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Theoretical Study of the Low-Aspect-Ratio Wing in Uniform Shear Flow* (The Case of the Free Stream Velocity Varying in the Spanwise Direction)

By Kuninobu MORITA**

The characteristics of the finite low-aspect-ratio wing in uniform shear flow is analyzed on the basis of the assumption that the disturbance velocities are small in comparison with the undisturbed flow velocity and that the width of the wing increases gradually in the chordwise direction. This assumption is analogous to that in the Jones' theory for the low-aspect-ratio wing in uniform potential flow.

As the result, it is found that the aerodynamic forces acting on the wing are strongly influenced by the velocity gradient of the free shear flow; when the zero-velocity point of the free shear flow comes infinitely closer to the wing tip, the total lift and total drag coefficients become about 20 percent smaller than the values of the wing in uniform potential flow.

If the free stream velocity gradient tends to zero (uniform velocity distribution), the present theory agrees perfectly with the Jones' one.

1. Introduction

There are many cases in which the relative fluid flow toward blades of axial-flow type fans, blowers and pumps, or toward wings of airplanes are not uniform. For instance, those towards the running blades of axial-type hydrodynamical machines are obviously not uniform in the spanwise direction. In the case of helicopter-rotor, the state is also similar to this. Problem of this kind can be called generally "the problem of the characteristics of the wings in the inviscid shear flow". Lately, this problem has been investigated by many researchers.

We may classify this kind of investigations made to date into the following two categories: one treats the case in which the relative air velocity varies perpendicular to the blade surface, and the other the case in which the velocity changes in the spanwise direction. Concerning the former case, Tsien⁽¹⁾, Sowyrda⁽²⁾ and Jones⁽³⁾ have theoretically studied the characteristics of various airfoil sections. On the other hand, Honda⁽⁴⁾ has analyzed, on the assumption of small perturbation, the characteristics of the wing between parallel planes, toward which the fluid flows with a spanwise-nonuniform velocity distribution. Namba and Asanuma^{(5) (6)} also treat

this kind of problems taking account of the fluid compressibility. In their investigations, they analyzed the characteristics of the wing, whose both tips touch two parallel side-planes. Their analyses are very useful when we try to predict the flow behaviors in the axial-flow type blowers, compressors, pumps and the like. By means of their theories, however, we can not analyze the flow-characteristics around the blades of helicopter-rotors, the non-pressure type axial-flow fans, and the like, because one tip or both tips of their blades are completely exposed to the surrounding flow field.

On the characteristics of the finite wings in shear flow, von Kármán and Tsien⁽⁷⁾ made an analytical investigation by means of the lifting-line theory. The lifting line theory, however, is based on the assumption that the wing sections at each spanwise station may have the same characteristics as those of the two dimensional wing section which is placed in the flow at the so-called "effective angle of attack". Therefore, this theory is applicable only to the wing with large aspect-ratios.

Recently, low-aspect-ratio wings or blades are being used in many axial-flow type fans, blowers, pumps, and further, in airplanes and so forth. Therefore, it becomes important to obtain the characteristics of low-aspect-ratio wings in shear flow. For instance, the problems of circuitous flight or

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** Research Fellow, Mechanical Engineering Research Laboratory, Hitachi Ltd., Adachi-ku, Tokyo.

yawing motion of an airplane which has a slender delta wing can not be analyzed with validity by the lifting-line theory.

In this paper, in order to solve these problems, the characteristics of a low-aspect-ratio wing in uniform shear flow with the velocity varying linearly in the spanwise direction are analyzed theoretically. In general, for the purpose of obtaining most exact solutions for the wing characteristics, it seems natural to analyze the problem by the lifting surface theory. However, by the lifting surface theory, it is very difficult to find an analytic solution for a wing with an arbitrary planform: even for a very simple planform, only a numerical solution will be obtainable. Therefore, by the lifting-surface theory, we may not get the general relation between the wing characteristics and the wing planform.

