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Buckling of a Cylindrical Shell Reinforced by Spiral Stiffeners under External Pressure*

By Yoshiyuki YAMAMOTO**, Katsuhiko WATANABE***,
and Tokio NAKAMURA****

Buckling of a cylindrical shell reinforced by spiral stiffeners under external pressure was investigated theoretically and experimentally. Critical buckling pressure is determined by the energy method, and is almost identical with that for a cylindrical shell reinforced by ring stiffeners spaced with the same axial distance. The buckling pressure obtained by experiments is fairly below the theoretical one in the case of closely spaced stiffeners, and it is shown that plastic deformations due to initial imperfections cause such a decrease of the buckling pressure.

1. Introduction

A cylindrical shell subjected to external pressure is usually reinforced by stiffeners to prevent buckling failure. Not only ring stiffeners but also spiral ones are used for this purpose. The buckling of cylindrical shells reinforced by ring stiffeners has been investigated by many authors experimentally and theoretically^{(1)~(6)}, and the buckling strength can be predicted exactly. As for cylindrical shells reinforced by spiral stiffeners, few investigations have been conducted on their buckling strength. When the rigidity of stiffeners is sufficiently large, the buckling occurs in the shell plate alone and stiffeners remain undeformed. This kind of buckling is called shell buckling and is investigated experimentally and theoretically in this paper for a cylindrical shell reinforced by spiral stiffeners under external pressure.

Cylindrical shells used for experiments were

* Received 6th April, 1968.

** Professor, Faculty of Engineering, University of Tokyo.

*** Graduate Student, University of Tokyo.

**** Engineer, Toyo Rayon Co. Ltd., Nagoya.

fabricated by welding with mild steel plates as shown in Fig. 1. The shell plates are 1.6 mm in thickness, and are reinforced by spiral stiffeners of 5.0 mm in thickness and 25.0 mm in depth. The spiral angle α and the number of the spiral stiffeners m are also shown in Table 1. Experiments

Fig. 1 Test model

were conducted on a tank as shown in Fig. 2.

As soon as the water pressure arrived at a certain value, the pressure went down suddenly with a noise. This value of pressure q was adopted as the critical buckling pressure for the experiment, and the results are shown in Table 1. In the Table, nondimensional pressure $qR/2Et$ is used, where R is the radius, t is the thickness and E is Young's modulus of the plates.

In the appearance of the tested model, a dent as shown in Fig. 3 (the shaded part) was observed. Since the dent was formed by plastic deformation, we cannot determine the actual buckling pattern through the results of experiments. For reference,

Fig. 2 Tank

Table 2

Young's modulus	kg/cm ²	2.05×10^8
Yield stress	kg/cm ²	3.05×10^3
Tensile strength	kg/cm ²	3.75×10^3

the observed values of β^0 and l_2^0 (cf. Fig. 3), are shown in Table 1. Figure 4 is a photo of a tested model.

2. Theoretical analysis

2.1 Fundamental equations

Consider a thin cylindrical shell of thickness t and radius R , and assume that R/t is large enough (cf. Fig. 5). The orthogonal Cartesian coordinates, x and y , are taken in the axial and circumferential directions in the developed middle surface of the

Fig. 3

Fig. 4 Tested model ($\alpha=15^\circ$)

Table 1

Model number	Spiral angle α (degree)	Number of spiral m	Length L mm	Experiment			Small displacement theory			Finite displacement theory ($A_0/t=0.2$)		
				β^0 (degree)	l_2^0 mm	Buckling pressure ($\times R/2Et$)	β (degree)	l_2 mm	Buckling pressure ($\times R/2Et$)	β (degree)	l_2 mm	Buckling pressure ($\times R/2Et$)
1	9	1	300	12	94	2.92×10^{-4}	1.17	63.8	4.92×10^{-4}	1.42	65.4	2.90×10^{-4}
2	12	1	350	8	108	2.70×10^{-4}	1.17	72.8	3.66×10^{-4}	1.50	75.0	2.58×10^{-4}
3	15	1	450	7	130	2.32×10^{-4}	1.42	83.3	2.91×10^{-4}	1.58	86.1	2.37×10^{-4}
4	18	1	550	6	142	2.07×10^{-4}	1.42	91.9	2.42×10^{-4}	2.00	94.3	2.18×10^{-4}
5	21	1	600	5	134	1.94×10^{-4}	1.17	99.5	2.08×10^{-4}	2.00	103.6	2.01×10^{-4}
6	21	2	350	11	116	2.93×10^{-4}	2.42	73.6	4.23×10^{-4}	3.00	75.6	2.73×10^{-4}
7	24	2	400	7	121	2.59×10^{-4}	2.58	80.6	3.67×10^{-4}	3.25	82.9	2.52×10^{-4}
8	27	2	400	4	138	2.40×10^{-4}	2.83	89.6	3.23×10^{-4}	3.00	92.2	2.37×10^{-4}

cylindrical shell, respectively. The coordinate z is taken in the normal direction to the middle surface. The displacements U, V, W at a generic point in the shell can be expressed in terms of the displacements $\bar{U}, \bar{V}, \bar{W}$ in the middle surface as follows:

$$\left. \begin{aligned} U &= \bar{U} - z \frac{\partial \bar{W}}{\partial x} \\ V &= \bar{V} - z \frac{\partial \bar{W}}{\partial y} \\ W &= \bar{W} \end{aligned} \right\} \dots\dots\dots (1)$$

