534. 1. 015

A Dividing Analysis of the Principal Vibrations of Multi-Degree-of-Freedom Linear Vibrating Systems*

By Michio KURODA**

"A dividing differential equation method" to analyze the principal vibrations of multi-degree-of-freedom linear vibrating systems is proposed. By this method, an overall system is divided into several partial systems in order to make the principal vibrations of the overall system correspond one-to-one and continuously to those of the partial systems, and a first-order differential equation with an independent variable expressing the degree of connectivity among the partial systems is introduced. The theory is developed for studying the natural frequencies and the normal modes of vibration simultaneously and also for another case when only the natural frequencies are treated. A way of setting up an auxiliary system is presented. The numerical computations of the principal vibrations of an overall system can easily be executed by making use of a suitable numerical method and an electronic computer, the programming for which is very easy.

1. Introduction

This paper will deal with several linear vibrating systems and one large vibrating system which is equal to a combination of those systems by linear springs; those shall be called hereafter "the partial systems" and "the overall system" respectively.

Now suppose that an overall system is in the state of a principal vibration. If all the spring constants of the springs by which the partial systems are connected with one another were slightly decreased, the natural frequency and normal mode of vibration would be slightly changed as well. If these spring constants were decreased little by little until finally they vanished, the overall system would at last become a simple gathering of the partial systems among which no connectivity exists; consequently, the natural frequency would gradually shift to a certain natural frequency of a certain partial system, and the normal mode of vibration would lead at last to the situation where only one partial system was acting as a principal vibration, with all the other partial systems at a standstill.

Conversely, let us suppose that an overall system is at first just a gathering of partial systems and that a certain partial system is acting as a principal vibration, with all the other partial systems remaining stationary. If all the spring constants of

the connecting parts were increased, little by little, from zero to given values, there would eventually be one certain principal vibration of a given overall system.

One of the main purposes of this paper is to propose a dividing differential equation method created on the basis of the above reasoning to analyze the principal vibrations of multi-degree-of-freedom linear vibrating systems. This method may be briefly described as follows. The overall system under consideration is divided into several partial systems to make the principal vibrations of the overall system correspond one-to-one to those of the partial systems, and a first-order differential equation with an independent variable expressing the degree of connectivity among the partial systems is introduced. Then we solve, numerically, an initial value problem of the differential equation.

We shall assume that the system is conservative and shall consider small vibrations about a configuration of a stable equilibrium of the system. Accordingly, the kinetic energy, T , and the potential energy, U , of the system are, respectively, given by⁽¹⁾:

$$2T = \dot{X}^T M \dot{X}, \quad 2U = X^T K X \quad \dots\dots\dots (1)$$

where X denotes the column vector whose components are n generalized coordinates, n being the degree of freedom of the overall system, where $\dot{X}$ denotes the derivative of X with respect to t , t being the time; where X^T means the transpose of X , and where M and K are both real symmetric

* Received 30th July, 1968.

** Associate Professor, College of Technology, Seikei University, Musashino.

matrixes of order n . In these expressions, M should be positive definite⁽²⁾ because $T > 0$ if $\dot{X} \neq 0$; K should also be positive definite because at $X = 0$ U takes the minimal value of zero.

The equations of motion of the free vibrations of the overall system are obtained by substituting the expression (1) into Lagrange's equations of motion⁽³⁾ as follows:

$$M \frac{d^2 X}{dt^2} + KX = 0 \quad \dots\dots\dots(2)$$

By putting:

$$X = A \cos(\omega t + \alpha) \quad \dots\dots\dots(3)$$

the following equation concerning the natural frequencies and amplitudes of the principal vibrations is obtained:

$$(K - \omega^2 M)A = 0 \quad \dots\dots\dots(4)$$

Setting the determinant of the coefficient matrix equal to zero yields the frequency equation:

$$|K - \omega^2 M| = 0 \quad \dots\dots\dots(5)$$

From the theory of matrix⁽⁴⁾, it is possible to choose an appropriate real nonsingular matrix, C , such that:

$$\left. \begin{aligned} C^T M C &= E \text{ (} E: \text{unit matrix)} \\ C^T K C &= D \text{ (} D: \text{real diagonal matrix)} \end{aligned} \right\} \quad \dots\dots\dots(6)$$