In the present research, the wing characteristics are theoretically analyzed on the basis of the Jones' finite wing theory⁽⁶⁾ which can give sufficiently accurate solutions for the low-aspect-ratio wings. Outline of the Jones' theory will be described in the next chapter. Here, only basic concept of the Jones' theory is interpreted: the flow field in a plane which is normal to the flight direction of a low-aspect-ratio pointed wing changes its pattern only slightly in the streamwise direction. Therefore, this flow pattern can be considered similar to that of the two-dimensional case. The Jones' theory is based on the assumption that this flow pattern is supposed to be two-dimensional. On such assumption, the wing characteristics can be obtained comparatively easily.

2. The equations of motion and the boundary conditions

In the first place, the Jones' theory on which the present research is based will be summarized. Let us consider a low-aspect-ratio pointed wing in a uniform flow with the velocity U_0 and placed at the angle of attack α , as shown in Fig. 1. The flow pattern in the plane (π -plane) normal to the x -axis may be considered to be nearly two-dimensional because it varies only very slightly in the chordwise

Fig. 1 A low-aspect-ratio wing and the π -plane

direction. Therefore, this flow pattern is similar to that of a two-dimensional flow caused by a flat plate, which proceeds in the direction normal to its surface with velocity U_0 . As the flow proceeds downstream, the local wing breadth $2b$ increases. Then, the flow momentum J in the π -plane, which is equal to $\pi\rho U_0\alpha b^2$, increases with time. The lift force per unit chord, $\bar{l}_x(x)$ is equal to a rate of increase of the momentum,

$$\bar{l}_x(x) = 2\pi\rho U_0^2\alpha b \frac{db}{dx} \dots\dots\dots (1)$$

$$\text{since } \frac{dJ}{dt} = U_0 \frac{dJ}{dx}.$$

When the local wing breadth $2b$ increases in the chordwise direction, $\bar{l}_x(x)$ is positive. On the other hand, as can be seen in Eq. (1), when $2b$ decreases, $\bar{l}_x(x)$ becomes negative: a part of the wing, where the local wing breadth decreases along x -axis, is expected to develop a negative lift force theoretically. However, the actual scale of the flow pattern in the π -plane aft of the maximum local wing breadth is not reduced but remains constant. The momentum J of the flow also remains constant. Therefore, this part of the wing develops no lift. Above-mentioned is the summary of the Jones' theory.

Now, let us take the Cartesian coordinate system (x', y', z') as shown in Fig. 2. An on-coming free flow is parallel to the x' -axis. In the present research, since we deal with uniform shear flow only, the free flow velocity U can be expressed by a linear function of y' alone. The wing is placed in the $x'y'$ -plane at a small angle of attack. We are going to neglect thickness of the wing in the present research. The fluid is considered to be incompressible and inviscid. Let u, v and w denote the perturbation velocities in the x', y' and z' -directions, respectively, and are small compared with the free flow velocity $U(y')$. The pressure p denotes the perturbed pressure.

The equations of motion and the equation of continuity will be written as follows on the assumption of small perturbation.

$$\begin{cases} U \frac{\partial u}{\partial x'} + v \frac{dU}{dy'} + \frac{1}{\rho} \frac{\partial p}{\partial x'} = 0 \dots\dots\dots (2) \end{cases}$$

$$\begin{cases} U \frac{\partial v}{\partial x'} + \frac{1}{\rho} \frac{\partial p}{\partial y'} = 0 \dots\dots\dots (3) \end{cases}$$

$$\begin{cases} U \frac{\partial w}{\partial x'} + \frac{1}{\rho} \frac{\partial p}{\partial z'} = 0 \dots\dots\dots (4) \end{cases}$$

$$\frac{\partial u}{\partial x'} + \frac{\partial v}{\partial y'} + \frac{\partial w}{\partial z'} = 0 \dots\dots\dots (5)$$

where ρ denotes the density of the fluid.

