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The Vibration of a Cylindrical Shell with Built-in-Free Ends*

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Using the Lagrangian denoted in the previous paper, we have investigated the vibration of a cylindrical shell with built-in-free ends except the axisymmetric vibration. From the minimum conditions of the Lagrangian which is a quadratic form of the unknown boundary values, we show how to obtain the frequency equations and the relations of the boundary values. The frequencies of the two cylindrical shells with radius-thickness ratios of 0.2887×10^2 and 0.2887×10^3 are obtained. The experiments are carried out and their results are compared with the theoretical ones. The vibration of a cylindrical shell which resembles that of a bar with built-in-free ends is also shown and compared with the vibration of the bar and that of the shell in other modes.

1. Introduction

Many reports on vibrations of a cylindrical shell under various boundary conditions have already been published by Warburton⁽¹⁾, Forsberg⁽²⁾ and others⁽³⁾, but we haven't had any report on those of a cylindrical shell with built-in-free ends yet.

In this paper, except the axisymmetric vibrations, we have investigated the vibrations of a cylindrical shell with built-in-free ends, using the Lagrangian of a cylindrical shell introduced in our previous paper⁽⁴⁾. From the minimum conditions of the Lagrangian which is a quadratic form of the unknown boundary values, we obtain the frequency equations and the relations of the boundary values. As numerical examples, we obtain the frequencies of the two cylindrical shells with radius-thickness ratios of 0.2887×10^2 and 0.2887×10^3 for various modes of vibration. The displacements in the radial direction for some modes of vibration are shown. The experimental results are compared with the theoretical ones. The vibrations of a cylindrical shell with two nodal generators are also investigated to be compared with the bending vibrations of a bar with built-in-free ends and the vibrations of a cylindrical shell in other modes.

2. The vibrations of a cylindrical shell with built-in-free ends

Let us consider the vibrations of a cylindrical shell with built-in-free ends (which is hereafter called a built-in-free cylindrical shell for short). The shell is taken to be of length $2l$, radius R and thickness h . Take the center of the shell as the origin, and the angular coordinate θ is taken in the circumferential direction as shown in Fig. 1. We take the coordinate axis x' along the generator on the middle surface and take the coordinate z directed inward along the radius. Why we take the origin

Fig. 1

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as mentioned above is to be in accord with use of the Lagrangian which has been reported before⁽⁴⁾ (in general, it is convenient to take the origin at the clamped or free end of a cylindrical shell when we only consider the vibrations of a built-in-free cylindrical shell). Let x denote the non-dimensional length x'/R . The symbols p , t , $1-\zeta$, γ , D and g denote circular frequency, time, Poisson's ratio, specific weight, bending rigidity of the plate and acceleration of gravity respectively. The displacements of an arbitrary point on the middle surface in the direction x' , θ and z are $(u, v, w)\sin pt$ respectively. We take non-dimensional quantities α , β and μ as follows:

$$\alpha^4 = \gamma h p^2 R^4 / g D, \quad \beta = 12 R^2 / h^2, \quad \mu = l / R$$

We assume that it is possible to expand the values of u , v , w and $\partial w / \partial x$ at both ends ($x = \pm \mu$) of a cylindrical shell into Fourier series, that is,

$$\left. \begin{aligned} 2u|_{x=\pm\mu} &= \sum_{n=1}^{\infty} (g_n \pm g_n') \cos n\theta, \\ 2v|_{x=\pm\mu} &= \sum_{n=1}^{\infty} (\mp p_n - p_n') \sin n\theta, \\ 2w|_{x=\pm\mu} &= \sum_{n=1}^{\infty} (\pm a_n + a_n') \cos n\theta, \\ 2\mu \frac{\partial w}{\partial x} \Big|_{x=\pm\mu} &= \sum_{n=1}^{\infty} (c_n \pm c_n') \cos n\theta \end{aligned} \right\} \dots\dots\dots (1)$$

where the Fourier coefficients a_n , a_n' , c_n , c_n' , g_n , g_n' , p_n and p_n' are the boundary values. The other coefficients b_n , b_n' , d_n , d_n' , h_n , h_n' , q_n and q_n' in equations (7) in Reference (4) corresponding to Eqs. (1) are omitted as the same result is obtained. In this paper, the axisymmetric vibration will not be considered.