Then, if we transform the coordinates from X to Y by the relation:

$$X = CY \quad \dots\dots\dots(7)$$

we obtain:

$$2T = \dot{Y}^T \dot{Y}, \quad 2U = Y^T D Y \quad \dots\dots\dots(8)$$

where the diagonal elements of D should all be positive real numbers, because $Y^T D Y$ is a positive definite quadratic form. Let these elements be denoted by $\omega_i^2 (0 < \omega_1^2 \leq \omega_2^2 \leq \dots \leq \omega_n^2)$. Rewriting Eq. (2) in terms of Y , we obtain:

$$\frac{d^2 Y}{dt^2} + D Y = 0 \quad \dots\dots\dots(9)$$

For the overall system to be in the state of a principal vibration:

$$Y = B \cos(\omega t + \alpha) \quad \dots\dots\dots(10)$$

it is obvious that a necessary and sufficient condition is $\omega^2 = \omega_i^2 (i = 1, 2, \dots, n)$. This means that the roots of the frequency equation (5) are

$$\omega^2 = \omega_i^2 (i = 1, 2, \dots, n).$$

With regard to the rank of matrix, the following relation holds⁽⁵⁾:

$$\begin{aligned} \text{rank}(K - \omega^2 M) &= \text{rank}\{C^T (K - \omega^2 M) C\} \\ &= \text{rank}(D - \omega^2 E) \quad \dots\dots\dots(11) \end{aligned}$$

Therefore, if ω^2 is a root of Eq. (5) with multiplicity p , the rank of $K - \omega^2 M$ becomes $n - p$ and the vectors which satisfy Eq. (4) form a p -dimensional vector space⁽⁶⁾. In other words, if the natural frequency is a simple root of the frequency equation, the corresponding A 's form a one-dimensional vector space, so that the normal mode of vibration, defined as the ratios of the components of A , is uniquely

determined; but if such is not the case, the corresponding normal mode of vibration is not uniquely determined.

2. First-order differential equations concerning the natural frequencies and amplitudes of the principal vibrations

In order to develop our theory, we shall assume that the partial systems are only statically coupled with one another. Let the number of partial systems be denoted by r ; the degree-of-freedom of the system i (the i th partial system), by n_i , and the column vector whose components are n_i generalized coordinates describing the motion of the system i , by X_i . Then we can substitute the following expressions into the equations obtained in the preceding chapter:

$$\left. \begin{aligned} X &= \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{bmatrix}, \quad M = \begin{bmatrix} M_1 & & \\ & M_2 & 0 \\ & 0 & \ddots \\ & & & M_r \end{bmatrix}, \quad K = K_0 + K_1 \\ K_0 &= \begin{bmatrix} K_{10} & & \\ & K_{20} & 0 \\ & 0 & \ddots \\ & & & K_{r0} \end{bmatrix}, \quad K_1 = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1r} \\ K_{21} & K_{22} & \dots & K_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ K_{r1} & K_{r2} & \dots & K_{rr} \end{bmatrix} \quad \dots\dots\dots(12) \end{aligned} \right\}$$

where M_i , K_{i0} , and $K_{ii} (i = 1, 2, \dots, r)$ are real symmetric matrixes of order n_i , where $K_{ij} (i \neq j)$ is the $n_i n_j$ matrix with $K_{ji}^T = K_{ij}$, and where K_1 and K_0 are taken in such a way that K_1 vanishes when the degree of connectivity among the partial systems becomes zero, while K_0 remains constant regardless of the connectivity.

Let us consider the system which results when K_1 is replaced by $sK_1 (0 \leq s \leq 1)$, thus continuously changing the degree of connectivity among the partial systems. Then the equation corresponding to Eq. (4) becomes:

$$(K_0 + sK_1 - \Omega M)A = 0 \quad \dots\dots\dots(13)$$

where $\Omega = \omega^2$. Since $K_0 + sK_1$ remains positive definite on an s interval $[0, 1]$, all the discussions

Fig. 1 Overall system composed of three partial systems

in the preceding chapter of the principal vibrations of the overall system may be applied to the overall system now introduced for an arbitrary value of s .

The frequency equation is:

$$|K_0 + sK_1 - \Omega M| = 0 \quad (14)$$

By expanding the determinant and by denoting the result newly by $f(s, \Omega)$, this equation may be rewritten as follows:

$$\left. \begin{aligned} f(s, \Omega) &= 0, \\ f(s, \Omega) &= {}_n l_0 \Omega^n + ({}_{n-1} l_1 s + {}_{n-1} l_0) \Omega^{n-1} \\ &\quad + ({}_{n-2} l_2 s^2 + {}_{n-2} l_1 s + {}_{n-2} l_0) \Omega^{n-2} + \dots + ({}_0 l_n s^n \\ &\quad + {}_0 l_{n-1} s^{n-1} + \dots + {}_0 l_0) \end{aligned} \right\} \quad (15)$$