The boundary conditions are shown as follows: at infinity,

$$(x'^2 + y'^2 + z'^2)^{\frac{1}{2}} \rightarrow \infty; u, v, w, p \rightarrow 0 \dots\dots\dots (6)$$

and at the wing surface

$$\left[\frac{w}{U}\right]_{\text{at the wing surface}} = -\alpha(x') \quad \dots\dots\dots(7)$$

where $\alpha(x')$ is the local slope of the wing at station x' , and is a function of x' only because we deal with only the wing whose spanwise cross section is parallel to the y' -axis. In the boundary conditions, the velocity v does not appear explicitly.

Eliminating u , w and p from Eqs. (2), (3), (4) and (5), we can get the following equation for v .

$$U(y') \left[\frac{\partial^2 v}{\partial x'^2} + \frac{\partial^2 v}{\partial y'^2} + \frac{\partial^2 v}{\partial z'^2} \right] - U''(y') v = 0 \quad \dots\dots\dots(8)$$

where $U''(y')$ denotes the second order derivative of $U(y')$ with respect to y' . Because we take uniform shear flow only into consideration, $U(y')$ becomes a linear function of y' . Therefore, $U''(y')$ is always equal to zero. Thus, Eq. (8) yields immediately

$$\frac{\partial^2 v}{\partial x'^2} + \frac{\partial^2 v}{\partial y'^2} + \frac{\partial^2 v}{\partial z'^2} = 0 \quad \dots\dots\dots(9)$$

Here, let us introduce the same assumption as that in the Jones' theory. In the flow field around the low-aspect-ratio wing, $\partial v / \partial x'$ is considered to be much smaller than both $\partial v / \partial y'$ and $\partial v / \partial z'$. Then, $\partial v / \partial x'$ can be neglected. On the basis of such assumption, Eq. (9) will be simplified as

$$\frac{\partial^2 v}{\partial y'^2} + \frac{\partial^2 v}{\partial z'^2} = 0 \quad \dots\dots\dots(10)$$

Thus, the independent variable x' disappears; the flow pattern in the plane normal to the x' -axis (π -plane) is similar to that of a two-dimensional flow. Therefore, in order to analyze the three-dimensional flow around the wing, it is sufficient to consider only the two-dimensional flow in the π -plane.

Then, let us introduce the new two-dimensional coordinate system (y , z) fixed in the π -plane. As shown in Fig. 2, let the y -axis be fixed on the wing, and let the origin of this coordinate system be located at the center of the local wing breadth.

Fig. 2 A low-aspect-ratio wing in uniform shear flow and system of coordinates

The following analysis will be made using this new coordinate system.

The new variables y , z are linearly transformed variables of y' , z' . Therefore, independent variables y' , z' which are used in all of the above equations can be exchanged for y , z ; and x' is regarded as a parameter.

Let station $y = -c$ denote the zero-velocity point of the free shear flow. Then, $U(y)$ will be given by the expression

$$U(y) = K(y+c) \quad \dots\dots\dots(11)$$

where K is constant.

Now, we shall solve Eq. (10) under the boundary condition expressed in Eqs. (6) and (7). Meanwhile, in order to get the boundary condition for v at the wing surface, we should obtain the relation between v and w . Eliminating p from Eqs. (3) and (4), we can get the relation

$$\frac{\partial}{\partial z} [Uv] - \frac{\partial}{\partial y} [Uw] = 0 \quad \dots\dots\dots(12)$$

Integration of this equation from $y = -c$ to y yields

$$U(y)w(y, z; x') = \int_{-c}^y U(\eta) \frac{\partial}{\partial z} \{v(\eta, z; x')\} d\eta \quad \dots\dots\dots(13)$$

If we put $z \rightarrow 0$ in Eq. (13) and substitute it into Eq. (7), the boundary condition for v at the wing surface is obtainable.

