Investigations of the Aerodynamic Characteristics of the Shock Tubes*
(Part 2, On the Formation of Shock Waves)

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To explain the mechanism of the shock formation in a simple shock tube, the multi-stage model was presented, which improved White's theory, and some numerical calculations were carried out. Simple and White's theories correspond respectively to the case of zero or one stage of the present multi-stage model. The shock Mach number calculated by the multi-stage model is always larger than that obtained by the White's model. The difference between them becomes large as the diaphragm pressure ratio increases.

Further in order to express the variation of the shock Mach number along the tube a function \( I(\xi) \) was proposed. By using this function, the phenomenon that shocks stronger than that predicted by the simple theory would possibly happen was discussed.

1. Introduction

Actual shock tube flow is, as pointed out in the previous paper, not always consistent with the ideal flow predicted by the simple theory. Especially on the flow near the diaphragm, the opening time of the diaphragm exerts serious influence. In the previous paper, the formation process of a shock front was observed optically near the diaphragm, and the functional relationship between various factors influencing the diaphragm opening time and the shock formation distance have been obtained by using dimensional analysis. As a quantitative method to estimate the effect of the diaphragm opening time on the characteristics of a shock tube, White suggested the "formation from compression" model. The details will be described in the next article. Although his model made an epoch from the point of view that the finite rupture time of the diaphragm was taken into account in the shock formation process, it could not explain the phenomenon of acceleration of the shock front during its formation process, which always occurs in the actual flow. In addition, although the shock Mach numbers calculated by his model agree better with the experimental values than the theoretical values, it has been observed that, at an extremely large diaphragm pressure ratio, the experimental shock Mach number was still larger than the value predicted by his model. For example, in the case where helium and oxygen were used as the high and low pressure gases respectively and the diaphragm pressure ratio was \( 5 \times 10^3 \), the experimental shock Mach number was 11.5, though the value predicted by White's model was 10.4. In the case where the mixed gas of oxygen, hydrogen and helium was used as the high pressure gas and air was used as low pressure gas, the experimental shock Mach number for the diaphragm pressure ratio of \( 3 \times 10^6 \) was about 40, though the value predicted by his model was about 30.

On the other hand the accelerated piston theory in which the shock is formed from a progressive compression wave, conforms closely with the one-dimensional shock formation theory. However the piston theory is an initial value problem with an unknown boundary, and so the analytical solution can not be obtained without the path of the piston, which corresponds to the locus of the contact surface on the (time)-(distance) diagram of shock tube flow. For example, Pillow found an analytical solution for the shock produced by a uniformly accelerated piston. Also Winter found graphically the position and strength of the shock by the piston-operated compressor by using the method of characteristics. But this kind of flow is essentially different from

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the shock tube flow, and cannot be applied to the shock formation problem in the shock tube.

In this paper, the modified White's model in which the shock front was generated and accelerated by the successive compression waves during the formation process, was proposed. And some examples of the numerical calculations by this shock formation model were shown. Further, in order to express the variation of the shock Mach number along the tube, a function was proposed, which was defined as the ratio in percentage of the deviation of the actual shock Mach number from the simple theory to the value predicted by the simple theory. By using this function, the phenomenon that the measured shocks are stronger than that predicted by the simple theory may be generally elucidated.

Nomenclature used in this paper are the same as those used in the previous paper.

2. White's model

In order to estimate the effect of the opening time of the diaphragm on the shock tube flow, White made the following new assumption instead of that described in the simple theory of the previous paper, while retaining the assumptions (1) and (2).

(3) After successive compression waves are generated by the gradual opening of the diaphragm, the individual elements of compression waves will all overtake the leading wave at the same point of the tube and a shock front will be formed.

Since the gas velocity is not the same behind a shock wave as behind a compression wave having the same pressure ratio, a reflected wave is required to satisfy the boundary conditions if the shock is formed as he assumes. It is readily shown that this reflected wave must be an upstream-facing expansion wave and so that the transmitted shock wave formed under his assumption (3) has a lower pressure ratio than the original compression waves. This proposed model of the shock tube flow is depicted in Fig. 1. A thin solid line in Fig. 1 (b) represents the compression wave, and the other lines are the same as in Fig. 1 (b) in the previous paper. Also C_s, C_e and R-H in Fig. 1 (c) represent the relations of simple compression, expansion and Rankine-Hugoniot respectively.

