infty} \left( A_n r^2 + B_n \right) r^{n+1} + (C_n r^2 + D_n) r^{2n} \cos m \theta \right)$$

where $A_n$, $A_n$, $B_n$, $C_n$, $D_n$ and $D_n$ ($n=1, 2, \cdots$) are unknown coefficients to be determined by the following two sets of boundary conditions:

$$\sigma_r = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \phi}{\partial r} \right) \left( \frac{E}{1+\nu} \right) = 0; \quad r=1$$

$$\tau_{r\phi} = -\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \phi}{\partial \phi} \right) \left( \frac{E}{1+\nu} \right) = 0; \quad r=\alpha$$

Substituting the potential of thermo-elastic displacement (33) and the Airy’s stress function (42) into the boundary conditions, we get

$$A_n = \frac{1}{2} (1-\alpha^2) \left( 2 \alpha - 1 \right) \alpha^{n+1} e^{-2 \alpha \log \alpha} + m (2 \alpha - 1) \alpha^{2n+1} \log \alpha$$

$$D_n = \alpha^{n+1} e^{-2 \alpha \log \alpha} - m (2 \alpha - 1) \alpha^{2n+1} \log \alpha$$

and $A_n$, $B_n$, $C_n$ and $D_n$ are determined by solving the simultaneous equations (47)~(50):

$$\left( mn-2 \right) \left( mn+1 \right) \alpha^{mn} A_{n'} + mn \left( mn-1 \right) \alpha^{mn-1} B_{n'} + mn+2 \left( mn-1 \right) \alpha^{mn-2} C_{n'} + mn+1 \left( mn-1 \right) \alpha^{mn-3} D_{n'} = I_{mn}$$

$$\left( mn-2 \right) \left( mn+1 \right) \alpha^{mn} A_{n'} + mn \left( mn-1 \right) \alpha^{mn-1} B_{n'} + mn+2 \left( mn-1 \right) \alpha^{mn-2} C_{n'} + mn+1 \left( mn-1 \right) \alpha^{mn-3} D_{n'} = J_{mn}$$

$$\left( mn-2 \right) \left( mn+1 \right) \alpha^{mn} A_{n'} + mn \left( mn-1 \right) \alpha^{mn-1} B_{n'} + mn+2 \left( mn-1 \right) \alpha^{mn-2} C_{n'} + mn+1 \left( mn-1 \right) \alpha^{mn-3} D_{n'} = K_{mn}$$

$$\left( mn-2 \right) \left( mn+1 \right) \alpha^{mn} A_{n'} + mn \left( mn-1 \right) \alpha^{mn-1} B_{n'} + mn+2 \left( mn-1 \right) \alpha^{mn-2} C_{n'} + mn+1 \left( mn-1 \right) \alpha^{mn-3} D_{n'} = L_{mn}$$

where

$$I_{mn} = \left( mn-2 \right) \alpha^{mn} a_{n'} - \left( mn+2 \right) \alpha^{mn} b_{n'} + \left( mn-2 \right) \alpha^{mn-1} c_{n'}$$

$$J_{mn} = \left( mn-2 \right) \alpha^{mn} a_{n'} + \left( mn+2 \right) \alpha^{mn} b_{n'} + \left( mn-2 \right) \alpha^{mn-1} c_{n'}$$

$$K_{mn} = \left( mn-2 \right) \alpha^{mn} a_{n'} - \left( mn+2 \right) \alpha^{mn} b_{n'} - \left( mn-2 \right) \alpha^{mn-1} c_{n'}$$

$$L_{mn} = \left( mn-2 \right) \alpha^{mn} a_{n'} + \left( mn+2 \right) \alpha^{mn} b_{n'} - \left( mn-2 \right) \alpha^{mn-1} c_{n'}$$