The strain components in the direction of the middle surface are given by

$$\left. \begin{aligned} \epsilon_x &= \bar{\epsilon}_x - z \frac{\partial^2 \bar{W}}{\partial x^2} \\ \epsilon_y &= \bar{\epsilon}_y - z \frac{\partial^2 \bar{W}}{\partial y^2} \\ \gamma &= \bar{\gamma} - 2z \frac{\partial^2 \bar{W}}{\partial x \partial y} \end{aligned} \right\} \dots\dots\dots (2)$$

where $\bar{\epsilon}_x, \bar{\epsilon}_y$ and $\bar{\gamma}$ are the strains in the middle surface and are expressed in the forms

$$\left. \begin{aligned} \bar{\epsilon}_x &= \frac{\partial \bar{U}}{\partial x} + \frac{1}{2} \left(\frac{\partial \bar{W}}{\partial x} \right)^2 \\ \bar{\epsilon}_y &= \frac{\partial \bar{V}}{\partial y} + \frac{1}{2} \left(\frac{\partial \bar{W}}{\partial y} \right)^2 + \frac{W}{R} \\ \bar{\gamma} &= \frac{\partial \bar{U}}{\partial y} + \frac{\partial \bar{V}}{\partial x} + \frac{\partial \bar{W}}{\partial x} \frac{\partial \bar{W}}{\partial y} \end{aligned} \right\} \dots\dots\dots (3)$$

Elimination of $\bar{U}$ and $\bar{V}$ from Eqs. (3) yields the condition of compatibility in the middle surface

$$\frac{\partial^2 \bar{\epsilon}_x}{\partial y^2} + \frac{\partial^2 \bar{\epsilon}_y}{\partial x^2} - \frac{\partial^2 \bar{\gamma}}{\partial x \partial y} = \frac{1}{R} \frac{\partial^2 W}{\partial x^2} + \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 - \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \dots\dots\dots (4)$$

The stress resultants N_x, N_y , and N_{xy} should satisfy the following condition of equilibrium in the middle surface:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0. \dots\dots\dots (5)$$

The stress resultants which satisfy Eqs. (5) can be expressed in terms of an appropriate stress function F ;

$$N_x = \frac{\partial^2 F}{\partial y^2}, \quad N_y = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \dots\dots\dots (6)$$

Fig. 5

The strains in the middle surface can be expressed in the form as

$$\left. \begin{aligned} \bar{\epsilon}_x &= \frac{1}{Et} (N_x - \nu N_y) = \frac{1}{Et} \left(\frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} \right) \\ \bar{\epsilon}_y &= \frac{1}{Et} (N_y - \nu N_x) = \frac{1}{Et} \left(\frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{\partial y^2} \right) \\ \bar{\gamma} &= \frac{2(1+\nu)}{Et} N_{xy} = -\frac{2(1+\nu)}{Et} \frac{\partial^2 F}{\partial x \partial y} \end{aligned} \right\} \dots\dots\dots (7)$$

where E and ν are Young's modulus and Poisson's ratio. Substitution of $\bar{\epsilon}_x, \bar{\epsilon}_y$, and $\bar{\gamma}$ from Eqs. (7) into Eqs. (4) yields the condition of compatibility

$$\frac{1}{Et} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 F = \frac{1}{R} \frac{\partial^2 W}{\partial x^2} + \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 - \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \dots\dots\dots (8)$$

Consider a certain equilibrium state S_0 of the dynamical system consisting of the reinforced shell and external pressure. Without loss of generality, it can be assumed that the potential energy Π and its first variation vanish in the equilibrium state. Let u, v , and w be the additional displacements from the equilibrium configuration to a generic configuration. If terms of higher order are ignored, the potential energy Π is expressed in a quadratic term of displacement, and consists of three quadratic expressions Π_e, Π_σ , and Π_q :

$$\Pi = \Pi_e + \Pi_\sigma + \Pi_q \dots\dots\dots (9)$$

Π_e is of the same form as the strain energy due to the displacements (u, v, w) . Π_σ is the work done by the stress in the equilibrium state S_0 on the quadratic part of the additional strain, and Π_q is the quadratic part of the work done by the external pressure q on the decrease of the volume of the shell. The three contributions Π_e, Π_σ , and Π_q are given by

$$\begin{aligned} \Pi_e &= \frac{1}{2Et} \iint \left[\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right)^2 + 2(1+\nu) \right. \\ &\quad \times \left. \left\{ \left(\frac{\partial^2 F}{\partial x \partial y} \right)^2 - \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 F}{\partial y^2} \right\} \right] dx dy \\ &\quad + \frac{D}{2} \iint \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{w}{R^2} \right)^2 + 2(1-\nu) \right. \\ &\quad \times \left. \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial^2 w}{\partial y^2} - \frac{w}{R^2} \right) \right\} \right] dx dy \dots\dots\dots (10) \end{aligned}$$

$$\begin{aligned} \Pi_q &= q \iint \left[\frac{1}{2} w \frac{\partial u}{\partial x} + \frac{1}{2} w \frac{\partial v}{\partial y} + \frac{1}{2} \frac{w^2}{R} \right. \\ &\quad \left. + \frac{1}{2} \frac{v^2}{R^2} \right] dx dy \dots\dots\dots (11) \end{aligned}$$

$$\begin{aligned} \Pi_\sigma &= \iint \left[N_x \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + N_y \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right. \\ &\quad \left. + N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy \dots\dots\dots (12) \end{aligned}$$

where D is $Et^3/[12(1-\nu^2)]$. As the external pressure acts on all sides of the shell, the stress resultants in the state S_0 are given approximately by

$$N_x = -qR/2, \quad N_y = -qR, \quad N_{xy} = 0. \dots\dots\dots (13)$$

Substituting N_x , N_y , N_{xy} from Eq. (13) into Eq. (11), we have

$$\Pi_\sigma = -q \int \int \left[\frac{R}{4} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{R}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] dx dy. \quad (14)$$