In this equation, f , $\partial f / \partial s$, and $\partial f / \partial \Omega$ are all continuous functions of (s, Ω) . Then, with the exceptions of the singular points satisfying:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial \Omega} = 0 \quad (16)$$

the set of points (s, Ω) satisfying Eq. (15) forms a continuous curve on the s - Ω plane. The curve can be expressed as⁽⁷⁾:

$$\left. \begin{aligned} \text{If } \frac{\partial f}{\partial \Omega} \neq 0, \Omega &= \Omega(s), \\ \frac{d\Omega}{ds} &= - \left(\frac{\partial f}{\partial s} \right) / \frac{\partial f}{\partial \Omega} \end{aligned} \right\} \quad (17)$$

That is, Ω and $d\Omega/ds$ are uniquely determined by Eq. (17) on an interval, I , appropriately taken in the region of s where the natural frequency is a simple root of the frequency equation.

Now let us put:

$$N = K_0 + sK_1 - \Omega M \quad (18)$$

If we take such a point (s, Ω) as $s \in I$ and $\Omega = \Omega(s)$, rank $N = n-1$ at this point. Therefore, it is clear that there exists at least one ${}_i N_{-j}$ satisfying $|{}_i N_{-j}| \neq 0$, ${}_i N_{-j}$ being the submatrix obtained from N by removing its i th row and j th column. By the continuity of $|{}_i N_{-j}|$ in s , we can determine an interval, I_0 , such that $|{}_i N_{-j}(s)| \neq 0 (s \in I_0 \subset I)$. Let the j th component of A be denoted by a_j ; the column vector obtained from A by removing its j th component, by ${}_j A$, and the column vector obtained from the j th column, N_j , of N by removing its i th component, by ${}_i N_j$. Then on I_0 , ${}_j A$ is uniquely determined from Eq. (13) in the following form:

$${}_j A = -{}_i N_{-j}^{-1} {}_i N_j a_j \quad (19)$$

It would be reasonable to assume that $a_j =$ constant on I_0 , because we are now interested in the normal modes of vibration. On this assumption, the (19) relation states that the normal mode of vibration or each amplitude of the principal vibration is uniquely determined. The functions, $\Omega = \Omega(s)$ and ${}_j A = {}_j A(s)$, are continuously differentiable because of the fact that $f(s, \Omega)$ is such.

In Eq. (13), by formally assuming $\Omega = \Omega(s)$ and

$A = A(s)$, and by differentiating the equation with respect to s , we obtain:

$$N \frac{dA}{ds} - \frac{d\Omega}{ds} MA + K_1 A = 0 \quad (20)$$

Then, if it is assumed that $a_j =$ constant on I_0 , as has been mentioned above, either Eq. (13) or Eq. (20) may be regarded as a system of n simultaneous equations in the n unknowns, Ω and ${}_j A$. Solving the relation (20) for the derivatives yields:

$$\left[\begin{array}{c} \frac{d{}_j A}{ds} \\ \frac{d\Omega}{ds} \end{array} \right] = -[N_{-j} - MA]^{-1} K_1 A \quad (21)$$

where N_{-j} denotes the submatrix obtained from N by removing its j th column. The solutions of this differential equation on I_0 under the given initial conditions should necessarily be identical with the $\Omega(s)$ in the (17) expression and with ${}_j A(s)$ in the (19) expression.

When Eq. (15) is introduced in addition to Eq. (20), it would be more advantageous for the purposes of our analysis to make use of the following equations:

$$\left. \begin{aligned} \frac{d\Omega}{ds} &= - \left(\frac{\partial f}{\partial s} \right) / \frac{\partial f}{\partial \Omega} \\ \frac{d{}_j A}{ds} &= \frac{d\Omega}{ds} {}_i N_{-j}^{-1} {}_i M A - {}_i N_{-j}^{-1} {}_i K_1 A \end{aligned} \right\} \quad (22)$$

where ${}_i M$ denotes the submatrix obtained from M by removing its i th row and where ${}_i K_1$ denotes the submatrix obtained from K_1 by removing its i th row.

If $I_0 \supset [0, 1]$, by solving an initial value problem of the differential equation (21) or (22) under the given initial values, $\Omega(0)$ and ${}_j A(0)$, the natural frequency and the normal mode of vibration of the given overall system are obtained.