λ_{np} are the roots of the equation (5), ϕ_{np} and ψ_{np} are the roots of the simultaneous equations (6) in Reference (4) respectively. The suffix p denotes any number of 1, 2, 3 and 4. Introducing the notation $\xi_{np} = \lambda_{np} \mu$, we express the formulas of A_n , A_n' , A_{np} , A_{np}' , C_{np} , C_{np}' , G_{np} , G_{np}' , P_{np} and P_{np}' in equations (9) in Reference (4). These quantities are independent of boundary values and are commonly used in the analysis of all the other cylindrical shells under various boundary conditions. In the vibrations of a cylindrical shell fixed at $x = -\mu$ and free at $x = \mu$, from Eqs. (1), the boundary conditions are obtained as follows;

$$\left. \begin{aligned} a_n &= a_n', \quad c_n = c_n', \\ g_n &= g_n', \quad p_n = p_n' \end{aligned} \right\} \dots\dots\dots (2)$$

The notations k_{np} , k_{np}' and Z_n are introduced as follows;

$$\left. \begin{aligned} k_{np} &= a_n A_{np} + c_n C_{np} + g_n G_{np} + p_n P_{np}, \\ k_{np}' &= a_n A_{np}' + c_n C_{np}' + g_n G_{np}' + p_n P_{np}' \end{aligned} \right\} \dots\dots\dots (3)$$

$$\begin{aligned} Z_n &= \frac{n^2 \zeta}{\mu} a_n c_n + \frac{n(-1+3\zeta)}{2\mu} c_n p_n + \frac{n\beta(-2+3\zeta)}{4} g_n p_n - \frac{\beta(1-\zeta)}{2} a_n g_n \\ &+ \sum_{p=1}^4 k_{np} \left[-\frac{1}{\mu^3} a_n \xi_{np}^3 \cosh \xi_{np} + \frac{1}{\mu^3} c_n \xi_{np}^2 \sinh \xi_{np} \right. \\ &\left. + \frac{n(1+\zeta)}{\mu} a_n \phi_{np} \xi_{np} \cosh \xi_{np} + \frac{\beta \zeta}{2\mu} p_n \psi_{np} \xi_{np} \cosh \xi_{np} + \frac{\beta}{\mu} g_n \phi_{np} \xi_{np} \sinh \xi_{np} \right] \dots\dots\dots (4) \end{aligned}$$

Replacing k_{np} by k_{np}' and interchanging $\sinh$ and $\cosh$ in Z_n , we get Z_n' . The Lagrangian L of a built-in-free cylindrical shell in a period of its vibration, and the displacements u , v and w are given from Reference (4) as follows:

$$\begin{aligned} -L / \frac{\pi D}{2R^2} &= \sum_{n=1}^{\infty} (Z_n + Z_n') \dots\dots\dots (5) \\ \left. \begin{aligned} u &= \sum_{n=1}^{\infty} \sum_{p=1}^4 \phi_{np} (k_{np} \cosh \lambda_{np} x + k_{np}' \sinh \lambda_{np} x) \cos n\theta \\ -v &= \sum_{n=1}^{\infty} \sum_{p=1}^4 \psi_{np} (k_{np} \sinh \lambda_{np} x + k_{np}' \cosh \lambda_{np} x) \sin n\theta \\ w &= \sum_{n=1}^{\infty} \sum_{p=1}^4 (k_{np} \sinh \lambda_{np} x + k_{np}' \cosh \lambda_{np} x) \cos n\theta \end{aligned} \right\} \dots\dots\dots (6) \end{aligned}$$