In the π -plane, the flow on the "lower surface" (the pressure side) of the wing is separated, and each flow advances toward both tips. On the contrary, the flow on the "upper surface" (the suction side) of the wing concentrates in the middle of the section from both tips. Hence, there must be a velocity discontinuity for v across the spanwise cross section. The solution of the two-dimensional Laplace's equation whose solution has discontinuity across the region $-b \leq y \leq b$ will be obtained by the doublet distributed continuously in that region. Hence, v is,

$$v(y, z; x') = \frac{z}{2\pi} \int_{-b}^b \frac{\varphi(\xi)}{(y-\xi)^2 + z^2} d\xi \quad \dots\dots\dots(14)$$

where $\varphi(\xi)$ denotes the distributed doublet along the spanwise cross section, and symbol $\int$ denotes the Cauchy principal value of integral. $\varphi(\xi)$ should be determined so that v may satisfy the boundary condition at the wing surface. By substituting Eq. (14) into (13), we obtain

$$U(y)w(y, z; x') = -\frac{U(y)}{2\pi} \int_{-b}^b \frac{(y-\xi)\varphi(\xi)}{(y-\xi)^2 + z^2} d\xi + \frac{K}{4\pi} \int_{-b}^b \varphi(\xi) \log \left[\frac{(y-\xi)^2 + z^2}{(c+\xi)^2 + z^2} \right] d\xi \quad \dots\dots\dots(15)$$

Because v and w obtained thus vanish at infinity, there they satisfy the boundary condition. Putting $z \rightarrow 0$ in Eq. (15) and substituting it into Eq. (7), which shows the boundary condition at the wing

surface, we can get the equation

$$-K(y+c)^2\alpha = -\frac{(y+c)}{2\pi} \int_{-b}^b \frac{\varphi(\xi)}{y-\xi} d\xi + \frac{1}{2\pi} \int_{-b}^b \varphi(\xi) \log \left| \frac{y-\xi}{c+\xi} \right| d\xi \dots\dots\dots(16)$$

since

$$U(y) = K(y+c)$$

This is an integral equation whose unknown function is $\varphi(\xi)$. If Eq.(16) is solved and, at the same time, $\varphi(\xi)$ is used in Eqs. (14) and (15), the perturbation velocities v, w which satisfy the set of the boundary conditions will be obtainable. In the next chapter, the hydrodynamic forces acting on the wing will be expressed by using $\varphi(\xi)$. Furthermore, in chapter 4, we shall solve the integral equation.

3. Expressions of the hydrodynamic forces

In order to get the hydrodynamic forces acting on the wing, the pressure on its surface should be obtained. In the irrotational flow field, when each velocity component is acquired, we can easily get the pressure by the Bernoulli theorem. In uniform shear flow, however, the total pressure has a different value on each stream line. Consequently, in such a problem, the Bernoulli theorem seems not so effective. For such a reason, we shall derive the pressure p from Eq. (3).

Integrating Eq. (3) with respect to y from $-\infty$ to y , we can acquire the following expression,

$$p(y, z; x') = -\rho \frac{\partial}{\partial x'} \left[\int_{-\infty}^y U(\eta) v(\eta, z; x') d\eta \right] \dots\dots\dots(17)$$

The pressure had a different value between on the upper surface and on the lower surface of the wing. This difference in pressure across the wing surface develops a lift force. As was discussed in the preceding chapter, v also has discontinuity across the wing surface $z=0$, $-b \leq y \leq b$. So, let us write p and v at the wing surface as follows:

$$z \rightarrow +0; \begin{cases} p \rightarrow p_+ \\ v \rightarrow v_+ \end{cases}, \quad z \rightarrow -0 \begin{cases} p \rightarrow p_- \\ v \rightarrow v_- \end{cases}$$

Assuming z tends to ± 0 in Eq. (17), the following expressions are given

$$\left. \begin{aligned} p_+ &= -\rho \frac{\partial}{\partial x'} \left[\int_{-\infty}^y U(\eta) v_+ d\eta \right] \\ p_- &= -\rho \frac{\partial}{\partial x'} \left[\int_{-\infty}^y U(\eta) v_- d\eta \right] \end{aligned} \right\} \dots\dots\dots(18)$$

Then, the pressure difference $\Delta p = p_- - p_+$ is

$$\Delta p(y; x') = -\rho \frac{\partial}{\partial x'} \left[\int_{-\infty}^y U(\eta) (v_- - v_+) d\eta \right] \dots\dots\dots(19)$$