Thus the stresses in both solid and hollow cylinders can be calculated by the following expressions:
\[ \sigma_r = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{E}{1+\nu} \phi \right) \right), \]
\[ \sigma_\theta = \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{E}{1+\nu} \phi \right) \right), \]
\[ \sigma_z = \frac{1}{r^2} \frac{\partial^2}{\partial r^2} \left( \frac{E}{1+\nu} \phi \right), \]
\[ \sigma_{r\theta} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{E}{1+\nu} \phi \right) \right), \]
\[ \tau_{r\theta} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{E}{1+\nu} \phi \right) \right), \]

6. Numerical results and considerations

The numerical calculations are performed for the case of a solid cylinder by using the results of foregoing analysis. Some typical results are plotted in Figs. 3 to 14. Considerations for the results of temperature and stress fields will be summarized as follows:

Fig. 3 Temperature distribution along the radius \( \phi=0 \)

Fig. 4 Temperature distribution along the radius \( \phi=\pi/m \)

Fig. 5 Cross section of a solid cylinder with a ring of point heat sources (\( m=6 \))

Fig. 6 Distribution of \( \sigma_r \) along the radius \( \phi=0 \) (\( \nu=0.3 \))

Fig. 7 Distribution of \( \sigma_\theta \) along the radius \( \phi=0 \) (\( \nu=0.3 \))

Fig. 8 Distribution of \( \sigma_z \) along the radius \( \phi=\pi/m \) (\( \nu=0.3 \)).
(i) Temperature field

To clarify the relationship between temperature distribution and stress distribution, the values of dimensionless temperature divided by \( W/2\pi\alpha \) were calculated along the radius \( \phi = 0 \) (radius intersecting the point heat source) and the radius \( \phi = \pi/m \) (intermediate radius between two point heat sources). The results are shown in Figs. 3 and 4. Temperature gradient is largest at the vicinities of point heat sources where the high thermal stresses are

Fig. 9 Distribution of \( \sigma_t \) along the radius \( \phi = \pi/m (\nu = 0.3) \)

Fig. 10 Distribution of \( \sigma_t \) along the periphery of the cylinder (\( \nu = 0.3 \))

Fig. 11 Distribution of \( \sigma_t \) along the periphery of the cylinder (\( \nu = 0.3 \))

Fig. 12 Distribution of \( \sigma_t \) along the periphery of the cylinder (\( \nu = 0.3 \))

Fig. 13 Distribution of \( \sigma_t \) along the periphery of the cylinder (\( \nu = 0.3 \))

Fig. 14 Distribution of \( \sigma_t \) along the periphery of the cylinder (\( \nu = 0.3 \))
generated. It becomes smaller as the number of point heat sources increases. The value of temperature at a point heat source diverges to positive infinity. Moreover the values of temperature in the cylinder become high uniformly as the number of point heat sources increases.

(ii) Stress field

The values of dimensionless stresses divided by $EaW/8\pi\lambda$ were calculated along the radii $\phi=0, \phi=\pi/m$ and the periphery of the cylinder with $\nu=0.3$. The results are shown in Figs. 6 to 14. Figures 6 and 7 show the variations of $\sigma_\phi$ and $\sigma_\rho$ along the radius $\phi=0$ with the number of point heat sources $m$ as parameter.

(1) It is observed from these figures that the values of $\sigma_\rho$ and $\sigma_\phi$ become divergent to negative infinity at the point heat sources.

(2) It should be noted that the largest stress next to these stresses is $\sigma_\phi$ generated on the surface of the cylinder, and it becomes larger as the number of point heat sources increases.

(3) The values of $\sigma_\rho$, being similar to that of $\sigma_\phi$, become larger as the number of point heat sources increases.

(4) It is remarkable that the position where the value of $\sigma_\phi$ changes its sign moves outwards a little on account of mutual interference of point heat sources.

(5) The maximum value of $\sigma_\phi$ appears at the points where the radii passing through the point heat sources intersect the periphery of the cylinder. It becomes larger as the position of point heat sources moves more outwards for the case of a small number of point heat sources, and also has tendency to become larger when the position of point heat sources moves towards the center of the cylinder for the case of a large number of point heat sources.

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