We obtain the frequency equations and the relations of the boundary values from the minimum conditions of the Lagrangian (5) which is the quadratic form of the boundary values:

$$\partial L / \partial (a_n, c_n / \mu, g_n, p_n) = 0 \dots\dots\dots (7)$$

3. Numerical examples

In the numerical calculations, the Poisson's ratio is taken to be 0.3 in the cases of $\beta = 10^4$ and $\beta = 10^6$. As for the results calculated in this paper, λ_1 , λ_2 , λ_3 and λ_4 being positive real numbers, the roots λ_{np} ($p=1, 2, 3, 4$) are given in the forms, $\pm \lambda_1$, $\pm i \lambda_2$, $\pm (\lambda_3 \pm i \lambda_4)$. The lower limit values of α^4 / β for the roots λ_{np} in such forms are shown in Table 1. They perfectly or almost agree with the values of the Love's equation⁽⁵⁾,

$$\alpha^4 / \beta = \frac{n^2(n^2 - 1)^2}{\beta(n^2 + 1)}$$

which express the non-dimensional frequencies of an infinitely long cylindrical shell. The relations between α^4/β and μ for $\beta=10^4$ in the cases of $n=2, 3, 4$ are shown in Fig. 2. It is shown in Fig. 2 that the value of μ for the large value of α^4/β varies very little according to various values of n . The same holds in the case in which n becomes larger or β varies. The results for the small values of α^4/β will be stated later.

The differences among the frequencies of the cylindrical shells with various boundary conditions are shown in Fig. 3. That is, the relations between μ and α^4/β in the case of $\beta=10^4$ and $n=2$ are shown about the three cylindrical shells: one with built-in-free ends, another with both built-in ends and the third with both free ends. In the case of the cylindrical shells with both ends clamped, free or freely supported whose boundary conditions are symmetric about the middle cross section, the coefficients of the boundary values in Eq. (7) have such forms as $(K_1 \sin \lambda_2 \mu + K_2 \cos \lambda_2 \mu)$ in the range where $\tanh \lambda_1 \mu$ and $\tanh \lambda_3 \mu$ are approximately equal to 1 (constants K_1 and K_2 being independent of μ). For the above-mentioned reason, the arrangement of the frequency curves $\alpha^4/\beta - \mu$ is approximately equidistant in the range where $\lambda_1 \mu$ and $\lambda_3 \mu$ are large. On the other hand, the coefficients of the boundary values in Eq. (7) for a built-in-free cylindrical shell are given in the form,

$$(K_1 \sin 2\lambda_2 \mu + K_2 \cos 2\lambda_2 \mu) / \Delta_n \Delta_n'$$

Table 1 Lower limit values of α^4/β

n	$\beta=10^4$	$\beta=10^6$
2	0.000 637 40	0.000 007 26
3	0.005 667 89	0.000 057 60
4	0.021 074 33	0.000 209 10
5	0.055 267 07	0.000 512 92
6	0.119 045 32	0.001 181 76
7	0.225 605 06	0.002 256 94
8	0.390 540 53	0.003 906 93
9	0.631 844 63	0.006 320 93
10	0.969 909 05	0.009 702 92

Fig. 2 Relations between the length and the frequency of a built-in-free cylindrical shell for $\beta=10^4$ in the cases of $n=2, 3, 4$ (The modes are from the first to the fourth)

Therefore, except the first frequency curve, the second one of a built-in-free cylindrical shell is in the middle between the frequency curves $\alpha^4/\beta - \mu$ of the symmetric vibrations of the cylindrical shell with both built-in ends and the cylindrical shell with both free ends, and the third frequency curve is between the frequency curves of the antisymmetric vibrations of the cylindrical shell with both built-in ends and the cylindrical shell with both free ends and the same holds true of the higher frequency curves. In the axisymmetric vibrations, almost all the frequency curves $\alpha^4/\beta - \mu$ are in equidistant arrangement, but as to the vibrations for $n \geq 2$, the frequency curves of higher degrees than the third or the fourth are in almost equidistant arrangement. The displacements u , v and w are obtained from Eqs. (6). The values of a_2 , c_2 , g_2 and p_2 are determined by substituting the roots μ of the frequency equations into Eq. (7).

The radial displacements w of a built-in-free cylindrical shell in the case of $n=2$ and $\beta=10^4$ are denoted in Fig. 4. The plotted curves correspond to the points for $\alpha^4/\beta=0.1, 0.5, 0.9$ on the curves

Fig. 3 Comparison among the curves $(\alpha^4/\beta) - \mu$ of the three cylindrical shells: one with built-in-free ends and the others with built-in ends or free ends

$n=2, \beta=10^4$
The left end of the abscissa axis corresponds to the built-in end and the right end corresponds to the free end.