Ignoring the small terms in Eq. (12), we can rewrite the expression for Π_q as

$$\Pi_q = q \int \int \left[\frac{1}{2} \frac{w^2}{R} \right] dx dy. \quad (15)$$

In the following, the deformation of a cylindrical shell with initial deflection will be discussed. The total displacements U , V , and W at a generic point in the shell can be expressed as

$$U = u_0 + u, \quad V = v_0 + v, \quad W = w_0 + w, \quad (16)$$

where u_0 , v_0 and w_0 are the initial displacements and u , v , and w are the additional displacements. Denoting the quantity in the middle surface with a bar, we have the strains in the middle surface:

$$\left. \begin{aligned} \bar{\epsilon}_x &= \frac{\partial \bar{u}}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \bar{\epsilon}_y &= \frac{\partial \bar{v}}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial w_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 + \frac{w}{R} \\ \bar{\gamma} &= \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w_0}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial w_0}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{aligned} \right\} \quad (17)$$

By the similar way, the condition of compatibility can be expressed in the form as

$$\frac{1}{Et} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 F^* = 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w_0}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{R} \frac{\partial^2 w}{\partial x^2}, \quad (18)$$

where F^* is the stress function corresponding to the strains given by Eqs. (17). The potential energy Π^* can be obtained approximately in the form similar to Eq. (9):

$$\Pi^* = \Pi_e^* + \Pi_\sigma^* + \Pi_q^*, \quad (19)$$

where Π_e^* , Π_σ^* and Π_q^* are given by

$$\begin{aligned} \Pi_e^* &= \frac{1}{2Et} \int \int \left[\left(\frac{\partial^2 F^*}{\partial x^2} + \frac{\partial^2 F^*}{\partial y^2} \right)^2 + 2(1+\nu) \left\{ \left(\frac{\partial^2 F^*}{\partial x \partial y} \right)^2 - \frac{\partial^2 F^*}{\partial x^2} \frac{\partial^2 F^*}{\partial y^2} \right\} \right] dx dy \\ &\quad + \frac{D}{2} \int \int \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{w}{R^2} \right)^2 + 2(1-\nu) \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial^2 w}{\partial y^2} - \frac{w}{R^2} \right) \right\} \right] dx dy \quad (20) \end{aligned}$$

$$\Pi_\sigma^* = -q \int \int \left[\frac{R}{4} \left\{ \left(\frac{\partial w}{\partial x} \right)^2 + 2 \frac{\partial w}{\partial x} \frac{\partial w_0}{\partial x} \right\} + \frac{R}{2} \left\{ \left(\frac{\partial w}{\partial y} \right)^2 + 2 \frac{\partial w}{\partial y} \frac{\partial w_0}{\partial y} \right\} \right] dx dy \quad (21)$$

$$\Pi_q^* = q \int \int \left(\frac{1}{2} \frac{w^2}{R} + \frac{w w_0}{R} \right) dx dy. \quad (22)$$

The energy method will be used as a method of approach to the present problem. As the first step, the displacement pattern w should be assumed in a rational form with unknown parameters. Substitution of the assumed displacement w into the condition of compatibility yields the equation to

determine the stress function F , and the expression for the potential energy can be determined in terms of the parameters in the expression of w . The equilibrium state, therefore, can be determined by the principle of minimum potential energy. Since the stiffeners are not in the circumferential direction, it is convenient that the coordinates, ξ and η , are introduced on the middle surface as shown in Fig. 6. The coordinates (x, y) and (ξ, η) are related by

$$\begin{aligned} x &= \xi \cos \beta + \eta \sin \alpha \\ y &= \xi \sin \beta + \eta \cos \alpha \end{aligned} \quad (23)$$

where α is the value of spiral angle. For the sake of convenience, it is assumed that the displacement pattern w is expressed in the following form;

$$w = -A \sin \frac{\pi \xi}{l_1} \sin \frac{\pi \eta}{l_2}, \quad (24)$$

where A is an amplitude, l_1 and l_2 are the half wave lengths of the pattern in the ξ - and η -directions. Let m be the number of stiffeners. Then the width l_0 of unstiffened part of the shell in the

Fig. 6

axial direction is given by

$$l_0 = 2\pi R \tan \alpha / m. \quad (25)$$

Then l_1 and l_2 are given by

$$l_1 = l_0 \cos \alpha / \cos(\alpha + \beta), \quad l_2 = n l_0, \quad (26)$$

where n is a positive constant. The displacement given by Eq. (24) satisfies the condition that the shell is simply supported along the stiffeners, and can be rewritten as

$$w = \frac{A}{2} \{ \cos(C_+ x + D_+ y) - \cos(C_- x + D_- y) \}, \quad (27)$$

where

$$\left. \begin{aligned} C_{\pm} &= \frac{\pi}{\cos(\alpha + \beta)} \left(\frac{\cos \alpha}{l_1} \mp \frac{\sin \beta}{l_2} \right) \\ D_{\pm} &= -\frac{\pi}{\cos(\alpha + \beta)} \left(\frac{\sin \alpha}{l_1} \mp \frac{\cos \beta}{l_2} \right) \end{aligned} \right\} \quad (28)$$

2.2 Critical buckling pressure

For the sake of simplicity, it is assumed that the length of cylindrical shell is infinitely large and the buckling is not influenced by both ends. In order to determine the critical buckling pressure, nonlinear terms in the fundamental equations can be ignored, and the condition of compatibility

becomes as follows;

$$\frac{1}{Et} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 F = \frac{1}{R} \frac{\partial^2 w}{\partial x^2} \quad (29)$$

