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On Stress-Strain Relation of Mild Steel in Unloading and Inverse Loading for Bending Analysis of Thin Circular Plate*

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Uniaxial tension-compression tests on plate specimens of mild steel were carried out in order to obtain the characteristics of stress-strain relation to be used for analysing unloading and inverse loading processes of thin plates elasto-plastically bent by lateral load. Results of the experiment were approximated fairly well with an analytical expression on the unloading coordinate in which the effect of plastic pre-strain was taken into consideration. Moreover, influence of plastic pre-strain on the relation or the Bauschinger effect was examined in detail.

1. Introduction

It is well known that mild steel having a stress-strain curve which consists of elastic and non-hardening plastic ranges together with a discrete yield point in initial loading process shows a continuous non-linear stress-strain curve in the unloading and subsequent inverse loading processes from plastic state. The latter curve, though nearly linear with the same slope as in the initial loading in the unloading range, deviates from the linear relation in the inverse loading process at a less magnitude of stress than that at the beginning of unloading. Such a trend is generally called the Bauschinger effect and has been investigated by many workers because of the importance of the effect in the investigations of residual deformation and stress after metal forming and of strength of materials under repeated loading and especially in low-cycle fatigue.

In many cases, according to the original definition of the Bauschinger effect, the effect has been evaluated by taking the state at the beginning of inverse loading as a reference. Recently, Талыпов⁽¹⁾ considered yield surfaces in stress space to describe deformations under general combined loading whose stress trajectories had sharp corners or reversal and obtained the following expression for the yield surface after plastic deformation of initially isotropic materials:

$$(\sigma_{ij}' - a_{ij}')(\sigma_{ij}' - a_{ij}') = \frac{1}{4} \{ \sqrt{\sigma_{i1}' \sigma_{i1}'} (1 + \lambda) \}^2 \quad \dots\dots\dots (1)$$

and

$$a_{ij}' = \sigma_{i1}'(1 - \lambda)/2 \quad (i, j = 1, 2, 3), \quad \dots\dots\dots (2)$$

where σ_{i1}' and σ_{ij}' denote the stress deviators at the point previously loaded and on the yield surface, respectively, a_{ij}' are the coordinates of the center of yield surface, and $\lambda = \sigma_{is2}/\sigma_{is1}$; σ_{is1} and σ_{is2} express the yield stresses in the reloading and inverse loading processes, respectively, after perfect unloading from plastic state designated by σ_{i1} . By the way,

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λ corresponds to the Bauschinger effect factor when σ_{i1} coincides with σ_{i1} . As the result of experiment, he found that the value of λ depends on the Lodé parameter μ and varies as a quadratic function of the effective plastic pre-strain $\bar{\epsilon}_e^p$ as far as $\bar{\epsilon}_e^p$ is less than 2~3 percent and λ then approaches asymptotically to a constant value for further increase of $\bar{\epsilon}_e^p$. In other words, the effect of $\bar{\epsilon}_e^p$ on the configuration of stress-strain curve becomes saturated for $\bar{\epsilon}_e^p$ more than 2~3 percent. Талыпов performed some analyses⁽²⁾ only for this saturated state. However, the Bauschinger effect itself should be considered as a special case, in which the stress trajectory reverses, of the general theory of plasticity concerning complex loading having arbitrary shapes of trajectory in the stress space, and the theory⁽³⁾ proposed by Ильюшин seems to be the most systematic one for such general problems.

Recently, Z. Mróz⁽⁴⁾ considered the problem of anisotropic work-hardening in the deformation along the generic stress trajectory in general complex loading. In his treatment, he considered the Bauschinger effect and its analogous phenomena as deformations which correspond to a discrete kink or reversal of loading path. He said that his concept of "field of work-hardening moduli" made it possible to illustrate schematically the concept of general theory of plasticity by approximating the non-linear hardening curve with a piece-wise linear curve and that the Bauschinger effect was able to be explained by using this concept.

In such a treatment of general theory of plasticity, the state at the beginning of inverse loading has not a definite meaning because it corresponds to only a point on the unloading curve starting from the beginning of unloading. Moreover, since the stress distribution in the actual deformation is not uniform and a perfectly unloaded state in each element can not correspond to the perfectly unloaded state of the whole body due to the existence of residual stress, it is meaningless to consider the beginning of inverse loading as the reference for analysing problems of unloading and inverse loading.