Curves	A: $\alpha^4/\beta=0.1$	$\mu=0.546 68$
	B: $\alpha^4/\beta=0.1$	$\mu=1.389 73$
	C: $\alpha^4/\beta=0.5$	$\mu=0.225 90$
	D: $\alpha^4/\beta=0.5$	$\mu=0.635 68$
	E: $\alpha^4/\beta=0.9$	$\mu=0.140 56$
	F: $\alpha^4/\beta=0.9$	$\mu=0.386 68$

Fig. 4 Displacements w in the radial direction of a built-in-free cylindrical shell

$\alpha^4/\beta - \mu$ of the first and the second vibrations in Fig. 3. In the numerical calculation, the magnitude of the displacement is measured by defining a_2 , c_2 and g_2 for $p_2=1$, but these curves are plotted with respective units well-adapted for graphical representation in a graph. As shown in Fig. 4, there is no nodal circle about the displacements w corresponding to any points of the first frequency curves $\alpha^4/\beta - \mu$ in Fig. 3, and the number of nodal circles about the second frequency curve is one.

Next, we evaluate the frequencies of a built-in-free cylindrical shell of $\beta=10^6$ in the cases of $n=1, 2, 3, \dots, 8, 9, 10$ from the first mode to the fourth or the sixth mode, respectively. They show the same tendency as that in Fig. 2. In case α^4/β is in the range from 0.2 to 1.0, in the region between the frequency curves for $n=2$ and $n=8$ lie the frequency curves for the other values of n .

The frequency curves in the range of small values of α^4/β are shown in Fig. 5 in which $\sqrt{\alpha^4/\beta}$ is employed as ordinate to magnify the reduced scale. The same tendency as that in Fig. 5 is

The frequency curves in the range of small values of α^4/β are shown. The curves approach the value of $\sqrt{\alpha^4/\beta}$ given in Table 1 as μ becomes larger. Comparison between the experimental results and the theoretical ones is also made.

Fig. 5 Relations between the length and the frequency of a built-in-free cylindrical shell for the first mode of vibration in the case of $\beta=10^6$

The value of μ in the case of $\beta=10^6$ becomes smaller than that in $\beta=10^4$ as α^4/β becomes larger.
Fig. 6 Frequencies of a built-in-free cylindrical shell in the cases of $\beta=10^4$ and $\beta=10^6$

evident in the case of $\beta=10^4$. In the range of $\sqrt{\alpha^4/\beta} \geq 0.1$, the value of μ decreases gradually in the order, $n=2, 3, 4, \dots, 8$, but it increases only slightly when $n=9$ and $n=10$. (The frequency curves for $n=9, 10$ are omitted in order to avoid being indistinguishable from one another.) The decrease of α^4/β causes the increase of μ , and then μ becomes infinitely large at the value of α^4/β given in Table 1. The frequencies of $n=8$ at $\mu=1$ are much the same as those of an infinitely long cylindrical shell, and so are the frequencies of $n=2$ in the range of $\mu > 5$. The envelope of the frequency curves which shows the least frequency is important in practical application. The least frequency will take place in the frequencies for $n=2$ in the range of $\mu > 4.3$, for $n=3$ in $\mu > 2.2$, for $n=4$ in $\mu > 1.2$, for $n=6$ in $\mu > 0.6$, for $n=7$ in $\mu > 0.4$, and for $n=8$ in $\mu > 0.04$. The least frequency will not take place in the frequencies for $n=5$ in all the range of μ . The vibration of $n=1$ in which the value of μ is greater than that of $n \geq 2$ is not interesting from the standpoint of practical application, and it will be treated in chapter 5.

The influence of the change of β on the frequency is shown in Fig. 6. There is little difference between the values of μ in the cases of $\beta=10^4$ and $\beta=10^6$ when α^4/β is small. But the value of μ in the case of $\beta=10^6$ becomes smaller than that in the case of $\beta=10^4$ as α^4/β becomes larger.