Now, because there must be no discontinuity for v in the region $|y| > b$, that is, the region not on the wing surface, the relation $v_- - v_+ = 0$ must hold in this region $|y| > b$. Therefore, integrand of

Eq. (19) vanishes in the integration range between $\eta = -\infty$ and $\eta = -b$, and hence

$$\Delta p(y; x') = -\rho \frac{\partial}{\partial x'} \left[\int_{-b}^y U(\eta) (v_- - v_+) d\eta \right] \dots\dots\dots(20)$$

Furthermore, putting $z \rightarrow \pm 0$ in Eq. (14), the following relation is obtained.

$$v_{\pm} = \left[\frac{z}{2\pi} \int_{-b}^b \frac{\varphi(\xi)}{(y-\xi)^2 + z^2} d\xi \right]_{z=\pm 0} = \pm \frac{1}{2} \varphi(y) \dots\dots\dots(21)$$

From this relation,

$$v_- - v_+ = -\varphi(y)$$

can be derived. Then, Δp is written as

$$\Delta p(y; x') = \rho \frac{\partial}{\partial x'} \left[\int_{-b}^y U(\eta) \varphi(\eta) d\eta \right] \dots\dots\dots(22)$$

In the region $y > b$, Δp should be equal to zero independently of x' . Then, the following relation should hold.

$$\int_{-b}^b U(\eta) \varphi(\eta) d\eta = 0 \dots\dots\dots(23)$$

The solution of the integral equation, $\varphi(\xi)$, should satisfy this equation as well.

Since the pressure difference between on the upper and on the lower surface of the wing has been obtained, the total lift force, the spanwise lift distribution, etc. will be acquired, if Δp is integrated over the surface of the wing.

In the first place, let us derive the chordwise lift distribution $l_z(x')$ acting on the wing which has an arbitrary planform as shown in Fig. 3. This can be easily obtained when Δp is integrated from a point A to A' with respect to y .

$$l_z(x') = \rho \frac{d}{dx'} \left[\int_{-b}^b \left\{ \int_{-b}^y U(\eta) \varphi(\eta) d\eta \right\} dy \right] \dots\dots\dots(24)$$

This expression is equivalent to Eq. (1), which is derived from the Jones' theory. As mentioned in the chapter 2, the lift acts only on the part of the wing where the bracketed value of integral increases in the streamwise direction. And the part of the wing where the bracketed value decreases in the

Fig. 3 Path of integration for pressure difference Δp

streamwise direction develops no lift. Then, let us call the spanwise section at which the bracketed value reaches maximum, the "effective trailing edge", and affix a subscript "e" to all the physical and geometrical values. The "effective trailing edge" of a small cambered wing corresponds to the chordwise section at which the local wing breadth reaches maximum.

The total lift force L can be obtained by integrating $l_x(x')$ from the leading edge to the "effective trailing edge".

$$L = \int_{(\text{the leading edge, } b=0)}^{(\text{the effective trailing edge})} l_x(x') dx' \\ = \rho \int_{-b_e}^{b_e} \left\{ \int_{-b_e}^y U(\eta) \varphi_e(\eta) d\eta \right\} dy \dots \dots \dots (25)$$

Thus, we can conclude that the total lift is determined when only the conditions of the effective trailing edge are given, and that the lift is independent of the wing planform.

It can easily be seen from Eq. (25) that the bracketed value $\int_{-b_e}^y U(\eta) \varphi_e(\eta) d\eta$ corresponds to the spanwise lift distribution, $l_y(y)$.

$$l_y(y) = \rho \int_{-b_e}^y U(\eta) \varphi_e(\eta) d\eta \dots \dots \dots (26)$$

The same result as is given in Eq. (26) is obtainable even if we integrate $\Delta p(y; x')$ along the chord.