Substitution of w from Eq. (27) into Eq. (29) yields a equation for F . A particular solution F_0 of the resulting equation is obtained as

$$F_0 = -\frac{EtA}{2R} \left\{ \left(\frac{C_+}{C_+^2 + D_+^2} \right)^2 \cos(C_+ x + D_+ y) - \left(\frac{C_-}{C_-^2 + D_-^2} \right)^2 \cos(C_- x + D_- y) \right\}, \quad (30)$$

and the general solution F is given by the sum of F_0 and a harmonic function F_1 . The stress function F is determined by the boundary conditions for (u, v) or (N_x, N_y, N_{xy}) . The lateral rigidity of the stiffeners for the displacements in the middle surface is small, and the buckling pattern u, v, w should be the periodic functions in the ξ - and η -directions. The particular solution F_0 satisfies the condition of periodicity, and then the solution F is given by

$$F = F_0. \quad (31)$$

Introducing Eqs. (27) and (31) into Eqs. (10), (14) and (15), we have the expressions Π_e and $\Pi_o + \Pi_q$:

$$\begin{aligned} \Pi_e &= \frac{A^2 l_1 l_2}{16R^2} \cos(\alpha + \beta) Et \left[\left(\frac{C_+^2}{C_+^2 + D_+^2} \right)^2 + \left(\frac{C_-^2}{C_-^2 + D_-^2} \right)^2 \right] \\ &\quad + \frac{A^2 l_1 l_2}{16R^2} \cos(\alpha + \beta) D \left[R^2 \{ (C_+^2 + D_+^2)^2 + (C_-^2 + D_-^2)^2 \} + \frac{2}{R^2} + 2 \{ (D_+^2 + D_-^2) + \nu(C_+^2 + C_-^2) \} \right] \quad (32) \end{aligned}$$

$$\Pi_o + \Pi_q = -q \frac{A^2 l_1 l_2}{32R} \cos(\alpha + \beta) [R^2 \{ (C_+^2 + C_-^2) + 2(D_+^2 + D_-^2) \} - 4]. \quad (33)$$

The total potential energy Π is given by

$$\Pi(A, n, \beta, q) = \Pi_e + \Pi_o + \Pi_q = (B_3 - B_4 q) A^2, \quad (34)$$

where

$$\begin{aligned} B_3 &= \frac{l_1 l_2}{16R^2} \cos(\alpha + \beta) Et \left[\left\{ \left(\frac{C_+^2}{C_+^2 + D_+^2} \right)^2 + \left(\frac{C_-^2}{C_-^2 + D_-^2} \right)^2 \right\} \right. \\ &\quad \left. + \frac{t^2}{12(1-\nu^2)} \left[R^2 \{ (C_+^2 + D_+^2)^2 + (C_-^2 + D_-^2)^2 \} + \frac{2}{R^2} + \{ (D_+^2 + D_-^2) + \nu(C_+^2 + C_-^2) \} \right] \right] \quad (35) \end{aligned}$$

$$B_4 = \frac{l_1 l_2}{32R} \cos(\alpha + \beta) [R^2 \{ (C_+^2 + C_-^2) + 2(D_+^2 + D_-^2) \} - 4]. \quad (36)$$

The potential energy Π is a function of A, n, β and q , and an equilibrium state is obtained by the condition that Π takes an extremum value. It then follows that

$$\frac{\partial \Pi}{\partial A} = \frac{\partial \Pi}{\partial n} = \frac{\partial \Pi}{\partial \beta} = 0. \quad (37)$$

It can be assumed for the analysis of buckling that A takes a nonvanishing value. Substituting Π from Eq. (34) into Eq. (37), we obtain

$$B_3 - B_4 q = 0, \quad \frac{\partial B_3}{\partial n} - \frac{\partial B_4}{\partial n} q = 0, \quad \frac{\partial B_3}{\partial \beta} - \frac{\partial B_4}{\partial \beta} q = 0. \quad (38)$$

The critical pressure q_{cr} is determined by the condition that Eqs. (38) hold simultaneously. In reality, the critical buckling pressure q_{cr} is conveniently obtained by the following direct method:

$$q_{cr} = \min_{n, \beta} q(n, \beta), \quad (39)$$

where $q(n, \beta)$ is obtained as the solution of the first of Eqs. (38) in the following form;

$$\begin{aligned} q(n, \beta) &= \frac{B_3}{B_4} = \frac{2Et}{R} \frac{1}{R^2 \{ (C_+^2 + C_-^2) + 2(D_+^2 + D_-^2) \} - 4} \left[\left\{ \left(\frac{C_+^2}{C_+^2 + D_+^2} \right)^2 + \left(\frac{C_-^2}{C_-^2 + D_-^2} \right)^2 \right\} \right. \\ &\quad \left. + \frac{t^2}{12(1-\nu^2)} \left[R^2 \{ (C_+^2 + D_+^2)^2 + (C_-^2 + D_-^2)^2 \} + \frac{2}{R^2} + 2 \{ (D_+^2 + D_-^2) + \nu(C_+^2 + C_-^2) \} \right] \right] \quad (40) \end{aligned}$$

The extremum condition of Eq. (40) yields the equations

$$\left. \begin{aligned} \frac{\partial q}{\partial n} = \frac{\partial}{\partial n} \left(\frac{B_3}{B_4} \right) &= \frac{\frac{\partial B_3}{\partial n} - \frac{\partial B_4}{\partial n} q}{B_4} = 0 \\ \frac{\partial q}{\partial \beta} = \frac{\partial}{\partial \beta} \left(\frac{B_3}{B_4} \right) &= \frac{\frac{\partial B_3}{\partial \beta} - \frac{\partial B_4}{\partial \beta} q}{B_4} = 0 \end{aligned} \right\} \dots\dots\dots(41)$$

It is evident that these equations are equivalent to the second and third of Eqs. (38), respectively.