4. Experimental investigation

As the measuring range of the instruments in our laboratory is not wide enough to take measurements of high frequencies, we could not make experiments sufficiently. The outline will be described as follows. The cylindrical shells ($\beta=10^6$) used in the experiments were made of steel sheet with 0.4 mm or 0.8 mm thickness and brazed or welded along the joint. And then we dressed the shell on a concrete block and tightened it with a steel band. Six strain gauges were plastered at equal intervals near the free end of the cylindrical shell, and two were provided near the center of the shell. We measured the frequencies by means of oscillograph or synchroscope and conjectured the mode of vibration from the phase differences of the eight strain gauges. The vibration would take place when a small electromagnet excited by a frequency oscillator reacted to it, or a rubber hammer hit it. More than two

sorts of vibrations often arose when we excited the shell. However, owing to damping, the high mode of vibration happened to disappear after a short time, and we were left only with the first mode of vibration. We obtained the data at such a time. Only one kind of vibration was rarely recorded from the beginning, but when we excited the shell again by the above recorded frequency, or struck the same part of the shell, a resultant wave was recorded. We could obtain the frequencies of the individual waves constituting the resultant wave, but we could not precisely deduce the phase difference.

The results of the experiments, too, are shown in Fig. 5. From a consideration of the phase differences between the strain gauges of the free end and those of the center, we find that all the frequencies that we can measure are those of the first mode. It can be seen from Fig. 5 that almost all the experimental values are on the envelope of the least frequencies. We could deduce the mode shape when n was small, but we could only decide it with inaccuracy for $n \geq 5$ for all our carefulness.

5. Vibration in the case of $n=1$

The frequency curves $\sqrt{\alpha^4/\beta} - \mu$ of a built-in-free cylindrical shell in the case of the Poisson's ratio 0.3, $\beta=10^6$ and $n=1$ are shown in Fig. 7. It is seen from this figure that μ in the case of $n=1$ is greater than that in $n \geq 2$ at a definite value of α^4/β , so that this vibration may be supposed to be of minor importance for practical application. It seems that this is the reason why there are very few reports on the vibrations in the case of $n=1$ under the other boundary conditions, too. But it is necessary for us to show at least one example of numerical calculation in the case of $n=1$, therefore let us briefly discuss such a sort of vibration of a built-in-free cylindrical shell.

The value of μ in the case of $n=1$ is greater than that in $n \geq 2$ at a definite value of α^4/β .
Fig. 7 Relations between the length and the frequency of a built-in-free cylindrical shell in the case of $\beta=10^6$ and $n=1$

The vibration of a built-in-free cylindrical shell for $n=1$ is similar to that for $n \geq 2$; the first frequency does not but the second frequency lies between the frequencies of symmetric vibrations of a cylindrical shell with both built-in ends and that with both free ends, and the third frequency lies between those of antisymmetric vibrations of these cylindrical shells, and then the same may be said of still higher frequencies. As shown in Fig. 7, the relation between the length and frequency of the shell is similar to that in the case of $n \geq 2$, too.

On the other hand, the main difference between the vibration of the built-in-free cylindrical shell for $n=1$ and that for $n \neq 1$ is described as follows.

If the axis of the vibrating cylindrical shell is defined as a locus of centroid of each cross section of the shell deformed due to vibration, the vibration for $n=1$ is supposed equivalent to a bending deflection of the axis. Figure 8 indicates comparison between the vibration of a built-in-free cylindrical shell for $n=1$ and that of a built-in-free bar with the same cross sectional area as that of the shell. Introducing the notation

$$f = \frac{\gamma S p^2 (2l)^4}{EI_z g} \quad (8)$$

where S and I_z denote cross sectional area and geometrical moment of inertia of the bar respectively; and then assuming that the difference between the inner and outer radius of the cylindrical shell is very small, and denoting its mean radius and thickness by R and h respectively, we have

$$I_z = \pi R h^3 \beta / 12, \quad S = 2\pi R h$$

Hence, Eq. (8) becomes

$$f = \frac{\alpha^4}{\beta} \frac{32\mu^4}{\zeta(2-\zeta)} \quad (9)$$

The results calculated for $\zeta=0.7$ (the Poisson's ratio 0.3) are plotted in Fig. 8. In the case of a