The difference between the simple theory and the White's model lies in that, in Fig. 1 (c), the expansion from the state c is along c→S in simple theory, whereas it is along c→e in the White's model. Therefore, which gives the larger shock Mach number at the same diaphragm pressure ratio depends on the tangents at the state c in Fig. 1 (c) of the equations representing the states c→S and c→e. Let us denote these tangents as

\[ A = \frac{d(p/p_s)}{d(u/a)} \text{Simple} \quad \text{and} \quad B = \frac{d(p/p_s)}{d(u/a)} \text{White} \]

then, after some calculations, we obtain

\[ \lambda = \left( \frac{A}{B} \right) = 1 \]

From this it will be obvious that whether \( \lambda \) is larger than 1 or not depends on the initial condition. For example, in the case where \( \tau_1 = \tau_4 \) and \( a_1 = a_4 \), it becomes

\[ \lambda = \left( \frac{p_4}{p_1} \right)^{\frac{\tau_4-1}{\tau_4}} \]

and so, irrespective of the diaphragm pressure ratio, the shock Mach number calculated by the White's model is larger than the value predicted by the simple theory.

3. Observation of the shock formation process by the pressure-transducer detector

Measurements of the strength of shock front produced by bursting the diaphragm in 38 mm square tube were done by using the strain pressure gauges mounted flush with the tube wall near the diaphragm position. The working conditions were the same as those in the previous paper. The typical oscillograms are represented in Fig. 2, which give pressure-time
records at $x=55$ mm for the various diaphragm pressure ratios. In this figure, $\Delta P_f$ shows the discontinuous increase of the pressure due to the shock front, and the subsequent gradual increase of the pressure $\Delta P_e$ is probably due to the subsequent successive compression waves. Behind the compression waves, the pressure remains nearly constant. As is evident from Fig. 2, the proportion of $\Delta P_f$ to the total pressure increase $\Delta P_e$ ($=\Delta P_f+\Delta P_e$) decreases with an increasing diaphragm pressure ratio. This is caused by the fact that, as described in the previous paper, the shock formation distance $x_f$ increases with an increasing diaphragm pressure ratio. At $x=x_f$, $\Delta P_e$ becomes zero and $\Delta P_f$ is equal to $\Delta P_e$. The experimental values of $\Delta P_f$ obtained from the oscillograms are plotted against $x$ in Fig. 3. The reason why $\Delta P_f$ becomes maximum near $P_{di}=2$, as represented in Fig. 3, is evident from the relation between the diaphragm pressure ratio $P_{di}$ and the pressure rise due to a shock front ($P_2-P_1$) in the case where $P_1$ remains constant. (See Fig. 4) Figure 3 shows that $\Delta P_f$ increases rapidly with an increasing $x$ near the diaphragm, until it reaches the maximum value, and thereafter it decreases monotonously. It was observed that $x$ where $\Delta P_f$ became maximum was nearly equal to the shock formation distance obtained from the experimental values on the shock speed. From these considerations it may be confirmed that in the formative stage of a shock front, at first an extremely weak shock front is generated by a series of compression waves which are produced by the bursting diaphragm and it is accelerated by the successive compression waves. This was one of the conclusions in the previous paper.

4. Multi-stage model

It was pointed out so far that one of the main differences between the formation process of the shock front in the actual shock tube flow and those assumed in the simple theory or the White's model was in the fact that the shock front was accelerated after generated in the actual flow. The flow pattern at the formative stage of the shock front estimated from the oscillograms (Fig. 2) will be depicted as Fig. 5, on the assumption of one-dimensional flow. However this kind of flow cannot be analysed, as described previously, without both the initial conditions and the boundary conditions being given (for example, the locus of the contact surface in $x-t$ diagram).

To take the acceleration phenomenon of the shock front during its formative stage into account, we introduce the following assumption. A series of compression waves produced by the bursting diaphragm are divided into a finite number of groups of compression waves and the individual elements of compression waves belonging to each group coalesce at the same point of the tube. In this way, the shock front is generated by the first group of coalescent compression waves and the subsequent
successive groups accelerate it stepwise. According to this assumption, the shock Mach number can be calculated only by the initial conditions. We propose to call this formation process of a shock front the multi-stage model. The simple and the White's model correspond respectively to the case of zero and one stage of the present multi-stage model. The flow pattern obtained by the infinite-stage model will become very similar to that represented in Fig. 5, and will be expected to be analogous to the formation process in the actual shock tube flow.