According to the flow chart in Fig. 7, the critical pressure is numerically determined, and the results are shown in Table 1, Fig. 15 and Fig. 16.

From them, it can be seen that the critical pressure decreases with the increase of the spiral angle and also with the decrease of the number of spiral stiffeners. In Figs. 15 and 16 is also shown the critical pressure for a cylindrical shell reinforced by ring stiffeners with the corresponding space l_0 . It can be confirmed from these figures that the critical pressure for a cylindrical shell reinforced by spiral stiffeners is almost identical with that for the same shell reinforced by ring stiffeners with the corresponding space.

The critical pressure obtained by the above theory is far beyond the experimental critical pressure, and this discrepancy widens with the decrease of the spiral angle. In order to improve the theory, deflections due to initial deflection and the effects of plastic deformation will be taken into account in the following section.

$$F = -\frac{EtA}{2R} \left\{ \left(\frac{C_+}{C_+^2 + D_+^2} \right)^2 \cos(C_+x + D_+y) - \left(\frac{C_-}{C_-^2 + D_-^2} \right)^2 \cos(C_-x + D_-y) \right\} + \frac{EtA^2}{8} (C_+D_- - C_-D_+)^2 \left[\frac{\cos\{(C_+ + C_-)x + (D_+ + D_-)y\}}{\{(C_+ + C_-)^2 + (D_+ + D_-)^2\}^2} + \frac{\cos\{(C_+ - C_-)x + (D_+ - D_-)y\}}{\{(C_+ - C_-)^2 + (D_+ - D_-)^2\}^2} \right] \dots\dots\dots(42)$$

The corresponding expressions for Π_e and $\Pi_\sigma + \Pi_q$ become as follows;

$$\begin{aligned} \Pi_e &= \frac{A^2}{16R^2} Et \cos(\alpha + \beta) l_1 l_2 \left\{ \left(\frac{C_+^2}{C_+^2 + D_+^2} \right)^2 + \left(\frac{C_-^2}{C_-^2 + D_-^2} \right)^2 \right\} - \frac{A^3}{12\pi^2 R} Et \cos(\alpha + \beta) l_1 l_2 (C_+D_- - C_-D_+)^2 \\ &\quad \times \left[\left(\frac{C_+^2}{C_+^2 + D_+^2} + \frac{C_-^2}{C_-^2 + D_-^2} \right) \left\{ \frac{1}{(C_+ + C_-)^2 + (D_+ + D_-)^2} + \frac{1}{(C_+ - C_-)^2 + (D_+ - D_-)^2} \right\} \right. \\ &\quad \left. - (1 + \nu)(C_+D_- - C_-D_+)^2 (C_+D_- - C_-D_+)^2 \left\{ \frac{C_+^2}{(C_+^2 + D_+^2)^2} + \frac{C_-^2}{(C_-^2 + D_-^2)^2} \right\} \right. \\ &\quad \left. \times \left\{ \frac{1}{\{(C_+ + C_-)^2 + (D_+ + D_-)^2\}^2} + \frac{1}{\{(C_+ - C_-)^2 + (D_+ - D_-)^2\}^2} \right\} \right] \\ &\quad + \frac{A^2 D}{16R^2} \cos(\alpha + \beta) l_1 l_2 \left[R^2 \{(C_+^2 + D_+^2)^2 + (C_-^2 + D_-^2)^2\} + \frac{2}{R^2} + 2\{(D_+^2 + D_-^2) + \nu(C_+^2 + C_-^2)\} \right] \dots\dots\dots(43) \end{aligned}$$

$$\Pi_\sigma + \Pi_q = -\frac{A^2}{32R} q \cos(\alpha + \beta) l_1 l_2 [R^2 \{(C_+^2 + C_-^2) + 2(D_+^2 + D_-^2)\} - 4] \dots\dots\dots(44)$$

The total potential energy Π is then given by

$$\Pi = \Pi_e + \Pi_\sigma + \Pi_q = B_1 A^4 - B_2 A^3 + (B_3 - B_4 q) A^2, \dots\dots\dots(45)$$

where B_3 and B_4 are given by Eqs. (35) and (36), respectively, and B_1 and B_2 are as follows;

$$B_1 = \frac{Et}{256} \cos(\alpha + \beta) l_1 l_2 (C_+D_- - C_-D_+)^4 \left[\frac{1}{\{(C_+ + C_-)^2 + (D_+ + D_-)^2\}^2} + \frac{1}{\{(C_+ - C_-)^2 + (D_+ - D_-)^2\}^2} \right] \dots\dots\dots(46)$$

$$\begin{aligned} B_2 &= \frac{Et}{12\pi^2 R} \cos(\alpha + \beta) l_1 l_2 (C_+D_- - C_-D_+)^2 \left[\left(\frac{C_+^2}{C_+^2 + D_+^2} + \frac{C_-^2}{C_-^2 + D_-^2} \right) \left\{ \frac{1}{(C_+ + C_-)^2 + (D_+ + D_-)^2} \right. \right. \\ &\quad \left. \left. + \frac{1}{(C_+ - C_-)^2 + (D_+ - D_-)^2} \right\} - (C_+D_- - C_-D_+)^2 \left\{ \frac{C_+^2}{(C_+^2 + D_+^2)^2} + \frac{C_-^2}{(C_-^2 + D_-^2)^2} \right\} \right. \\ &\quad \left. \times \left\{ \frac{1}{\{(C_+ + C_-)^2 + (D_+ + D_-)^2\}^2} + \frac{1}{\{(C_+ - C_-)^2 + (D_+ - D_-)^2\}^2} \right\} \right] \dots\dots\dots(47) \end{aligned}$$