The present authors⁽⁵⁾ have reported a paper on the residual deformation of a thin circular plate of mild steel after perfect unloading from the elasto-plastic deformation under a uniform lateral load, in which the Masing model is used as unloading stress-strain relation according to the isotropic hardening theory. This was done for the purpose of obtaining the result in which the Bauschinger effect is not considered at all as an extreme case, and on the other hand, because the Bauschinger effect was supposed to be not so large as far as the plastic pre-strain is small. As was expected, the result of

experiment coincided well with that of the analysis for small plastic prestrain and the agreement became worse as the pre-strain increased. Though the disagreement could appear from various causes, the Bauschinger effect was supposed as the major part of them. In order to discuss the influence of the Bauschinger effect on the analytical result, the results of analysis in which the effect is and is not taken into consideration are to be compared. However, the results of studies on the Bauschinger effect published so far can not be introduced directly into this analysis because they are not formulated conveniently for the unloading analysis in the bending of thin plates.

In this paper, therefore, a method is proposed to formulate the Bauschinger effect suitably for the above purpose and an analytical expression for stress-strain relation of mild steel in the unloading and subsequent inverse loading processes is obtained with a sufficiently high accuracy and in a convenient form for complex calculations. Though the formulas proposed by Талыпов and Mróz on the basis of the flow theory are inconvenient to apply for the usual bending analysis of thin plates, the expression herein proposed is convenient for the use in the bending analysis of thin plates in which in combination with Kirchhoff's hypothesis parameters expressing the integrated effect of stress through thickness in the plate are introduced.

2. Stress-strain relation in unloading and subsequent inverse loading processes

In the elasto-plastic bending problem of thin plates, the advantage of the flow theory becomes meaningless as the distributions of strain components along thickness are linear according to Kirchhoff's hypothesis. On the other hand, the deformation theory may be applicable to the plastic layer because of the comparatively strong restriction from the elastic layer. In treating the unloading process with the use of the deformation theory, an unloading coordinate system is usually introduced in which the origin is located at the beginning of unloading and the stress and strain axes are selected in the opposite sense to those of the loading coordinate. When the state of deformation of a plate at the beginning of unloading is characterized by the effective strain, according to Kirchhoff's hypothesis, the state is defined by parameters expressing deformation of the middle plane of the plate and terms proportional to the coordinate along the thickness. In other words, the value of pre-strain in each element is known from the preceding calculation.

In the bending analysis of thin plates, the equilibrium conditions for the bending moment and membrane force which are considered as integrated

effects of stresses distributed along the thickness and the compatibility conditions which hold between the components of displacement of middle plane and those of strain are used. When the relation between stress and strain is expressed by an analytical form, components of bending moment and membrane force are found by integrating the stress components and their moments through the thickness with the use of Kirchhoff's hypothesis and the Hencky relation.

When the Hencky relation

$$\sigma_{ij}' = \frac{2\sigma_e}{3\epsilon_e} \epsilon_{ij}', \quad \epsilon_{ii} = \alpha \sigma_{ii} \quad (3)$$

is used in the unloading and inverse loading processes, it is necessary to obtain a relation $\sigma_e = \sigma_e(\epsilon_e)$ for these processes. In Eq. (3), σ_{ij}' and ϵ_{ij}' , $\sigma_{ii}/3$ and $\epsilon_{ii}/3$, σ_e , ϵ_e and α denote the stress and strain deviators, the mean stress and strain, the effective stress, strain and the coefficient of volumetric strain, respectively. The relation $\sigma_e = \sigma_e(\epsilon_e)$ is expressed by a continuous non-linear curve even for mild steel, and its shape depends on Lodé's parameter μ and the plastic pre-strain, as pointed out by Талыпов. Since the amount of pre-strain at the beginning of unloading is less than 2~3 percent in the bending problem of thin plate considered here, the effect of pre-strain is supposed to exist in this case.

In view of the effect of Lodé's parameter a uniaxial tension-compression test may be used as an approximate calibration of the stress-strain relation, since the central part of the plate suffering the most severe deformation has the parameter value near -1 or 1 in the bending problem of thin circular plates. Therefore, the stress-strain curve in the unloading and subsequent compression processes is determined for each prescribed tensile pre-strain, and these curves are approximated by a polynomial relation between unloading stress and strain in which the coefficients also are expressed as functions of plastic pre-strain. As the amounts of plastic pre-strain in plate bending are known as the result of calculation for the loading process, the unloading stress-strain relation for each element is completely determined with this approximate expression. There-

fore, components of membrane force and bending moment in the unloading process can be found by integrating the unloading stress components.