In the present research, we define the total kinematic energy of the flow in the π -plane at far down-stream, as the "induced drag". This definition for the induced drag is the same as that in the ordinary wing theory. Let the perturbation velocities v, w in the π -plane at far down-stream, that is, the Trefftz plane, be v_∞, w_∞ respectively. Then, the induced drag is given by the form.

$$D_i = \frac{\rho}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [v_\infty^2 + w_\infty^2] dy dz \dots \dots \dots (27)$$

Now, assuming that $\partial p / \partial x'$ tends to zero at far downstream, we get a kind of the potential function,

$$\phi = -\lim_{x' \rightarrow \infty} \left[\int_{-\infty}^{x'} \frac{p}{\rho} dx' \right] \dots \dots \dots (28)$$

This function is the same as the one Honda⁽⁴⁾ has defined in his research. From Eqs. (3) and (4), the following equations are derived.

$$v_\infty = \frac{1}{U} \frac{\partial \phi}{\partial y} \dots \dots \dots (29)$$

$$w_\infty = \frac{1}{U} \frac{\partial \phi}{\partial z} \dots \dots \dots (30)$$

The equation which ϕ should satisfy is

$$\frac{\partial}{\partial y} \left[\frac{1}{U^2} \frac{\partial \phi}{\partial y} \right] + \frac{1}{U^2} \frac{\partial^2 \phi}{\partial z^2} = 0 \dots \dots \dots (31)$$

Substituting Eqs. (29) and (30) into Eq. (27), we change the surface integral into a line integral by

Green's theorem. As ϕ satisfies Eq. (30), D_i is reduced to the following expression:

$$D_i = -\frac{\rho}{2} \left[\int_{-b_e}^{b_e} \frac{\phi_+}{U^2} \frac{\partial \phi_+}{\partial z} dy - \int_{-b_\infty}^{b_\infty} \frac{\phi_-}{U^2} \frac{\partial \phi_-}{\partial z} dy \right] \dots \dots \dots (32)$$

where, b_∞ denotes a half width of the wake, and ϕ_+ and ϕ_- denote ϕ just above the wake and that just below the wake. As shown in Eq. (30), $\frac{1}{U} \frac{\partial \phi}{\partial z}$ corresponds to w_∞ , and is continuous all over the flow field. Now, ϕ is

$$\phi(y, z) - \phi_0 = \int_{-\infty}^y U(\eta) v_\infty(\eta, z) d\eta \dots \dots \dots (33)$$

from Eq. (29).

And, $\phi_0 = \phi(-\infty; z)$: finite value.

Because the velocity $v(y, z)$ has discontinuity only across the wake, $-b_\infty \leq y \leq b_\infty$, the following relation will be derived from the above expression.

$$\phi_+ - \phi_- = \int_{-b_\infty}^y U(\eta) \{v_\infty(\eta, +0) - v_\infty(\eta, -0)\} d\eta \dots \dots \dots (34)$$

So far, we have considered the flow in the Trefftz plane. On the assumption of a "slender wing", the flow in the Trefftz plane is equivalent to that in the plane at the effective trailing edge. Therefore, the velocity components, width of the wake, etc. in the Trefftz plane are equal to those in the plane at the effective trailing edge. Thus, $v_\infty(y, +0) - v_\infty(y, -0)$ can be replaced by $\varphi_e(y)$.

$$\text{And, } \frac{1}{U} \frac{\partial \phi_+}{\partial z} = \frac{1}{U} \frac{\partial \phi_-}{\partial z} = w_\infty(y, 0) = w_e(y, 0)$$

is equal to $-U(y) \alpha_e$ from the boundary condition at the wing surface. Finally, the induced drag D_i given in Eq. (32) becomes

$$D_i = \frac{\rho}{2} \alpha_e \int_{-b_e}^{b_e} \left[\int_{-b_e}^y U(\eta) \varphi_e(\eta) d\eta \right] dy \dots \dots \dots (35)$$

The bracketed value $\frac{\rho}{2} \alpha_e \int_{-b_e}^y U(\eta) \varphi_e(\eta) d\eta$ is the spanwise distribution of the induced drag, $d_i(y)$,

$$d_i(y) = \frac{\rho}{2} \alpha_e \int_{-b_e}^y U(\eta) \varphi_e(\eta) d\eta \dots \dots \dots (36)$$

The relations between the induced drag D_i and the total lift L , and between the spanwise distribution of the induced drag $d_i(y)$ and that of the lift, $l_y(y)$ are

$$\left\{ \begin{array}{l} D_i = \frac{\alpha_e}{2} L \dots \dots \dots (37) \\ d_i(y) = \frac{\alpha_e}{2} l_y(y) \dots \dots \dots (38) \end{array} \right.$$

Now, using $\varphi(\xi)$ we have expressed the hydrodynamic forces acting on the wing. When the integral equation (16) is solved, we can get the concrete hydrodynamic forces. In the next chapter, the integral equation will be solved in the form of a Fourier series.

