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On the Self-Synchronization of Mechanical Vibrators*

(Part 3, Forced Frictional Vibrations)

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In this paper the forced frictional vibrations acting upon friction, which is generally expressed as a function of relative sliding velocities, are treated from the point of view of self-synchronization.

First of all, the vibrations are considered as approximately a combination of two simple harmonic vibrations, one with the frequency of free vibration, the other with the frequency of excitation.

The motion is considered as a vibration with the frequency of excitation with slowly varying amplitude and phase, and a certain zone of synchronization is determined next.

Further an experiment is conducted to verify the result of analysis.

1. Introduction

Free self-excited vibrations caused by dry friction which is generally expressed as a function of relative sliding velocities between two solid bodies have been investigated by Watari⁽¹⁾ and Takano⁽²⁾. In the paper prepared by Watari experimental results obtained from dry friction show good coincidence with theoretical results. In the paper by Takano the theoretical results have been shown as derived by approximating characteristics of friction, to rectilinear segments.

A kind of the response of the entrained vibration has been obtained in experiments, conducted by Ogasawara⁽³⁾, on the forced frictional vibrations during which the surface pressure between two solid bodies changes periodically.

This paper deals with frictional vibrations during which the periodic external force acting upon a vibrating mass is in the same direction as that of motion of the mass from the point of view of self-synchronization.

The mechanism of this phenomenon has connection with certain vibrations in mechanical engineering, for instance, in machine tool flutter caused by cutting unbalanced metals and the vibrations produced by friction drive.

It is assumed that the velocity of the base is always larger than that of the mass in the theoretical

analysis. The friction force varying with the velocity of slipping of the mass relative to the base is expanded in the Taylor series in order to investigate the combination of two simple harmonic vibrations and the response of the entrained vibration. Further an experiment is conducted to verify the results of analysis.

2. Equation of motion

Let us consider the mechanical system shown in Fig. 1. A mass m rests on a rough belt which moves with a constant speed V , and is attached to a rigid support through a linear spring of spring constant k and the external periodic force $E_c \sin pt$ is applied to the mass. The position of the mass is assumed to be fixed at its distance z from the point at which the spring is neither stretched nor compressed. Applying the Newton's equation to the system in Fig. 1, the equation of motion is

$$m\ddot{z} + kz = F(V - \dot{z}) + E_c \sin pt \quad \dots\dots\dots (1)$$

It is convenient to introduce a new variable x , replacing z , by the relation

Fig. 1 Mechanical system capable of executing forced self-excited vibrations

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$$z - X = F(V)/k$$

The differential equation for x is then,

$$m\ddot{x} + kx = F(V - \dot{x}) - F(V) + E_c \sin pt \dots (2)$$

If the friction force as a function of relative velocity is assumed as shown in Fig. 2, the problem could be solved in pieces which then are combined by matching boundary conditions.

In this analysis, the function $F(V - x)$ may be expanded in the Taylor series in the range of $V - x > 0$. The result is:

$$F(V - \dot{x}) - F(V) = -\dot{x}F_v' + \frac{\dot{x}^2}{2}F_v'' - \frac{\dot{x}^3}{6}F_v''' \dots (3)$$

$$\text{where } U = V - \dot{x}, \quad F_v' = \left. \frac{d}{dU} F(U) \right|_{U=V}, \\ F_v'' = \left. \frac{d^2}{dU^2} F(U) \right|_{U=V}, \quad F_v''' = \left. \frac{d^3}{dU^3} F(U) \right|_{U=V}$$

3. Beat

3.1 Solution

If a non-linear system has a self-excited vibration of frequency ω , then when the system is subjected to an input of a slightly different frequency p , we expect the simultaneous occurrence of both p and ω , and, through nonlinear interaction, the beat frequency $|p - \omega|$.