In order to calculate the shock Mach number according to this model, in addition to the assumptions (1) and (2) described in the simple theory of the previous paper, we further simplify the calculation by assuming that the mutual interaction between the compression wave by bursting diaphragm and the derivative rarefaction wave or derivative contact surface originated during the formative stage of the shock front can be neglected. This assumption is considered to be reasonable as the strengths of the derivative rarefaction wave and the contact surface become weak with an increasing number of stages in the multi-stage model.

**4.1 Generation of shock front**

In order to simplify the explanation, consider a two-stage model whose flow pattern is shown in Fig. 6. A thick solid line, a thin solid line, a wave line and a broken line in Fig. 6 (b) mean respectively the same as in Fig. 1 (b). And C, C, and R-H in Fig. 6 (c) have the same meanings as in Fig. 1 (c) respectively. In Fig. 6, the state c represents the state just behind the first coalescent compression wave, and the state e is the state just behind the shock front generated by the first coalescent compression wave. A contact surface is derived from the point M in Fig. 6 (b), where the compression waves of the first group coalesce. And in the states e and e' which occur in front and in rear of the contact surface respectively, the gas velocity and the pressure are the same, but the temperature is different. The shock front strength generated thus may calculated by assuming the point 4' appropriately in Fig. 6 (c), and using the same calculating method for the White's model.

**4.2 Acceleration of the shock front**

All subsequent compression waves will accelerate the shock front. Let the point where the second coalescent compression wave overtake the shock front be indicated by N as shown in Fig. 6 (b). Shock front will be accelerated by this coalescent compression wave. In this way the shock Mach number for the given diaphragm pressure ratio will reach the final value, which we define as "final shock Mach
number", whose convergency will be discussed later.

The points S, W and 2 denoted in Fig. 6 (c) represent respectively the shock strengths calculated by the simple theory, the White's model and the two-stage model for the same diaphragm pressure ratio. In order to compare the White's model with the two-stage model, let us calculate the ratio of the tangents of the equations representing the changes of state $f \rightarrow W$ and $f \rightarrow 2$ at the state $f$ in Fig. 6 (c). If we denote

$$C = \frac{d(p/a_0)}{d(u/a_0)}_{Ma_{11}}$$

after some calculations, we obtain

$$\mu = \frac{|B|}{|C|} \frac{\alpha_f}{\alpha_{f'}} \left( \frac{\rho_f}{\rho_{f'}} \right)^{\frac{1}{2}} \frac{1}{M} \left( \frac{\gamma - 1}{2} \right)^{\frac{1}{2}} \left( \frac{T_1 - 1}{2} \right)^{\frac{1}{2}} \frac{1}{M} \left( \frac{T_1 - 1}{2} \right)^{\frac{1}{2}} \text{(5)}$$

Let the shock Mach number generated by the first coalescent compression wave be denoted by $M$. Rearranging Eq. (5) by using $M$ yields

$$\mu = \left( \frac{2}{T_1 + 1} \right)^{\frac{1}{2}} \left( \frac{\gamma - 1}{2} \right)^{\frac{1}{2}} \left( \frac{T_1 - 1}{2} \right)^{\frac{1}{2}} \frac{1}{M} \left( \frac{T_1 - 1}{2} \right)^{\frac{1}{2}} \left( \frac{T_1 - 1}{2} \right)^{\frac{1}{2}} \text{(6)}$$

This equation shows that $\mu = 1$ at $M = 1$ and when $M > 1$, $\partial \mu / \partial M = 0$. Therefore, we may conclude that the shock Mach number formed by the two-stage model will be always larger than that by the White's model at the same diaphragm pressure ratio.

So far the numerical calculations for the two-stage model were explained. But similarly any-stage model may be calculated by using the method of characteristics.

4-3 Final shock Mach number

As described previously, when a certain diaphragm pressure ratio is given, the final shock Mach number calculated by the multi-stage model will become larger with an increasing number of stages. Then the question will be raised: whether the final shock Mach number will converge or not if the number of stages approaches infinity. The answer to this equation is given as follows: As the characteristic C, shown in Fig. 6 (c) has always negative tangent, the final shock Mach number cannot become larger than the Mach number corresponding to the state U shown in Fig. 6 (c). This means that the final shock Mach number converges and approaches a finite value, though it becomes larger with an increasing number of stages.

5. Variation of the shock Mach number along the tube

The main differences between the actual flow and the simple theory may come from the following five points:

(1) imperfect diaphragm rupture,
(2) the effect due to the viscosity and heat conductivity of the working gas, i.e. wall boundary layer growth,
(3) mixing of hot and cold gases in contact region\textsuperscript{(13)},
(4) radiation heat transfer of the working gas\textsuperscript{(7)},
(5) real gas effect of the working gas.