Besides, if n natural frequencies are all simple roots of the frequency equation on the interval $[0, 1]$, and if there is present such an I_0 as $I_0 \supset [0, 1]$ associated with each natural frequency, n sets of differential equations of the above form can be introduced. In this case, by setting

$$s=0, X_i = A_i \cos(\omega t + \alpha)$$

in Eq. (13), we obtain:

$$\left[\begin{array}{ccc} (K_{10} - \Omega M_1) A_1 & & 0 \\ & (K_{20} - \Omega M_2) A_2 & \\ 0 & & (\bar{K}_{r0} - \Omega M_r) A_r \end{array} \right] = 0 \quad (23)$$

This represents the principal vibrations of the overall system when $s=0$. Now, if A_i is not zero, $|K_{i0} - \Omega M_i|$ must be zero, while $|K_{j0} - \Omega M_j| \neq 0$ ($j \neq i$) makes it necessary that $A_j = 0$ ($j \neq i$) on the basis of our assumption. This is what we have

referred to above as a one-to-one and continuous correspondence between the principal vibrations of the overall system and those of the partial systems.

3. A dividing analysis of the frequency equation

We have thus far considered the natural frequencies and the normal modes of vibration simultaneously, and have investigated the correlations between the change in s and the change in the principal vibrations. However, it is already clear that the treatment of the natural frequencies may be separated from that of the normal modes of vibration.

Rewriting the first equation in Eq. (22) gives:

$$\left. \begin{aligned} \frac{d\Omega}{ds} &= \frac{\varphi(s, \Omega)}{\psi(s, \Omega)} \\ \varphi(s, \Omega) &= -[{}_{n-1}l_1\Omega^{n-1} + (2{}_{n-2}l_2s \\ &\quad + {}_{n-2}l_1)\Omega^{n-2} + \dots + \{n_0l_ns^{n-1} \\ &\quad + (n-1)l_0\Omega^{n-1} + \dots + {}_0l_1\}] \\ \psi(s, \Omega) &= n_1l_0\Omega^{n-1} + (n-1)({}_{n-1}l_1s \\ &\quad + {}_{n-1}l_0)\Omega^{n-2} + \dots + ({}_1l_{n-1}s^{n-1} + {}_1l_{n-2}s^{n-2} \\ &\quad + \dots + {}_1l_0) \end{aligned} \right\} \dots (24)$$

If only the natural frequency is a simple root of the frequency equation, this differential equation is necessary and sufficient for the function, $\Omega = \Omega(s)$, to be fixed.

In case the n natural frequencies are all different from one another on the s interval $[0, 1]$, the natural frequencies of the overall system are correlated with those of the partial systems one-to-one and continuously, according to this differential equation.

If we are trying to obtain not only the natural frequencies but also the normal modes of vibration, it would be possible for us to solve the differential equation (24) first, and then to solve the simultaneous algebraic equations in Ω which are the result of the substitution of the obtained Ω into Eq. (13).

4. The significance of the dividing differential equation method

Now let us consider the significance of this dividing analysis. Attention should first be paid to the dividing comprehension of the principal vibrations by correlating, one-to-one, the principal vibrations of an overall system with those of the partial systems composing the overall system. Though there are many ways of dividing an overall system into partial systems, we will choose two or three ways by which each of the partial systems has its own dynamical coherence inside the system. By doing so and by giving the correspondence between the principal vibrations of the overall system and those of the partial systems, it might be possible for us to examine the degree of connectivity among

the partial systems and to find some basis for reducing the given problem of solving a principal vibration of the overall system to that of solving a principal vibration of a certain partial system with less degrees of freedom.

Next, the method introduces a concept of the rate of change in the natural frequency or in the amplitudes of the principal vibration with respect to the change in the degree of connectivity among the partial systems. For example, it may sometimes be quite important to compute and examine the magnitudes of the rates of change expressed by the relations (21), (22), or (24), as in the cases of design modification of the dimensions of machines or instruments.

Thirdly, the significance of the method as a means of the numerical computation of the principal vibrations of an overall system must be emphasized. It would generally be quite troublesome to arrange a frequency equation such as Eq. (15) for a system with many degrees of freedom, but if we do not mind taking the trouble to do so, Eq. (24) together with Eq. (13) would be most practical from the viewpoints of accuracy and convenience in actual computation.

5. A way of setting up an auxiliary system

Until now we have considered the system which results from replacing $K = K_0 + K_1$ by $K_0 + sK_1$ in order to change the spring constants of the springs by which the partial systems are connected with one another continuously from zero to given values. In this chapter, however, we shall split the matrix, K_0 , which has nothing to do with the connecting parts, into two matrixes, $K_0 = K_{01} + K_{02}$. In addition, we shall take a proper matrix, K_2 , and consider the system which results when we replace K by the right-hand side of the following relation:

$$K = K_{01} + sK_{02} + sK_1 + (1-s)K_2 \dots \dots \dots (25)$$

In this case, when $s=0$, K does not become K_0 but, rather, becomes the sum of K_{01} , which is one part of K_0 , and K_2 , which we have newly chosen. Let this system for $s=0$ be called "the auxiliary system." It seems to us important to devise good ways of dividing K_0 and of determining K_2 so that we can easily compute the principal vibrations of the auxiliary system and so that the natural frequencies all may become simple roots of the frequency equation. The matrix, K , in the expression (25) is, of course, positive definite on the s interval $[0, 1]$. Therefore, differential equations of the same type as were introduced in Chapters 2 and 3 can be introduced in exactly the same manner.