Let $\tau_1 = \omega t$, and substitute this expression into Eq. (2) to obtain

$$x'' + x = -f_v' x' + \frac{f_v''}{2} x'^2 - \frac{f_v'''}{6} x'^3 + E \sin \zeta \tau_1 \dots (4)$$

$$\text{where } \omega^2 = k/m, \quad \zeta = p/\omega, \quad E = E_c/k, \\ x' = \frac{dx}{d\tau_1}, \quad x'' = \frac{d^2x}{d\tau_1^2}, \quad f_v' = \frac{\omega}{k} F_v', \\ f_v'' = \frac{\omega^2}{k} F_v'', \quad f_v''' = \frac{\omega^3}{k} F_v'''$$

The solution of Eq. (4) can be written as

$$\left. \begin{aligned} x &= a \cos(\tau_1 + \phi) + b \sin \zeta \tau_1 \\ x' &= -a \sin(\tau_1 + \phi) + b \zeta \cos \zeta \tau_1 \end{aligned} \right\} \dots (5)$$

Fig. 2 Damping force as a function of velocity

and a and ϕ are slowly varying functions of τ_1 , and b is constant. Namely,

$$b = \frac{E}{1 - \zeta^2} \dots (6)$$

Differentiating the first expression of Eq. (5) with respect to τ_1 and equating the second expression of Eq. (5) to the above expression gives

$$\cos(\tau_1 + \phi) \frac{da}{d\tau_1} - a \sin(\tau_1 + \phi) \frac{d\phi}{d\tau_1} = 0 \dots (7)$$

Substituting Eq. (5) into Eq. (4) gives

$$\begin{aligned} -\sin(\tau_1 + \phi) \frac{da}{d\tau_1} - a \cos(\tau_1 + \phi) \frac{d\phi}{d\tau_1} \\ = g(\tau_1, \zeta \tau_1) + E \sin \zeta \tau_1 - b \sin \zeta \tau_1 \\ + b \zeta^2 \sin \zeta \tau_1 \dots (8) \end{aligned}$$

where

$$\begin{aligned} g = -f_v' \{-a \sin(\tau_1 + \phi) + b \zeta \cos \zeta \tau_1\} \\ + \frac{f_v''}{2} \{-a \sin(\tau_1 + \phi) + b \zeta \cos \zeta \tau_1\}^2 \\ - \frac{f_v'''}{6} \{-a \sin(\tau_1 + \phi) + b \zeta \cos \zeta \tau_1\}^3 \end{aligned}$$

From Eq. (7) and (8),

$$\left. \begin{aligned} da/d\tau_1 &= -g(\tau_1, \zeta \tau_1) \sin(\tau_1 + \phi) \\ ad\phi/d\tau_1 &= -g(\tau_1, \zeta \tau_1) \cos(\tau_1 + \phi) \end{aligned} \right\} \dots (9)$$

Since it is assumed that a and ϕ are slowly varying, Eq. (9) may be averaged over one cycle as follows.

$$\left. \begin{aligned} \frac{da}{d\tau_1} &= \frac{-1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} g(\tau_1, \zeta \tau_1) \sin(\tau_1 + \phi) \\ &\quad \times d\tau_1 d(\zeta \tau_1) \\ &= -\frac{f_v'}{2} a - \frac{f_v''}{16} a^3 - \frac{f_v'''}{8} b^2 \zeta^2 a \\ a \frac{d\phi}{d\tau_1} &= \frac{-1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} g(\tau_1, \zeta \tau_1) \cos(\tau_1 + \phi) \\ &\quad \times d\tau_1 d(\zeta \tau_1) = 0 \end{aligned} \right\} \dots (10)$$

From the result of Eq. (10), ϕ is constant. The steady-state motion corresponds to $da/d\tau_1 = 0$. Hence Eq. (10) becomes

$$\left. \begin{aligned} a_0 &= 0 \\ a_0 &= \sqrt{-8 \frac{f_v'}{f_v'''} - 2b^2 \zeta^2} \end{aligned} \right\} \dots (11)$$

Real solutions of Eq. (11) exist if, and only if

$$b^2 \zeta^2 + \frac{4f_v'}{f_v'''} < 0,$$

the zero subscript denoting the steady state. Equation (10) may be integrated straightforward; thus

$$a^2 = \frac{-8 \frac{f_v'}{f_v'''} - 2b^2 \zeta^2}{1 - C \exp\left(f_v' + \frac{f_v'''}{4} b^2 \zeta^2\right) \tau_1}$$

where C is an integration constant.