3. Experiment

3.1 Specimens

Since the purpose of this experiment is to obtain the characteristic of stress-strain curve in the unloading and subsequent inverse loading processes as a calibration, the specimen was made of the mild steel plate from which bending test pieces of circular plate would be cut out. Then, the specimen shown in Fig. 1 was finished from the mild steel plate of 12 mm thick whose chemical compositions and mechanical properties are shown in Tables 1 and 2.

In order to avoid buckling of the specimen in compression test, a shorter length of the parallel part of specimen is preferable. Therefore, the minimum parallel length, where the uniform stress range measurable by the cemented wire resistance strain gauges existed, was determined by the photoelastic method.

As shown in Fig. 1, the gauges of 10 mm length and those of 3 mm length were cemented in the longitudinal and transverse directions on both surfaces of specimen, and those of 3 mm length were cemented in the transverse direction on both sides. As shown in Table 2, the material did not

Fig. 1 Shape and size of Specimen

Table 1 Chemical components of S15 C steel (%)

C	Si	Mn	P	S	Cu
0.13	0.28	0.43	0.013	0.012	0.32

Table 2 Mechanical properties of S15 C steel

No. of specimen	Young's modulus kg/mm ²	Upper yield point kg/mm ²	Lower yield point kg/mm ²	Poisson's ratio	Remarks
B-2	2.110 × 10 ⁴	32.40	30.10	0.278	com. pre-strain, } ten. pre-strain, } ⊥ to rolled direction
B-5	2.076	32.03	29.98	0.280	
A-2	2.075	33.07	30.99	0.278	ten. } ten. } ten. } ten. } Parallel to rolled direction
A-3	2.070	31.40	29.90	0.273	
A-4	2.120	32.40	30.11	0.284	
A-5	2.100	32.51	30.39	0.277	
mean	2.093	32.30	30.24	0.278	

show any apparent anisotropy though it has not been subjected to any heat treatment.

3.2 Specimen holders

Since the testing machine adopted could not be used for the tension and compression test in the same setting condition, the compression test was performed by resetting the specimen after perfect unloading from tension. Every specimen was required to be set on the testing machine in a constant state and the setting and removal of specimen should be as easy as possible. The holders described in the following were then used. In the tension test, the specimen was set on the machine with the holder shown in Fig. 2, in which the load was transmitted to the specimen ① through the plates ③ and the pins ④ from the plates ② fixed to the upper and lower chucks of the machine. In compression, the guide device shown in Fig. 3 was used in order to perform an accurate uniaxial test, in which compression was applied to the specimen ① with the pins ⑧ and the top and bottom surfaces of guide

① Specimen ② Plates ③ Side plates
④ Pins

Fig. 2 Holders for tension test

① Specimen ⑤ Guide piece ⑥ Guide piece
⑦ Needle bearings ⑧ Pins

Fig. 3 Guide device for compression test

pieces ⑤ and ⑥ were finished precisely parallel. The needle bearings ⑦ were used for reducing friction between the guide planes.

3.3 Experimental procedure

Five magnitudes of tensile pre-strain in the non-hardening plastic range were selected for the specimens. Before determining these magnitudes, the tensile effective pre-strain of 16.5×10^{-3} near the beginning of strain hardening and 9.8×10^{-3} in the middle of the non-hardening plastic range were selected first by the preliminary test, and the stress-strain curves for these pre-strain in the unloading and subsequent inverse loading processes were obtained. As these two curves almost agreed with each other, it was observed that there was no need to test for pre-strain lying between the above two values. Accordingly, the values of pre-strain 1.6, 4.7 and 7.1×10^{-3} were selected for further experiments. The first pre-strain is very near to the strain at the yield point.

For each specimen, the strain components in the three directions on both surfaces were measured at successive load levels and from their mean values on both surfaces the effective strain, Young's modulus, Poisson's ratio and the coefficient of volumetric strain were calculated.