In the analysis of actual problems by the method

explained up to the preceding chapter, if $s \neq 0$, multiple roots of the frequency equation for an overall system will hardly occur. However, if $s = 0$, the case is different. In other words, it can happen that there exist natural frequencies of the same values among those of the partial systems, especially when the overall system is divided into many partial systems with a few degrees of freedom. Even in such cases, though, we can easily avoid the appearance of the multiple roots by setting up an auxiliary system.

The way of setting up an auxiliary system the principal vibrations of which are known to us in relation to a given overall system those of which are unknown to us, is closely related to the contents of the author's lecture⁽⁸⁾ titled "A study of numerical computation of the root of an equation."

In the case of setting up an auxiliary system, if we try not to lose our fundamental intention of dividing an overall system into partial systems, the analysis can not be more than a mere numerical computation of the principal vibrations of the overall system, because we can not establish the correlations between the given overall system and the given partial systems as they are.

It would be possible to generalize further this way of setting up an auxiliary system in such a way that the matrix, M , is changed with a change in s .

6. Comparison with traditional methods of solving the principal vibrations

The ways of proceeding presented in this paper — of dividing an overall system into partial systems, of setting up an auxiliary system, and of introducing first-order differential equations, could not be found among the hitherto reported papers dealing with the principal vibrations of multi-degree-of-freedom linear vibrating systems.

A representative method for the numerical computation of the principal vibrations is the "diagonalization method" (Jacobi's method⁽⁹⁾). This is a quite orthodox method of completing the diagonalization of a matrix by repeating simple orthogonal transformations n times in succession. One great advantage of this method is that we can obtain n principal vibrations simultaneously. If we wish to obtain n principal vibrations at the same time by the dividing differential equation method, we have to solve n sets of first-order differential equations at the same time. Since, however, our problem is not a simple eigenvalue problem for a real symmetric matrix, the diagonalization method complicates the process of computation and a large number of computations are required. On the contrary, an initial value

problem of the first-order differential equation can easily be solved numerically, and the programming for an electronic computer is very easy.

As a method of seeking the principal vibration of the maximum order or of the minimum order, the "power method⁽¹⁰⁾" is the most popular. By it a given problem is reduced to an eigenvalue problem of $M^{-1}K$ or $K^{-1}M$, and iterative computation is executed to find the largest eigenvalue of the matrix and the corresponding eigenvector. If we try to find the principal vibration of the order other than the maximum or the minimum using the power method, we have to compute, in due order, every principal vibration by utilizing the orthogonality relation for the eigenvectors⁽¹¹⁾; this might cause a deterioration in the accuracy of the numerical computations. On the other hand, by the dividing differential equation method, since the order of the principal vibrations of an overall system corresponds directly to the order of the magnitudes of the natural frequencies of the partial systems, we can compute the principal vibration in question separately.

When the frequency equation is first introduced and the normal mode of vibration is then computed as occasion demands after solving the frequency equation, our problem is reduced to one of finding the roots of an algebraic equation of a higher degree. The numerical method of solving an initial value problem of the first-order differential equation (24) by the dividing differential equation method might be as useful as such well-known traditional methods as "Gräffe's method⁽¹²⁾," judging from the accuracy of the computations and the quantity of computations necessary.

7. Numerical computations

When a principal vibration of an overall system is to be solved, the solution can not be, in general, obtained in the form of an already-known function of s , with the exception of particular and simple systems, whichever equation we may choose from among Eqs. (21), (22), and (24). Therefore, some suitable numerical method has to be utilized to solve an initial value problem of the differential equation. Since the initial values necessary for the numerical computations are the solutions of the principal vibrations of the partial systems or those of an auxiliary system, they can easily be computed by setting the degrees of freedom of the partial systems few enough or by setting up an auxiliary system technically.