3.2 Stability of motions

Let η be a small perturbation and set $a = a_0 + \eta$. If we put the above expression Eq. (10) and use the steady-state Eq. (11), Eq. (10) becomes

$$\frac{d\eta}{d\tau_1} = -\frac{f_v'}{2}\eta - \frac{3}{16}f_v'''a_0^2\eta - \frac{f_v'''}{8}b^2\zeta^2\eta \dots (12)$$

The condition under which Eq. (12) is stable is

$$f_v' + \frac{3}{8}f_v'''a_0^2 + \frac{f_v'''}{4}b^2\zeta^2 > 0 \dots (13)$$

The results of the above discussion can be summed up in Table 1. Under the condition of No. 1, harmonic vibration with amplitude b and frequency p exists and in No. 2, the phenomenon of the beat exists as a combination of two simple harmonic components, one with amplitude a_0 and frequency ω and the other with amplitude b and frequency p .

4. Entrained vibrations

4.1 Solution of motion

The beat frequency disappears suddenly as soon as p reaches a certain zone of synchronization. In this interval there exists only one frequency p , and everything happens as if the original frequency ω were entrained by the variable frequency p .

Let $\tau_2 = pt$, and substitute this expression into Eq. (2) to obtain

$$\begin{aligned} \frac{d^2x}{d\tau_2^2} + x &= \frac{p^2 - \omega^2}{p^2}x - \left(\frac{dx}{d\tau_2}\right) \frac{p}{mp^2}F_v' \\ &+ \frac{1}{2}\left(\frac{dx}{d\tau_2}\right)^2 \frac{p^2}{mp^2}F_v'' - \frac{1}{6}\left(\frac{dx}{d\tau_2}\right)^3 \frac{p^3}{mp^2}F_v''' \\ &+ \frac{E_c}{mp^2} \sin \tau_2 \dots (14) \end{aligned}$$

Since p is nearly equal to ω in the regions of frequency entrainment, we may get the approximation as follows.

$$\left. \begin{aligned} \frac{p}{mp^2}F_v' &\doteq \frac{\omega}{m\omega^2}F_v' = \frac{\omega}{k}F_v' = f_v', \\ \frac{p^2}{mp^2}F_v'' &\doteq f_v'', \quad \frac{p^3}{mp^2}F_v''' \doteq f_v''', \quad \frac{E_c}{mp^2} \doteq E \end{aligned} \right\} \dots (15)$$

Thus we write

$$x'' + x = hx - x'f_v' + \frac{x'^2}{2}f_v'' - \frac{x'^3}{6}f_v''' + E \sin \tau_2 \dots (16)$$

where $h = (p^2 - \omega^2)/p^2$ detuning, $x' = dx/d\tau_2$, $x'' = d^2x/d\tau_2^2$. Since p is close to ω , the solution of Eq. (16) can be written as

$$\left. \begin{aligned} x &= A \cos(\tau_2 - \phi) \\ x' &= -A \sin(\tau_2 - \phi) \end{aligned} \right\} \dots (17)$$

where A and ϕ are slowly varying functions of τ_2 . By proceeding in much the same way as we proceeded in Section 3, we may determine the result as follows;

Table 1 Amplitude and conditions for stability of the beat

No.	Amplitude	Condition for stability
1	0	$4f_v' + f_v'''b^2\zeta^2 > 0$
2	$\sqrt{-8f_v'/f_v''' - 2b^2\zeta^2}$	$f_v''' > 0$ and $4f_v' + f_v'''b^2\zeta^2 < 0$

$$\cos(\tau_2 - \phi) \frac{dA}{d\tau_2} + A \sin(\tau_2 - \phi) \frac{d\phi}{d\tau_2} = 0 \dots (18)$$