In order to give the prescribed amount of pre-strain to the specimen, a small increment of tensile strain was applied carefully and successively after the lower yield point was reached. After reaching the equilibrium state in the proximity of the prescribed pre-strain, the strain components were measured and then the load was removed stepwise by a small amount and the strain components were measured at each stage. After perfect unloading of the tensile load, the specimen was taken off from the holder and set into the guide device for compression test. In the compression test, the load was applied slowly by a small amount, and the strain components in the equilibrium state were measured at each stage. In this process, the corresponding values of strain components on both surfaces of specimen coincided well and this indicated that there was no appreciable effect of bending on the specimen.

In order to confirm the reproducibility of the result, another test with a new specimen was performed for the value 16.5×10^{-3} of pre-strain. The result shown in Fig. 4 shows that the shape of this curve agreed well with that of the previous test.

4. Experimental results

The results obtained from the above experiments are summarized in Fig. 4. It may be concluded that any influence of friction between the guide pieces does not exist, because the inverse loading curve for the pre-strain value 1.6×10^{-3} shows in

the range of low compressive load almost the same slope as that of the initial elastic loading. If the influence of friction would exist, the former slope should become steeper than the latter.

From the experimental results shown in Fig. 4 are taken the relation between the unloading effective stress σ_e^* and strain ϵ_e^* and they are plotted

Fig. 4 Stress-strain curves of various pre-strain obtained by experiments in unloading and reverse loading after various pre-strains

Plots: experimental data,

Solid line: experimental formula after Eq. (4)

Fig. 5 Stress-strain curves on unloading coordinate

Fig. 6 Relation between plastic pre-strain and unloading strain

in Fig. 5. In order to eliminate the scattering in the lower yield stress of individual specimen, both axes are expressed in the dimensionless quantities $\sigma_r^* = \sigma_e^* / \sigma_Y$ and $\epsilon_r^* = \epsilon_e^* / (\sigma_Y / 3G)$, respectively, where σ_Y and G denote the lower yield stress and the modulus of rigidity. The chain-dot line indicates the stress-strain relation in the initial elastic loading.

As the plastic pre-strain is considered as the factor affecting the unloading stress-strain curve, the figure is expressed with the parameter $\bar{\epsilon}_e^p$. As easily seen in the figure, some curves deviate from the chain-dot line even before perfect unloading of tension ($\sigma_r^* = 1$), and this trend becomes more significant for larger $\bar{\epsilon}_e^p$. This fact shows also the inconvenience that may result from regarding the beginning of inverse loading as the reference state.

In order to eliminate the scattering in the experimental results, ϵ_r^* is plotted against $\bar{\epsilon}_e^p$ with the parameter σ_r^* in Fig. 6 and each curve is drawn to connect smooth by the corresponding points. Moreover, each curve in Fig. 5 is assumed to have a common asymptote $\sigma_r^* = 2$. By cross-plotting Fig. 6, the unloading curves with parameter $\bar{\epsilon}_e^p$ shown in Fig. 4 are drawn in Fig. 7. As shown in the latter figure, the variation of the shape of the curves is remarkable for small $\bar{\epsilon}_e^p$ while it is slight for $\bar{\epsilon}_e^p$ beyond 10×10^{-3} . This trend may correspond to the saturation called by ТАЛЫПОВ.

5. Analytical expression of experimental results

As mentioned above, the stress must be expressed by an analytical function of strain for the analysis of unloading deformation of plastically deformed plates. For this purpose, it is necessary to find a function $\sigma_r^*(\epsilon_r^*)$ which describes the curves shown in Fig. 7 accurately. As the analytical forms approximating the stress-strain curves, those of single power, hyperbolic, exponential functions and polyno-

Fig. 7 Stress-strain curves after elimination of scattering and experimental values

mial have been used so far. Among these forms, the first one can not approximate satisfactorily the actual curves for the range of strain involved in the plate bending, and the second and third also can not be used because of the difficulty in the analytical integration of stress components in the case of large deflection in which the membrane force must be taken into account. The following polynomial is, therefore, used here:

$$\sigma_r^* = a_1 \varepsilon_r^* + a_2 [\varepsilon_r^*]^2 + a_3 [\varepsilon_r^*]^3 + a_4 [\varepsilon_r^*]^4 \dots \dots \dots (4)$$