The "Runge-Kutta method⁽¹³⁾" seems to be the most appropriate one for our purposes among the variety of numerical methods available. Let our

problem be expressed by :

$$\frac{dZ}{ds} = F(s, Z) \dots\dots\dots (26)$$

in order to deal with the three differential equations, (21), (22), and (24), collectively, where Z is a vector whose components are the functions of s now in question and where the right-hand side of Eq. (26) is a vector whose components are the functions of s and Z . When the s interval $[0, 1]$ is equally divided by m , by putting $\delta s = 1/m$ and by denoting the increase in Z corresponding to the δs by δZ , one step of the computations using the Runge-Kutta method can be written as :

$$\left. \begin{aligned} \delta Z &= \frac{1}{6} (D_1 + 2D_2 + 2D_3 + D_4) \\ D_1 &= F(s, Z) \delta s \\ D_2 &= F\left(s + \frac{\delta s}{2}, Z + \frac{D_1}{2}\right) \delta s \\ D_3 &= F\left(s + \frac{\delta s}{2}, Z + \frac{D_2}{2}\right) \delta s \\ D_4 &= F(s + \delta s, Z + D_3) \delta s \end{aligned} \right\} \dots\dots\dots (27)$$

The magnitude of the error in the components of the vector, $Z(1)$, obtained as a result of m step computations as above is known to be nearly proportional to $1/m^4$ (14). It seems, in general, that $\Omega = \Omega(s)$ and ${}_j A = {}_j A(s)$ have higher-order derivatives the magnitudes of which are not very large.

Therefore, we can expect that our computation has enough accuracy for practical purposes if m is small.

By the "Euler method (15)," which might be said to be the simplest numerical method, one step of the computations is written as :

$$\delta Z = F(s, Z) \delta s \dots\dots\dots (28)$$

The magnitude of the error is known to be nearly proportional to $1/m$. Thus, when compared with that of the Runge-Kutta method, the accuracy of the computations decreases to a large extent if the quantity of the computations is the same. However, if we wish to investigate the variations in the principal vibrations with an increase in s in addition to obtaining the solutions of the principal vibrations of a given overall system, the Euler method might on occasion be a better one for the purposes of analysis.

8. Multi-degree-of-freedom systems of a chain type

Let us analyze the principal vibrations of an n degree-of-freedom system of a chain type shown in Fig. 2, applying the theories we have developed.

After introducing the simultaneous equations which correspond to Eq. (4), we obtain the following coefficient matrix,

$$N(1) = K - \Omega M :$$

$$N = \begin{bmatrix} -m_1\Omega + k_{01} + k_1 & -k_1 & & & 0 \\ -k_1 & -m_2\Omega + k_{02} + k_1 + k_2 & -k_2 & & \\ & -k_2 & \ddots & \ddots & \\ & & -k_{n-2} & -m_{n-1}\Omega + k_{0n-1} + k_{n-2} + k_{n-1} & -k_{n-1} \\ 0 & & & -k_{n-1} & -m_n\Omega + k_{0n} + k_{n-1} \end{bmatrix} \dots\dots\dots (29)$$

If the following equation holds :

$$\sum_{j=1}^{n-1} c_j N_j = 0 \dots\dots\dots (30)$$

the n th component of the left-hand side $= 0$ gives $c_{n-1} = 0$; then, the $n-1$ th component $= 0$ gives $c_{n-2} = 0$. Continuing in this fashion, it results that $c_j = 0$ ($j = 1, 2, \dots, n-1$) and that $n-1$ vectors, N_j ($j = 1, 2, \dots, n-1$), are linearly independent. This means that $\text{rank } N = n-1$; thus, the natural frequencies are all found to be simple roots of the frequency equation.

Next, in order to divide the overall system into more than two partial systems, let us choose more than one spring connecting the masses with one another; let us consider the system which results when these spring constants, k_i , are replaced by sk_i . In this case; if $s \neq 0$, it can also be easily verified that $\text{rank } N(s) = n-1$ and that $\Omega(s)$'s are all simple roots of the frequency equation.

When the constant, k_{0i} , is split into two parts, such as $k_{0i} = k_{0i1} + k_{0i2}$, in order to construct a system furnished by replacing k_{0i} by $k_{0i1} + sk_{0i2} + (1-s)k_{0i3}$ with a proper k_{0i3} , in addition to the replacement of k_i by sk_i , the natural frequencies still all become simple roots of the frequency equation when $s \neq 0$. For

Fig. 2 n degree-of-freedom system of a chain type

multi-degree-of-freedom systems of a chain type, the appearance of the multiple roots at $s=0$ can easily be avoided in this manner by setting up an auxiliary system.

Judging from the form of the (29) expression, it goes without saying that an auxiliary system can be constructed by subtracting proper springs from or adding them to the connecting parts without the appearance of the multiple roots.

By putting $x_i = a_i \cos(\omega t + \alpha)$ in order to examine the amplitudes of the principal vibrations, we obtain the following equation corresponding to Eq. (13):

$$\sum_{j=1}^n a_j N_j = 0 \quad \dots\dots\dots (31)$$

From this, a treatment similar to that with Eq. (30) shows that $a_1 \neq 0$ and $a_n \neq 0$ when $s \neq 0$.