If we substitute Eq. (17) into Eq. (16), Eq. (16) becomes,

$$-\sin(\tau_2 - \phi) \frac{dA}{d\tau_2} + A \cos(\tau_2 - \phi) \frac{d\phi}{d\tau_2} = G(\tau_2) \dots (19)$$

where

$$\begin{aligned} G(\tau_2) &= hA \cos(\tau_2 - \phi) + f_v' A \sin(\tau_2 - \phi) \\ &+ \frac{f_v''}{2} \{-A \sin(\tau_2 - \phi)\}^2 \\ &- \frac{f_v'''}{6} \{-A \sin(\tau_2 - \phi)\}^3 + E \sin \tau_2 \end{aligned}$$

Solving these simultaneous equations gives

$$\left. \begin{aligned} dA/d\tau_2 &= -G(\tau_2) \sin(\tau_2 - \phi) \\ Ad\phi/d\tau_2 &= G(\tau_2) \cos(\tau_2 - \phi) \end{aligned} \right\} \dots (20)$$

Since it was assumed that A and ϕ were slowly varying, Eq. (20) may be averaged over one cycle as follows.

$$\left. \begin{aligned} \frac{dA}{d\tau_2} &= \frac{-1}{2\pi} \int_0^{2\pi} G(\tau_2) \sin(\tau_2 - \phi) d\tau_2 \\ &= -\frac{f_v'}{2}A - \frac{f_v'''}{16}A^3 - \frac{E}{2} \cos \phi \\ \frac{d\phi}{d\tau_2} &= \frac{1}{2\pi A} \int_0^{2\pi} G(\tau_2) \cos(\tau_2 - \phi) d\tau_2 \\ &= \frac{h}{2} + \frac{E}{2A} \sin \phi \end{aligned} \right\} \dots (21)$$

The steady-state harmonic solutions corresponding to $(dA/d\tau_2) = 0$ and $(d\phi/d\tau_2) = 0$ are

$$\left. \begin{aligned} f_v'A_0 + \frac{f_v'''}{8}A_0^3 + E \cos \phi_0 &= 0 \\ hA_0 + E \sin \phi_0 &= 0 \end{aligned} \right\} \dots (22)$$

Eliminating ϕ_0 gives

$$\left(f_v' + \frac{f_v'''}{8}A_0^2\right)^2 + h^2 = \left(\frac{E}{A_0}\right)^2 \dots (23)$$

Figure 3 is obtained by plotting Eq. (23) for $f_v' = -0.8$ and $f_v''' = 0.8$ as the parameter of E .

4.2 Stability of motion

Let ξ and η be small perturbations and set $A = A_0 + \eta$, $\phi = \phi_0 + \xi$. If we substitute the above expressions into Eq. (21) and use the steady-state

Fig. 3 Response curves for harmonic vibrations

Eq. (22), Eq. (21) becomes

$$\left. \begin{aligned} \frac{d\eta}{d\tau_2} &= -\frac{f_v'}{2}\eta - \frac{3}{16}f_v'''A_0^2\eta + \frac{E}{2}(\sin\phi_0)\xi \\ \frac{d\xi}{d\tau_2} &= -\frac{E}{2A_0^2}(\sin\phi_0)\eta + \frac{E}{2A_0}(\cos\phi_0)\xi \end{aligned} \right\} \dots\dots\dots(24)$$

The solution of Eq. (24) is given by

$$\eta = C_1 e^{\lambda \tau_2}, \quad \xi = C_2 e^{\lambda \tau_2}$$

Thus, the characteristic equation of the system is given by

$$\lambda^2 + q\lambda + r = 0 \dots\dots\dots(25)$$

where $q = f_v' + \frac{f_v'''}{4}A_0^2$

$$r = \frac{3}{16^2}f_v'''^2\left(A_0^2 + \frac{16}{3}\frac{f_v'}{f_v'''}\right)^2 + \frac{h^2}{4} - \frac{f_v'^2}{12}$$