In the present multi-stage model, only the effect of (1) was taken into consideration, while the shock attenuation theory\textsuperscript{(18)} has been developed by taking only the effect (2) into consideration. At present the effects of (3), (4) and (5) on the shock Mach number cannot be estimated exactly, and those effects will be studied hereafter. Hence in order to discuss the variation of the shock Mach number along the tube by taking the effects of (1) and (2) into account, let us define $\Pi(x)$ as the ratio in percentage of the deviation of the actual shock Mach number $M_1$ from the value $M_{11a}$ obtained by the simple theory to $M_{11a}$, namely

$$\Pi(x) = 100(M_1 - M_{11a})/M_{11a} \text{(7)}$$

Let us further define $\Pi_1$ as the increment of the shock Mach number due to the imperfect diaphragm rupture and $\Pi_2$ as the decrement due to the viscous and heat conductive effects of working gases. Then $\Pi(x)$ may be written as the difference between $\Pi_1$ and $\Pi_2$, namely

$$\Pi = \Pi_1 - \Pi_2 \text{(8)}$$

The properties of $\Pi_1$ and $\Pi_2$ will be discussed later. Figure 7 shows the relation between $M_1/M_{11a}$ and $\Pi$ along the tube axis. Further, when the rarefaction wave derived during the formation process of a shock front arrives at the contact surface, the rarefaction wave is reflected, which will overtake the shock front and decelerate its speed\textsuperscript{(10)}. Above effect cannot be estimated without both initial and boundary conditions, and further the variation of the shock Mach number due to this effect may be small as compared with $\Pi_1$ and $\Pi_2$. Therefore it is neglected in this work.
5.1 Properties of $\Pi_1$

As shown in Fig. 7, $\Pi_1$ becomes constant at $x$ larger than $x_f$ and its value is, according to the multi-stage model, dependent only on the diaphragm pressure ratio, that is,

$$\Pi_1 = \Pi_1(P_d) \quad \text{at} \quad x \geq x_f \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots 
6.1 Effects of the number of stages

Although the shock Mach number $M_1$, calculated by the multi-stage model becomes larger with an increasing number of the stages $K$, $M_1$ always converges, and so it is not necessary to take $K$ so large in the practical calculations. Table 2 shows the effects of $K$ on $M_1$ for the case where air is used as high and low pressure gases. The cases of $K=0$ and 1 are equivalent respectively to the values calculated by the simple theory and the White's model and the others correspond to the number of stages where the Mach number of the shock, which is generated and accelerated stepwise by the successive groups of compression waves, increases 0.1 and 0.01 at a time. As evident from Table 2, for example, in the case where $P_{\text{in}}=6.33 \times 10^6$, the increases of $M_1$ are $1.89 \times 10^{-1}$% and $3.91 \times 10^{-3}$% for the variations of $K=1 \rightarrow 22$ and $K=22 \rightarrow 209$ respectively. Also in the case where $P_{\text{in}}=2.27 \times 10^6$, the increases of $M_1$ are $1.55 \times 10^{-3}$% and $3.91 \times 10^{-3}$% for the variations of $K=1 \rightarrow 48$ and $K=48 \rightarrow 465$ respectively. From these considerations, if $K$ is chosen larger than 50, the error of $M_1$ will be less than $10^{-3}$%, and the shock Mach number thus obtained can be approximately regarded as the value in the case of infinite-stage model. We chose such value of $K$ in the following calculations.

6.2 Calculations of $\Pi_1$

Let us denote the shock Mach numbers which are calculated by the simple, White's and multi-stage theories as $M_{\text{sh}}, M_{\text{sw}}$, and $M_{\text{sm}}$ respectively. The numerical calculations were carried out for various diaphragm pressure ratios, whose results are shown in Figs. 9 and 10. In these figures, He/Air, for example, denotes helium gas and air used as high and low pressure gases respectively. For all cases, $M_{\text{sh}}$ is larger than $M_{\text{sw}}$ and its difference becomes larger as $P_{\text{in}}$ increases. This is due to the fact that $\mu$ defined by Eq. (5) is always larger than 1 and, as evident from Eq. (6), $\mu$ increases monotonously with an increasing $M$. In the case where $\gamma_1=\gamma_4$ and $a_1=a_4$ (Air/Air), as pointed out previously, it may be concluded that the relation $M_{\text{sh}}>M_{\text{sw}}>M_{\text{sm}}$ exists for any diaphragm pressure ratio, because $\lambda$ defined by Eq. (2) is always larger than 1. On the other hand, in the case where $\gamma_1=\gamma_4$ and $a_1=a_4$ (H$_2$/Air, He/A), the relation $M_{\text{sh}}>M_{\text{sm}}>M_{\text{sw}}$ holds within the range of $P_{\text{in}}$ from 1 to a certain value. This fact may be explained as follows. From Eq. (2), $\lambda$ may be written as