Let us now give an example of actual numerical computations of a case with three degrees of freedom shown in Fig. 3. We shall divide the overall system into two partial systems, the system 1 and the system 2, at the dash-dot curve, thus obtaining the two following equations, (32) and (33), which correspond to Eqs. (20) and (24) respectively.

$$\begin{bmatrix} -m_1\Omega + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -m_2\Omega + k_2 + k_3 & -sk_3 \\ 0 & -sk_3 & -m_3\Omega + k_4 + sk_3 \end{bmatrix} \begin{bmatrix} \frac{da_1}{ds} \\ \frac{da_2}{ds} \\ \frac{da_3}{ds} \end{bmatrix} - \frac{d\Omega}{ds} \begin{bmatrix} m_1 a_1 \\ m_2 a_2 \\ m_3 a_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0 \quad \dots\dots\dots (32)$$

Fig. 3 Three-degree-of-freedom system

$$\left. \begin{aligned} \frac{d\Omega}{ds} &= -\frac{{}_2l_1\Omega^2 + {}_1l_1\Omega + {}_0l_1}{3{}_3l_0\Omega^2 + 2({}_2l_1s + {}_2l_0)\Omega + ({}_1l_1s + {}_1l_0)} \\ {}_2l_1 &= m_1(m_2 + m_3)k_3 \\ {}_1l_1 &= -m_1k_3(k_2 + k_4) - (m_2 + m_3)(k_1 + k_2)k_3 \\ {}_0l_1 &= (k_1k_2 + k_2k_4 + k_4k_1)k_3 \\ {}_3l_0 &= -m_1m_2m_3 \\ {}_2l_0 &= m_1m_2k_4 + m_2m_3(k_1 + k_2) + m_3m_1k_2 \\ {}_1l_0 &= -(m_1k_2k_4 + m_2(k_1 + k_2)k_4 + m_3k_1k_2) \end{aligned} \right\} \quad (33)$$

For the principal vibrations of the partial systems, we obtain:

$$\begin{aligned} \text{System 1: } & \begin{bmatrix} -m_1\Omega + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\Omega + k_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \\ \text{System 2: } & (-m_3\Omega + k_4)a_3 = 0 \end{aligned} \quad \dots\dots\dots (34)$$

Now let us consider a case where:

$$m_1 = m_2 = m_3 = m, \quad k_1 = k_2 = k_3 = k_4 = k \quad \dots\dots\dots (35)$$

This is representative of the cases where three masses and four spring constants have nearly the same values. By putting $\Omega_0 = k/m$ and $\sigma = \Omega/\Omega_0$, and by rearranging Eq. (33), we obtain:

$$\frac{d\sigma}{ds} = \frac{2\sigma^2 - 6\sigma + 3}{3\sigma^2 - (4s + 8)\sigma + 6s + 4} \quad \dots\dots\dots (36)$$

Fig. 4 Natural frequencies of the three-degree-of-freedom system

Table 1 Natural frequencies (σ) of the three-degree-of-freedom system computed by the Euler method (equally-dividing number = 50)

Corresponding partial Systems	System 1		System 2
Orders of the principal vibrations	First-order	Second-order	Third-order
Initial values	$(3 - \sqrt{5})/2$	$(3 + \sqrt{5})/2$	1
Exact solutions	$2 - \sqrt{2}$	$2 + \sqrt{2}$	2
Solutions by Eq. (36)	0.588 88	3.404 8	2.004 5
Solution by Eq. (36)	1.005 3	0.997 25	1.002 3
Exact solution			

By applying the Euler method to this differential equation, the computations executed by dividing equally the s interval $[0, 1]$ by 50 yield the results shown in Table 1 and Fig. 4.

Next, the results of the computations by the Runge-Kutta method to find the second-order natural frequency are shown in Table 2. Judging from the equally-dividing numbers and the accuracy of the solutions, the dividing differential equation method can be called an excellent method of numerical computation.