The stability condition is given by the Routh-Hurwitz criterion, that is

$$\left. \begin{aligned} 4f_v' + f_v'''A_0^2 > 0 \\ \frac{\left(A_0^2 + \frac{16}{3}\frac{f_v'}{f_v'''}\right)^2}{\left(\frac{8}{3}\frac{f_v'}{f_v'''}\right)^2} + \frac{h^2}{\left(\frac{f_v'}{\sqrt{3}}\right)^2} > 1 \end{aligned} \right\} \dots\dots\dots(26)$$

In Fig. 3 the region of unstable solutions is represented by hatching where Eq. (26) is not satisfied.

It is also noted that the boundary curve between the stable and the unstable regions shows the locus of vertical tangency on resonance curves.

4.3 Influence of initial conditions

To study the influence of initial conditions on the steady-state motion, it is necessary to obtain

Table 2 Classification of singular points (see Fig. 4)

No.	Amplitude	Phase	(A_0, ϕ_0)	Singular points
1	OL	$\angle LOH$	$(2.87, -93^\circ)$	Stable nodal point
2	OM	$\angle MOH$	$(2.38, -55^\circ)$	Unstable saddle point
3	ON	$\angle NOH$	$(1.47, -31^\circ)$	Unstable focal point

Fig. 4 Integral curves

the integral curves for the motion; this can be done by eliminating τ_2 from Eq. (21).

$$\frac{dA}{d\phi} = -\frac{f_v'A + (1/8)f_v'''A^3 + E\cos\phi}{h + (E/A)\sin\phi} \dots\dots\dots(27)$$

Integral curves as shown in Fig. 4 are obtained by the isocline method using a digital computer, using the same parameters as shown in Fig. 3, namely $f_v' = -0.8$, $f_v''' = 0.8$, $E = 1.0$, $h = 0.35$. Accordingly, amplitude may be triple-valued. The three points L, M and N in Fig. 3 correspond to singular points L, M and N in Fig. 4 respectively (see Table 2). By using standard techniques the singular points are easily classified as shown in Table 3. The region of these singularities is given in Fig. 5.

5. Regions of frequency entrainment

In this section the regions of frequency entrainment are investigated. The regions of frequency entrainment in the case where the driving frequency p increases are different from those in the case where the driving frequency decreases in experiments. Thus the former case is considered in this section.

If the detuning is h , we obtain the approximation of regions of frequency entrainment

Table 3 Classification of singular points (see Fig. 5)

No.	Region	Singular point
1	I	Stable nodal point
2	II	Unstable saddle point
3	III	Unstable nodal point
4	IV	Unstable focal point
5	V	Stable focal point

Fig. 5 Stability of harmonic vibrations

$$|h_b| = \frac{|p_b^2 - \omega^2|}{p_b^2} = \frac{2\omega|p_b - \omega|}{\omega^2} = \frac{\Delta\omega}{\omega}$$

thus

$$\Delta\omega = h_b \omega \dots\dots\dots (28)$$

where the suffix b denotes the boundary of regions of frequency entrainment. For sufficiently small values of E , the detuning h_b is obtained from the intersection of ellipse with response curve in Fig. 3 as follows;

$$h_b = -f_v' \sqrt{\left(\frac{A_b^2}{a_{0f}^2} - 1\right) \left(1 - 3\frac{A_b^2}{a_{0f}^2}\right)}$$

thus

$$\Delta\omega_s = h_b \omega \dots\dots\dots (29)$$

where $\Delta\omega_s$ is the region of frequency entrainment in this case. If E increases further, the detuning is obtained from the intersection between straight line and response curve in Fig. 3 as follows

$$h_{bL} = \sqrt{2\frac{E^2}{a_{0f}^2} - \frac{f_v'^2}{4}}$$

thus

$$\Delta\omega_L = h_{bL} \omega \dots\dots\dots (30)$$

where $\Delta\omega_L$ is the region of frequency entrainment. Figure 6 shows the detuning $\Delta\omega$ plotted against E . This curve is an approximately straight line.