$$\lambda = \frac{a_1}{a_4} \left( \frac{P_{\text{in}}}{P_{\text{sh}}} \right)^{\gamma_1-1}$$

where $a_1/a_4$ is constant and less than 1, while

<table>
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<th>$P_{\text{in}}$</th>
<th>$K$</th>
<th>$M_1$</th>
<th>$P_{\text{in}}$</th>
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\( \lambda = \frac{\tau_2}{\tau_1} \left( \frac{p_{21}}{p_{22}} \right)^{\frac{1}{\gamma_1-1}} \left( \frac{p_{e1}}{p_{e2}} \right)^{\frac{1}{\gamma_2-1}} \) becomes larger with an increasing \( P_1 \). Hence there will exist \( 0 < \lambda < 1 \) within the range of \( P_1 \) from 1 to a certain value. In the case where \( \gamma_1 = \gamma_2 \) and \( a_1 = a_2 \) (He/Air, H\(_2\)/A), we obtain from Eq. (2)

\[
\lambda = \frac{\tau_2}{\tau_1} \frac{a_2}{a_1} \left( \frac{p_{21}}{p_{22}} \right)^{\frac{1}{\gamma}} \left( \frac{p_{e1}}{p_{e2}} \right)^{\frac{1}{\gamma}} \tag{18}
\]

and according to \( \lambda \leq 1 \), the relation \( M_{10b} < M_{1w} < M_{1m} \) holds. Further, as evident from Fig. 9 and 10, in the case where \( (M_{1m})_{\text{max}} \) is small (for example, Air/Air), there is only small difference among the shock Mach numbers which are calculated by the simple, White’s and multi-stage theories, while in the other cases a large difference is observed among them, especially for the diaphragm pressure ratio larger than about \( 10^4 \). The shock Mach number considerably larger than the value predicted by the simple theory for the diaphragm pressure ratio larger than \( 10^6 \) was observed in the experiments by Nagamatsu et al.\(^{(3)}\). Hence the shock Mach number for a large diaphragm pressure ratio should be calculated by the present multi-stage model where the acceleration phenomenon of the shock front during its formative stage is considered.

Let \( \Pi \) calculated by the White’s and the multi-stage models be indicated by \( \Pi_{1w} \) and \( \Pi_{1m} \) respectively. That is,

\[
\Pi_{1w} = 100 \left( \frac{M_{1w} - M_{1m}}{M_{1m}} \right), \quad \Pi_{1m} = 100 \left( \frac{M_{1m} - M_{10b}}{M_{1m}} \right) \tag{19}
\]

Transforming \( M_{1w} \) and \( M_{1m} \) of Figs. 9 and 10 into \( \Pi_{1w} \) and \( \Pi_{1m} \) according to Eq. (19) and taking \( M_{10b} \) as an abscissa, we obtain Figs. 11 and 12. As \( \Pi \) calculated by the simple theory is zero, the abscissas in these figures correspond to the simple theory. The correlation between these theories will be recognized more clearly by this transformation. That is to say, for all cases except the case of Air/Air, \( \Pi_{1m} \) and \( \Pi_{1w} \) are zero at \( M_{10b} = 1 \), and they at first become negative with an increasing \( M_{1w} \). And after they become minimum at a certain \( M_{1w} \) they increase rapidly with an increasing \( M_{1w} \). For the same combination of working gases, \( M_{10b} \) where \( \Pi_{1m} \) becomes minimum at (or zero) is smaller than \( M_{1w} \) where \( \Pi_{1w} \) becomes minimum (or zero). For the case of various combinations of working gases, \( M_{10b} \) where \( \Pi_{1m} \) or \( \Pi_{1w} \) becomes minimum (or zero) increases with an increasing \( (M_{1w})_{\text{max}} \). The minimum value of \( \Pi_{1m} \) or \( \Pi_{1w} \) becomes smaller as \( (M_{1w})_{\text{max}} \) becomes larger.