Finally, the results of the computations by Eq. (32) to find the second-order natural frequency and the corresponding normal mode of vibration are shown in Table 3 and Fig. 5. The computations were carried out by the Euler method of equally dividing the s interval $[0, 1]$ by 100, using the following equations, which were introduced by putting $a_3 = \text{constant} = 1$ regardless of the s in Eq. (32):

$$\left. \begin{aligned} \frac{da_1}{ds} &= \frac{(a_2-1)}{\Delta} \{a_1(-\sigma+1) + a_2+1\} \\ \frac{da_2}{ds} &= \frac{(a_2-1)}{\Delta} \{a_1 + (a_2+1)(-\sigma+2)\} \\ \frac{d\sigma}{ds} &= \frac{(a_2-1)}{\Delta} \{(-\sigma+1)(-\sigma+2)-1\} \\ \Delta &= -s\{a_1 + (a_2+1)(-\sigma+2)\} \\ &\quad -(-\sigma+1)(-\sigma+2)+1 \end{aligned} \right\} \dots\dots (37)$$

9. Conclusions

(1) In this paper, the author has proposed "the dividing differential equation method," in which the principal vibrations of a multi-degree-of-freedom linear vibrating system are made to correspond one-to-one to those of the partial systems composing the overall system, and a first-order differential equation with an independent variable expressing the degree of connectivity among the partial systems has been introduced. The theories have been developed for a case where the natural frequencies and the normal modes of vibration are studied simultaneously and for another case where only the natural frequencies are considered.

(2) This analytical method can be utilized to study the correlations between the principal vibrations of the overall system and those of the partial systems, and to examine the influence of the change in the degree of connectivity on the principal vibrations of the overall system. Besides, this is a splendid numerical method for computing the

Table 2 Second-order natural frequency (σ) of the three-degree-of-freedom system computed by the Runge-Kutta method

Equally-dividing numbers	1	2
Solutions by Eq. (36)	2.0025	2.0007
Solution by Eq. (36)	1.0013	1.0004
Exact solution		

principal vibrations of the overall system.

(3) Next, under the title of "A way of setting up an auxiliary system," the author has presented a technical method of computing the principal vibrations of the overall system. It can be effectively used when there are some natural frequencies of the partial systems with the same values.

(4) The initial values for the numerical computations to find the principal vibrations of the overall system can easily be obtained by setting a sufficiently few degrees of freedom of the partial systems, or by setting up a suitable auxiliary system. We can then make a digital computer program for general use, the values of the system constants being given as input data. The process of computation is very simple, and the programming is quite easy.

(5) Finally, by applying this method to a multi-degree-of-freedom system of a chain type, and by carrying out actual computations on a three-degree-of-freedom system, we have demonstrated the usefulness of this method.

The dividing differential equation method forms, in a sense, a counterpart to "dividing successive computations concerning the forced vibrations,"^{(16) (17)} on which the present author reported before. He hopes that his readers will refer to those previous reports in addition to the present paper.

Table 3 Second-order principal vibration of the three-degree-of-freedom system obtained by the Euler method (equally-dividing number=100)

	a_1	a_2	σ
Initial values	0	0	1
Exact solutions	-1	0	2
Solutions by Eq. (37)	-1.0008	-0.00501	2.0050
Solution by Eq. (37)	1.0008	—	1.0025
Exact solution

Fig. 5 Second-order principal vibration of the three-degree-of-freedom system

An electronic computer MELCOM 1101 F installed at Seikei University was used for the computations. The author is grateful to Assistant I. Hashimoto and to members of the computer laboratory for their assistance.

The author wishes also to express his deep gratitude to the editors of JSME for the valuable advice they have given him in this study.

References

- (1) T. Yamanouchi: *General Dynamics*, (1959), p. 178, Iwanami-shoten.
- (2) H. Toyama: *Theory of Matrix*, (1952), p. 155, Kyoritsu-shuppan.
- (3) Ref. (1), p. 168.
- (4) S. Furuya: *Matrix and Determinant*, (1959), p. 115.
- (5) Ref. (2), p. 80.
- (6) M. Iyanaga and M. Sugiura: *Algebra for Applied Mathematicians*, (1960), p. 41, Iwanami-shoten.
- (7) T. Takagi: *An Introduction to Analysis*, (1968), p. 294, Iwanami-shoten.
- (8) M. Kuroda: *Preprint of Kansai Branch, Japan Soc. Mech. Engrs.*, No. 44-1 (1968), p. 70.
- (9) T. Akasaka: *Numerical Computations*, (1967), p. 323, Corona-sha.
- (10) T. Uno: *Numerical Computations for Computers*, (1963), p. 64, Asakura-shoten.
- (11) Ref. (9), p. 296.
- (12) A. Watari: *Mechanical Vibrations*, (1966), p. 67, Maruzen.
- (13) Ref. (9), p. 351.
- (14) Ref. (10), p. 150.
- (15) Ref. (9), p. 348.
- (16) M. Kuroda: *Bulletin of JSME*, Vol. 10, No. 37 (1967), p. 48.
- (17) M. Kuroda: *Preprint of Japan Soc. Mech. Engrs.*, No. 154 (1966), p. 25.