Fig. 7

2.3 Finite displacement theory

In order to clarify the behavior of actual shells under external pressure, the equilibrium will be analyzed with the aid of the nonlinear theory, and the nonlinear terms in Eq. (8) should then be retained. Substituting w from Eq. (27) into Eq. (8), we obtain the stress function in the form

In the small displacement theory developed in the preceding section, Π is given by the expression of the second order of the amplitude A , and the terms of more than second order in Eq. (45) are due to the nonlinear terms in the condition of compatibility. The equilibrium state to a certain value of the external pressure q is determined by the principle of minimum potential energy, and it follows that

$$\frac{\partial \Pi}{\partial A} = \frac{\partial \Pi}{\partial n} = \frac{\partial \Pi}{\partial \beta} = 0. \quad (48)$$

The first of Eqs. (48) can be written in a simple form as

$$\frac{\partial \Pi}{\partial A} = 4B_1 A^3 - 3B_2 A^2 + 2(B_3 - B_4 q) A = 0. \quad (49)$$

The equilibrium state is determined numerically by the direct method according to the flow chart given in Fig. 8. In Figs. 12 and 13 are shown the equilibrium curves ($A_0/t=0$) for the test models nos. 1 and 5. It is evident from these figures that the snap-through buckling phenomenon does not occur in the case of the present problem⁽⁷⁾.

2.4 Effect of initial deflections and plastic deformations

For the sake of convenience, it is assumed that the deflection W and the initial deflection w_0 are given in the following form;

$$W = -A \sin \frac{\pi \xi}{l_1} \sin \frac{\pi \eta}{l_2}, \quad (50)$$

$$w_0 = -A_0 \sin \frac{\pi \xi}{l_1} \sin \frac{\pi \eta}{l_2}. \quad (51)$$

The additional deflection w is given by

$$w = W - w_0 = -A \sin \frac{\pi \xi}{l_1} \sin \frac{\pi \eta}{l_2}. \quad (52)$$

According to Donnell's concept⁽⁶⁾, it is assumed that the pressure at which plastic deformations start is the buckling pressure of the shell with initial imperfection.

Stresses are given by the sum of the uniform membrane stresses and bending stresses. The largest stress in absolute value appears at the points

$$(\xi, \eta) = \left(\frac{l_1}{2}, \frac{l_2}{2} \right), \left(\frac{l_1}{2}, \frac{3l_2}{2} \right), \quad (53)$$

and the equivalent points of other bays. At the point $(l_1/2, l_2/2)$ the shell deflects inwards, and plastic deformations around this point does not make the shell unstable. The stresses at the point $(l_1/2, 3l_2/2)$ where the shell deflects outwards will be investigated hereafter. Since Tresca's law will be adopted in the present paper, the maximum normal stress component in absolute value should be considered, and it is approximately given by the normal stress in the ξ -direction which is perpendicular to the ξ -direction (cf. Fig. 9). Hence it is assumed that the shell buckles when the absolute value of the maximum normal stress in the ξ -direction arrives at the yield stress.

Let $(\sigma_\xi, \sigma_\zeta, \tau_{\xi\zeta})$ be the components of stresses with reference to the ξ - and ζ -axes, and let $(N_\xi, N_\zeta, N_{\xi\zeta})$ and $(M_\xi, M_\zeta, M_{\xi\zeta})$ be the corresponding stress resultants and couples. The stress components σ_ξ, σ_ζ and $\tau_{\xi\zeta}$ can be expressed in terms of the stress resultants and couples:

$$\left. \begin{aligned} \sigma_\xi &= \frac{12z}{t^3} M_\xi + \frac{N_\xi}{t} \\ \sigma_\zeta &= \frac{12z}{t^3} M_\zeta + \frac{N_\zeta}{t} \\ \tau_{\xi\zeta} &= \frac{12z}{t^3} M_{\xi\zeta} + \frac{N_{\xi\zeta}}{t} \end{aligned} \right\} \quad (54)$$

The stress resultants are given by (cf. Fig. 10)

Fig. 8 (q_0 should be larger than q_{cr})

Fig. 9

$$\left. \begin{aligned} N_{\xi} &= N_x \cos^2 \beta + N_y \sin^2 \beta = -Rq \left(\frac{1}{2} \cos^2 \beta + \sin^2 \beta \right) \\ N_{\zeta} &= N_x \sin^2 \beta + N_y \cos^2 \beta = -Rq \left(\frac{1}{2} \sin^2 \beta + \cos^2 \beta \right) \\ N_{\xi\zeta} &= -(N_x - N_y) \sin \beta \cos \beta = -\frac{Rq}{2} \sin 2\beta \end{aligned} \right\} \dots\dots\dots (55)$$

In reality, β being very small, $N_{\xi\zeta}$ can be considered to be zero. The stress couples are expressed in the form

$$\left. \begin{aligned} M_{\xi} &= M_x \cos^2 \beta + M_y \sin^2 \beta + 2M_{xy} \sin \beta \cos \beta \\ M_{\zeta} &= M_x \sin^2 \beta + M_y \cos^2 \beta - 2M_{xy} \sin \beta \cos \beta \\ M_{\xi\zeta} &= -\frac{1}{2}(M_x - M_y) \sin 2\beta + M_{xy} \cos 2\beta \end{aligned} \right\} \dots\dots\dots (56)$$

where

$$\left. \begin{aligned} M_x &= -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ M_y &= -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\ M_{xy} &= -D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \right\} \dots\dots\dots (57)$$