Denoting the difference between \( \Pi_{1m} \) and \( \Pi_{1w} \) for a certain diaphragm pressure ratio as \( \Delta \Pi \), that is,

\[
\Delta \Pi = \Pi_{1m} - \Pi_{1w} \tag{20}
\]

and calculating \( \Delta \Pi \) from Figs. 11 and 12, we obtain Fig. 13. We obtain the following results from this figure.

(1) For all cases, \( \Delta \Pi \) is zero at \( M_{10b} = 1 \) and increases rapidly with an increasing \( M_{1w} \). As \( M_{10b} \) approaches \( (M_{1w})_{\text{max}} \), the rate of increase of \( \Delta \Pi \) decreases.

(2) Only small discrepancy owing to the difference of the combinations of working gases is recognized when \( M_{10b} \) is less than about 3.
(3) For the case where \( \gamma_1 \) or \( \gamma_4 \) remains constant, \( \Delta P \) at the same value of \( M_{\text{th}} \) becomes smaller as \( (M_{\text{th}})_{\text{max}} \) becomes larger.

7. Comparison between the multi-stage model and the actual flow

The multi-stage model can explain well the formation process of the shock front by the bursting diaphragm in the shock tube, yet further investigations will be necessary on the following points:

(1) The burst of the diaphragm is actually a three-dimensional phenomenon. Hence the shock front generated by a bursting diaphragm is not a plane but a curved surface, and the contact surface becomes a cone-like contact region\(^{[1]}\). Contact region is assumed as one-dimensional in the multi-stage model.

(2) Generating rate of the successive compression waves during the bursting process of the diaphragm may vary with the variation of the time of the opening of the diaphragm. Hence the formation of the stage in the multi-stage model should be determined by taking above fact into consideration. Campbell et al.\(^{[9]}\) obtained the variation with time of the opening of aluminium diaphragm experimentally. However, since the data on other diaphragms are insufficient, we cannot discuss this effect in detail.

(3) Theory of the shock attenuation\(^{[3]}\) assumes that the bursting of the diaphragm is instantaneous, and so the shock Mach number \( M_i \) at the diaphragm location is equivalent to the value \( M_{\text{th}} \) predicted by the simple theory. Meanwhile in the actual flow, even if the working gas is assumed to be ideal, \( M_i \) is not equivalent to \( M_{\text{th}} \), where \( 0 \leq \gamma_i \leq \gamma_f \). In Eq. (8) we defined \( \Pi_1 \) and \( \Pi_2 \), which deny the assumptions (2) or (3) respectively in the simple theory. In the actual flow where both assumptions (2) and (3) do not hold good, we must consider the interference term between \( \Pi_1 \) and \( \Pi_2 \). This term will have the effect to decrease \( \Pi_2 \), but at present it cannot be estimated quantitatively.

8. Summary

To explain the mechanism of the shock formation in the simple shock tube, the multi-stage model is presented which improves White’s theory. Simple and White’s theories correspond respectively to the case of zero or one stage of the present multi-stage model. The flow pattern obtained by infinite-stage model is very analogous to the actual flow in a shock tube where the shock front is generated and accelerated by the successive compression waves. As some examples, theoretical shock Mach numbers obtained by simple, White’s and multi-stage theories by use of hydrogen, helium or air as driver gases and air or argon as driven gases are compared. At a constant diaphragm pressure ratio, the shock Mach number which is obtained by the multi-stage model is larger than the White’s model. The differences between them become larger as the diaphragm pressure ratio increases.

Further, the function \( \Pi(\alpha) \) which expresses the distribution of shock Mach number along the tube is proposed. \( \Pi(\alpha) \) is obtained from the difference between \( \Pi_i \) and \( \Pi_f \), where \( \Pi_i \) is the increment of the shock Mach number due to the imperfect diaphragm rupture and is calculated from multi-stage model, and \( \Pi_f \) is the decrement due to the viscous and the heat conductive effects of working gases and is calculated from the theory of shock attenuation. The maximum value of \( \Pi(\alpha) \) along the tube is obtained at the location where the shock is formed. The problem whether the maximum shock Mach number along the tube becomes larger than the value predicted by the simple theory or not corresponds to whether the maximum value of \( \Pi(\alpha) \) is larger than zero or not.

As the shock formation process in the shock tube is a complicated three-dimensional phenomenon, the shock Mach numbers calculated by the present multi-stage model will not necessarily be exact. However, using this model the shock tube flow may be explained more quantitatively than by the conventional theories.
References