6. Experimental results

6.1 Experimental method

Figure 7 is a photograph of the mechanical vibrator. The brass block C with weight 50 ~ 200 g is mounted on the fly wheel A made of mild steel with a radius 330.5 mm, which is driven through reduction gear by the motor B. The vibrating

Fig. 6 Driving amplitude and entrainment frequency

A: fly wheel B: motor C: block
D: flat spring E: D.C. motor
Fig. 7 Mechanical vibrator

system then consists of the mass of the block C and the flat spring D with stiffness 0.1 ~ 1.0 kg/cm. The connecting rod long eccentrically mounted on the fly wheel shaft of D.C. motor E drives the spring F, in simple harmonic motion. With this system the vibrator was run at various cycles per sec. The vibration was observed by means of resistance-wire strain gages cemented to the flat spring at point of maximum flexure. The driving frequency was measured by the stroboscope.

6.2 Regions of frequency entrainment

In the free self-excited vibrations of this frictional system, when the peripheral velocity of the fly wheel is larger than that of the block, it vibrates freely at a natural frequency with a constant amplitude.

When one applies to the block an oscillator oscillating, say, with frequency ω , an extraneous exciting force of frequency p , one observes a "beat" of the two frequencies.

If the frequency p approaches the frequency ω , the frequency of the beat decreases but this happens only up to a certain value of the difference $|p - \omega|$ after which the beat disappears suddenly and there remains only the frequency p . Figure 8 shows the difference $|p - \omega|$ plotted against p .

Concerning the beat frequency $|p - \omega|$, the experimental results coincide with the theoretical results, but on the other hand, the theoretical region of frequency entrainment is 2.51 c/sec and the experimental one is 2.5 c/sec.

It is tedious and difficult to determine the function of the frictional force. In this experiment f_v' and f_v''' are obtained from the amplitude a_{0f} of the free self-excited vibration and the maximum amplitude A_{0max} of the entrained vibration using Eq. (23) as follows:

$$f_v' = -\frac{E a_{0f}^2}{A_{0max}(A_{0max}^2 - a_{0f}^2)} \dots\dots\dots (31)$$

$$f_v''' = -(8f_v'/a_{0f}^2) \dots\dots\dots (32)$$

The present work provides an explanation why the theoretical region of frequency entrainment is

Fig. 8 Beat frequency and entrainment frequency

Fig. 9 Wave form

Fig. 10 Driving amplitude and region of entrainment frequency

larger than the experimental results. The detuning is obtained from the intersection between straight line $A_0^2 = (1/2)a_{0f}^2$ and response curve in Fig. 3 as mentioned in section 5, but in this experiment the amplitude at the boundary of the region of the entrainment frequency is larger than the theoretical result $(1/\sqrt{2})a_{0f}$.

As shown in Fig. 9 when the driving frequency p is 21.5 c/sec and 22.5 c/sec, the vibration with the frequency $\omega_1 = 21.42$ c/sec and $\omega_1 = 21.58$ c/sec occurs respectively. The present work provides no explanation as to why an entrained vibration with

nearly the same frequency ω_1 as the driving frequency p occurs outside the range of entrained frequency.

6.3 Relation between zone of synchronization and amplitude of driving force

Figure 10 shows the entrainment frequency Δf plotted against the range of driving amplitude $E = 0.5 \sim 3.0$ mm. The straight line denotes the theoretical results and the experimental results are shown by points. The latter coincides with the former quantitatively.

7. Conclusions

The results obtained may be summarized as follows:

- (1) The influence of initial conditions on the entrained vibration is clarified by the phase plane analysis and singular points are classified.
- (2) The theoretical zone of the entrainment frequency is derived. The theoretical results are larger than the experimental results.
- (3) The relation between zone of synchronization and amplitude of driving force is obtained both theoretically and experimentally.

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