The parallel lines to the abscissa axis denote the eigen values of a built-in-free bar.
Fig. 8 Comparison between the vibration of a built-in-free cylindrical shell for $n=1$ and that of a built-in-free bar

bar with built-in-free ends, the first four consecutive frequencies f are shown as

$$f = (1.875)^4, (4.694)^4, (7.855)^4, (10.996)^4 \dots \dots \dots (10)$$

We designate the values of μ against a definite value of α^4/β in Fig. 7 as $\mu_1, \mu_2, \mu_3, \mu_4, \dots, \mu_n, \dots$ from left to right, and denote the value obtained by substituting μ_n into Eq. (9) by f_n . We illustrate $f_1, f_2 \times 10^{-1}, f_3 \times 10^{-3}$ and $f_4 \times 10^{-4}$ instead of f_1, f_2, f_3, f_4 , for convenience's sake, in Fig. 8, and the values of Eq. (10) are denoted by parallel lines to the abscissa axis. It is evident from Fig. 8 that the frequency f_n of a built-in-free cylindrical shell approaches f when α^4/β becomes smaller, that is, μ becomes larger. And then, the vibration of a sufficiently long cylindrical shell with built-in-free ends may be considered as that of a built-in-free bar with the same cross sectional area. But the vibration of the shell is not to be considered as that of a bar when α^4/β becomes larger, i.e., μ becomes smaller.

The variations of the displacements w, v and the moments M_x and M_θ along the axis x of the shell for the first mode of vibration ($\mu=0.562301$) are shown in Fig. 9 in the case of $\alpha^4/\beta=0.25$ and $\beta=10^6$. The moments M_x and M_θ are expressed in the forms

$$M_x = -\frac{D}{R^2} \left[\frac{\partial^2 w}{\partial x^2} + (1-\zeta) \left(\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta} \right) \right]$$

$$M_\theta = -\frac{D}{R^2} \left[\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta} + (1-\zeta) \frac{\partial^2 w}{\partial x^2} \right]$$

The left end of the abscissa axis in Fig. 9 corresponds to the built-in end of the shell where w and v are zero, but M_x and M_θ are very large. The right end corresponds to the free end, in which w and v are maximum, M_x is zero and M_θ is small. Both M_x and M_θ in the middle of abscissa axis are

The left end of the abscissa axis corresponds to the built-in end and the right end corresponds to the free end. The moment M_θ which shows the distribution similar to that of M_x near the free end is not plotted in the range of $x > 0$.

Fig. 9 Displacements of a built-in-free cylindrical shell corresponding to $\alpha^4/\beta=0.25$ ($\mu=0.562301$) in the case of $n=1$ and $\beta=10^6$ (Distributions of the bending moments along the axis x of the cylindrical shell)

small. In Fig. 9, M_θ in the range of $x > 0$ which shows the change similar to that of M_x , is not plotted. The individual graduations of the ordinate axis are well adapted according to the respective curves of w, v, M_x and M_θ in order to make the figure easily readable. It is seen that

$$w_{\max} \doteq 1.38v_{\max} \text{ and } M_{x\max} \doteq 3.33M_{\theta\max}$$

The next two figures, Fig. 10 and Fig. 11 are added for reference. Figure 10 shows the distributions of M_x and M_θ corresponding to $\alpha^4/\beta=0.1, 0.5, 0.9$ ($\mu=0.546682, 0.225904, 0.140557$) in the case of the first mode of vibration of a built-in-free cylindrical shell for $n=2$ and $\beta=10^4$. Figure 11 denotes the frequency curve $\alpha^4/\beta-\mu$ for the axisymmetric vibration ($n=0$) of a built-in-free cylindrical shell in $\beta=10^6$. It will be seen from Fig. 11

Suffixes 1, 2, 3 correspond to $\alpha^4/\beta=0.1, 0.5, 0.9$ respectively.

Fig. 10 Distributions of the moments M_x and M_θ along the axis x in the case of the first mode of vibration of a built-in-free cylindrical shell ($\beta=10^4, n=2$)

The frequency of the axisymmetric vibration ($n=0$) is the greatest of all for a definite length of the cylindrical shell.

Fig. 11 Axisymmetric vibration of a built-in-free cylindrical shell ($\beta=10^6$)

that the frequency of the axisymmetric vibration ($n=0$) is greater than that of $n \neq 0$ for a definite length of the shell.

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