Therefore, M_{ξ} , M_{ζ} and $M_{\xi\zeta}$ evaluated at $(\xi, \eta) = \left(\frac{l_1}{2}, \frac{3l_2}{2} \right)$ become

$$\left. \begin{aligned} M_{\xi} &= \frac{\pi^2 DA}{\cos^2(\alpha + \beta)} \left\{ \frac{1 + \nu + (1 - \nu) \cos 2(\alpha + \beta)}{2l_1^2} + \frac{\nu}{l_2^2} \right\} > 0 \\ M_{\zeta} &= \frac{\pi^2 DA}{\cos^2(\alpha + \beta)} \left\{ \frac{1 + \nu - (1 - \nu) \cos 2(\alpha + \beta)}{2l_1^2} + \frac{1}{l_2^2} \right\} > 0 \\ M_{\xi\zeta} &= \frac{\pi^2 DA}{\cos^2(\alpha + \beta)} \frac{\sin 2(\alpha + \beta)}{l_1^2} > 0 \end{aligned} \right\} \dots\dots\dots (58)$$

We have the relation

$$M_{\zeta} > M_{\xi} > M_{\xi\zeta} \dots\dots\dots (59)$$

and $M_{\xi\zeta}$ can also be regarded to be zero. When σ_{ζ} evaluated at the point $(\xi, \eta, z) = (l_1/2, 3l_2/2, -t/2)$ arrives at the yield stress, the buckling occurs in the shell (cf. Fig. 11). The corresponding stress σ_{ζ} and stress couple M_Y become as follows;

$$\sigma_{\zeta} = 2 \left(Y + \frac{N_{\zeta}}{t} \right) \frac{z}{t} + \frac{N_{\zeta}}{t} \dots\dots\dots (60)$$

$$M_Y = \int_{-t/2}^{t/2} \sigma_{\zeta} z dz = \frac{t^2}{6} Y \left(1 + \frac{N_{\zeta}}{Yt} \right) = \frac{t^2}{6} Y \left\{ 1 - \frac{Rq \left(\frac{1}{2} \sin^2 \beta + \cos^2 \beta \right)}{Yt} \right\}, \dots\dots\dots (61)$$

where Y is the yield stress. Then the buckling pressure q is obtained by the equation

$$M_{\zeta} = M_Y \dots\dots\dots (62)$$

Hereafter the equilibrium state will be analyzed for shells which have initial deflections. The condition of compatibility becomes as follows;

$$\frac{1}{Et} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) F^* = \frac{1}{R} \frac{\partial^2 w}{\partial x^2} + \left(1 + 2 \frac{A_0}{A} \right) \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \dots\dots\dots (63)$$

The stress function F^* is obtained as a particular solution of (63):

$$\begin{aligned} F^* &= -\frac{EtA}{2R} \left\{ \left(\frac{C_+}{C_+^2 + D_+^2} \right)^2 \cos(C_+x + D_+y) - \left(\frac{C_-}{C_-^2 + D_-^2} \right)^2 \cos(C_-x + D_-y) \right\} \\ &\quad + \frac{EtA^2}{8} \left(1 + 2 \frac{A_0}{A} \right) (C_+D_- - C_-D_+)^2 \left[\frac{\cos \{ (C_+ + C_-)x + (D_+ + D_-)y \}}{\{ (C_+ + C_-)^2 + (D_+ + D_-)^2 \}^2} \right. \\ &\quad \left. + \frac{\cos \{ (C_+ - C_-)x + (D_+ - D_-)y \}}{\{ (C_+ - C_-)^2 + (D_+ - D_-)^2 \}^2} \right] \dots\dots\dots (64) \end{aligned}$$

From Eqs. (20), (21) and (22), Π_e^* and $\Pi_o^* + \Pi_q^*$ are given by

Fig. 10

$$\begin{aligned}
\Pi_e^* = & \frac{\cos(\alpha+\beta)}{2Et} \left[\left(\frac{EtA}{2R} \right)^2 \frac{l_1 l_2}{2} \left\{ \left(\frac{C_+^2}{C_+^2 + D_+^2} \right)^2 + \left(\frac{C_-^2}{C_-^2 + D_-^2} \right)^2 \right\} \right. \\
& + \frac{l_1 l_2}{2} (C_+ D_- - C_- D_+)^4 \left\{ \frac{EtA^2 \left(1 + 2 \frac{A_0}{A} \right)}{8} \right\}^2 \left\{ \frac{1}{\{(C_+ + C_-)^2 + (D_+ + D_-)^2\}^2} \right. \\
& + \frac{1}{\{(C_+ - C_-)^2 + (D_+ - D_-)^2\}^2} \left. - \frac{(Et)^2 l_1 l_2}{6\pi^2 R} A^3 \left(1 + 2 \frac{A_0}{A} \right) (C_+ D_- - C_- D_+)^2 \left(\frac{C_+^2}{C_+^2 + D_+^2} + \frac{C_-^2}{C_-^2 + D_-^2} \right) \right. \\
& \times \left\{ \frac{1}{(C_+ + C_-)^2 + (D_+ + D_-)^2} + \frac{1}{(C_+ - C_-)^2 + (D_+ - D_-)^2} \right\} \\
& + \frac{(Et)^2 l_1 l_2}{6\pi^2 R} A^3 \left(1 + 2 \frac{A_0}{A} \right) (1+\nu) (C_+ D_- - C_- D_+)^2 \left\{ \frac{C_+^2}{(C_+^2 + D_+^2)^2} + \frac{C_-^2}{(C_-^2 + D_-^2)^2} \right\} \\
& \times \left\{ \frac{1}{\{(C_+ + C_-)^2 + (D_+ + D_-)^2\}^2} + \frac{1}{\{(C_+ - C_-)^2 + (D_+ - D_-)^2\}^2} \right\} \\
& \left. + \frac{A^2 D}{16R^2} \cos(\alpha+\beta) l_1 l_2 \left[R^2 \{(C_+^2 + D_+^2)^2 + (C_-^2 + D_-^2)^2\} + \frac{2}{R^2} + 2\{(D_+^2 + D_-^2) + \nu(C_+^2 + C_-^2)\} \right] \right] \dots (65)
\end{aligned}$$