By using this relation, each curve in Fig. 7 can be approximated satisfactorily and the values of coefficients a_1 , a_2 , a_3 and a_4 obtained by the least square method are shown in Table 3. In the next place, the relation between the coefficients and $\bar{\varepsilon}_e^p$ are plotted in Fig. 8. When the coefficients a_1 , a_2 , a_3 and a_4 are expressed as functions of $\bar{\varepsilon}_e^p$ so as to fit the points plotted in Fig. 8, the unloading stress-strain relation is correlated with the plastic pre-strain as follows:

$$\sigma_r^* = a_1(\bar{\varepsilon}_e^p) \varepsilon_r^* + a_2(\bar{\varepsilon}_e^p) [\varepsilon_r^*]^2 + a_3(\bar{\varepsilon}_e^p) [\varepsilon_r^*]^3 + a_4(\bar{\varepsilon}_e^p) [\varepsilon_r^*]^4 \dots \dots \dots (5)$$

As the unloading curve for $\bar{\varepsilon}_e^p = 0$ follows inversely the initial elastic relation, the value of a_1 in this case should correspond to the elastic modulus for the initial elastic loading and a_2 , a_3 and a_4 should be zero. Therefore, the curves of a_2 , a_3 and a_4

Table 3 Values of coefficients a_i in Eq. (4)

$\bar{\varepsilon}_e^p$	a_1	a_2	a_3	a_4
0.4×10^{-3}	1.082	-0.742×10^{-1}	0.808×10^{-2}	-0.294×10^{-3}
3.5	1.260	-3.079	3.415	-1.430
5.9	1.215	-3.022	3.473	-1.505
8.6	1.166	-2.838	3.169	-1.316
15.2	1.061	-2.235	2.019	-0.655

Fig. 8 Relation between coefficients a_i in Eq. (5) and plastic pre-strain

must pass through the origin. In order to approximate these curves, the following polynomial of degree five is selected:

$$a_i = a_{i0} + \sum_{j=1}^5 a_{ij} (\bar{\varepsilon}_e^p)^j, \quad a_i = \sum_{j=1}^5 a_{ij} (\bar{\varepsilon}_e^p)^j \quad (i=2,3,4), \dots \dots \dots (6)$$

where the coefficients a_{ij} obtained by the least square method are shown in Table 4. In the table, $a_{i0}=1$ corresponds to the elastic constant in the initial loading. In Fig. 5, the curves drawn by using Eqs. (5), (6) and the values of a_{ij} in Table 4 are in good agreement with the experimental results. It should be noted that the values of coefficients in Table 4 have been determined so that the approximation is valid as far as $\varepsilon_e^* = \bar{\varepsilon}_e$.

6. Discussion

As shown in Fig. 5, the curves obtained by the above approximation agree well with the experimental results and the difference between them is less than 5 percent. In bending problems of thin plates, as each element is subjected to a different plastic pre-strain at the beginning of unloading, it will have a different strain in the unloading process. However, as the strain range where the error is large is very narrow in Fig. 5, the influence of the error on the membrane force and bending moment obtained by integrating the stress components and their moments along thickness is supposed to be fairly small.

In Fig. 5, the broken line indicates the curve for $\bar{\varepsilon}_e^p = 15.2 \times 10^{-3}$ in which the unloading range is approximated by the elastic unloading line and the inverse loading range is approximated by a logarithmic function⁽⁶⁾. On the other hand, the curve approximated by an exponential function is shown in Fig. 9. The accuracies of these two cases are not so good as that of the polynomial relation shown in Fig. 5.

In order to clarify the relation between the plastic pre-strain and the measures of the Bauschinger effect hitherto used, the Bauschinger effect factor⁽⁷⁾ $\lambda = (\sigma_e^* - \sigma_r) / \sigma_r$ for some values of ε_e^* and Buckley's Bauschinger strain⁽⁸⁾ $(\varepsilon_e^*)_B = \varepsilon_e^* - (\varepsilon_e^*)_0$ for several values of λ are plotted as functions of $\bar{\varepsilon}_e^p$ in Figs. 10 and 11, respectively. In the above

Table 4 Values of coefficients a_{ij} in Eq. (6)

$i \backslash j$	1	2	3	4
0	1.000	—	—	—
1	1.667×10^2	-2.037×10^2	2.136×10	-7.762×10^{-1}
2	-4.013×10^4	4.831×10^4	-4.663×10^3	1.379×10^2
3	3.930×10^6	-5.200×10^6	4.655×10^5	-1.071×10^4
4	-1.797×10^8	2.663×10^8	-2.275×10^7	4.206×10^5
5	3.170×10^9	-6.006×10^9	4.359×10^8	-6.918×10^6