$$\Pi_e^* + \Pi_q^* = - \frac{A^2 \left(1 + 2 \frac{A_0}{A} \right)}{32R} q \cos(\alpha+\beta) l_1 l_2 [R^2 \{(C_+^2 + C_-^2) + 2(D_+^2 + D_-^2)\} - 4] \dots (66)$$

The total potential energy Π^* is given in the form

$$\Pi^* = \Pi_e^* + \Pi_q^* + \Pi_q^* = B_1^* A^4 - B_2^* A^3 + (B_3^* - B_4^* q) A^2 - B_5^* q A, \dots (67)$$

where

$$\left. \begin{aligned} B_1^* &= B_1 \\ B_2^* &= B_2 - 4B_1 A_0 \\ B_3^* &= B_3 + 4B_1 A_0^2 - 2B_2 A_0 \\ B_4^* &= B_4 \\ B_5^* &= 2B_4 A_0 \end{aligned} \right\} \dots (68)$$

and B_1 , B_2 , B_3 and B_4 are given by Eqs. (46), (47), (35), and (36).

By the direct use of the principle of minimum potential energy, the equilibrium states can be determined for each pressure, and the buckling pressure is determined by Eq. (62). In Figs. 12

Fig. 11

Fig. 12 Equilibrium curve (Model number 1, $\alpha=9^\circ$)

Fig. 13 Equilibrium curve (Model number 5, $\alpha=21^\circ$)

and 13 are shown the pressure-deflection curves, and in Fig. 14 the relations between initial deflections and the buckling pressure. The theoretical buckling pressure is compared with the results of experiments in Figs. 15 and 16. Figure 14 shows that the buckling pressure decreases with the increase of initial deflections, and becomes constant for the value of A_0/t larger than 0.3. Figures 15 and 16 show that the theoretical buckling pressure for A_0/t

$=0.2$ is in agreement with the results of experiments. It is presumable that initial deflections of such order $A_0/t=0.2$ exist in the present test models because of workmanship.

3. Conclusion

The buckling pressure for a cylindrical shell reinforced by spiral stiffeners is determined by the small displacement theory but the results are fairly large as compared with the experimental results. To improve this weakness, the finite displacement theory for cylindrical shells with initial deflections is developed on the basis of the principle of minimum potential energy, and the buckling pressure is determined by the condition that plastic deformations yield buckling. In case the maximum initial deflection is 20% of the thickness, the theoretical buckling pressure is in agreement with the experimental results. Moreover, the critical buckling pressure by the small displacement theory is almost identical with that for a cylindrical shell reinforced by ring stiffeners spaced with the same axial distance.

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Fig. 14 Initial deflection vs. buckling pressure

Fig. 15 Experimental and theoretical values ($m=1$)

Fig. 16 Experimental and theoretical values ($m=2$)

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Discussion

N. YAMAKI (1) Donnell's basic equations for the finite deformation of a cylindrical shell with the initial deflection w_0 are given by

$$(1/Et)\nabla^4 F = 2w_{,xy}w_{0,xy} - w_{,xx}w_{0,yy} - w_{,yy}w_{0,xx} + w_{,xy}^2 - w_{,xx}w_{,yy} + (1/R)w_{,xx} \dots \dots \dots (i)$$

$$D\nabla^4 w - F_{,yy}(w+w_0)_{,xx} + 2F_{,xy}(w+w_0)_{,xy} - F_{,xx}(w+w_0)_{,yy} + (1/R)F_{,xx} + q = 0, \dots \dots \dots (ii)$$

where w is the additional deflection and F is Airy's stress function. I would like to know the relation between Eq. (ii) and Euler's equation for the variational problem given by Eq. (18).

(2) The influence of the boundary conditions at both ends seems to be significant in Fig. 3. I think that longer cylindrical shells should be used to compare with the theoretical solutions here obtained.

(3) By what formula did you obtain the critical buckling pressure of the cylindrical shell with ring stiffeners given in Figs. 15 and 16?

K. OSHIMA (4) What is the relation between the buckling wave length of a cylindrical shell with ring stiffeners and that of a cylindrical shell with spiral stiffeners?

(5) To formulate the yield condition, I suggest that equivalent stress $\sigma_e (= \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y})$ should be adopted rather than the largest principal stress σ_1 .

S. SUZUKI (6) If the buckling pattern w is assumed in an expression with more terms, more preferable critical buckling pressure can be obtained.

Authors' closure

(1) Equation (18) is obtained on the basis of the principle of minimum potential energy, and Eqs. (i) and (ii) cannot be obtained not as Euler's equation of Eq. (18), but as that for the variational problem obtained from Eq. (18) by Legendre transformation to the surface direction.

(2) We had conducted experiments with cylindrical shells about twice as long as the cylindrical shells in this paper, and the results were almost the same.

(3) It was obtained by the von Mises formula.

(4) They were almost the same.

(5) In the present theory, some approximate assumptions are adopted. Therefore the exactness in this point seems to be unnecessary.

(6) The results must be better. But calculation becomes much more complicated.