4. The solution of the integral equation and the hydrodynamic forces on the wing

In the present chapter, we shall try to solve the integral equation (16) under the condition of Eq. (23) and then acquire the hydrodynamic forces acting on the wing concretely.

The form of the integral equation (16) resembles those which are given in the usual thin wing theory. Hence, $\varphi(\xi)$ will be expressed in the form of a Fourier series, if a suitable variable change is made. Then, the following new variables ϕ and θ , and the dimensionless value λ are introduced here.

$y = b \cos \phi$, $\xi = b \cos \theta$, $\varphi(\xi) = \varphi(\theta)$, $c = \lambda b$... (39)
When such variables change is made, the integral equation (16) and Eq. (23) become

$$-Kb\alpha(\cos \phi + \lambda)^2 \\ = -\frac{(\cos \phi + \lambda)}{2\pi} \int_0^\pi \frac{\varphi(\theta) \sin \theta}{\cos \phi - \cos \theta} d\theta \\ + \frac{1}{2\pi} \int_0^\pi \varphi(\theta) \sin \theta \log \left| \frac{\cos \phi - \cos \theta}{\lambda + \cos \theta} \right| d\theta \quad \dots (40)$$

$$\int_0^\pi (\cos \theta + \lambda) \varphi(\theta) \sin \theta d\theta = 0 \quad \dots (41)$$

where $\lambda = c/b$ is larger than unity, because the zero-velocity point of an on-coming free shear flow is not on the surface of the wing.

Now, we shall solve the above integral equation supposing that the solution $\varphi(\theta)$ will be expressed in a Fourier series. Because $\varphi(\theta)$ is required to satisfy Eq. (41), the integrand $(\lambda + \cos \theta) \varphi(\theta) \sin \theta$ should be written by a Fourier cosine series. Then, $\varphi(\theta)$ is

$$\varphi(\theta) = -2\alpha Kc \frac{\lambda}{\lambda + \cos \theta} \frac{1}{\sin \theta} \sum_{n=1}^{\infty} A_n \cos n\theta \quad \dots (42)$$

Substituting $\varphi(\theta)$ given in Eq. (42) into Eq. (40), we obtain the algebraic equality

$$\sum_{n=1}^{\infty} A_n G_n(\phi) = \frac{(1 + 2a \cos \phi + a^2)^2}{2(1 + a^2)^2} \quad \dots (43)$$

where

$$G_n(\phi) \equiv \frac{\sin n\phi}{2 \sin \phi} + H_n(\phi) + \frac{(-a)^{n+1}}{1 - a^2} - Q_n \quad \dots (44)$$

$a = \lambda - \sqrt{\lambda^2 - 1}$ ($\lambda > 1$)
and, $H_n(\phi)$ and Q_n are

$$H_n(\phi) = \frac{1}{2\pi} \int_0^\pi \frac{\cos n\theta}{\lambda + \cos \theta} \log |\cos \phi - \cos \theta| d\theta \\ Q_n = \frac{1}{2\pi} \int_0^\pi \frac{\cos n\theta}{\lambda + \cos \theta} \log (\lambda + \cos \theta) d\theta \quad \dots (45)$$

Unknown coefficients A_n 's should be determined, so that Eq. (43) may be valid for an arbitrary value of ϕ . Now, $H_n(\phi)$ and Q_n for an arbitrary number n can not be integrated directly. However, by the following reduction formulae, we can obtain $H_n(\phi)$

and Q_n successively.

$$\left. \begin{aligned} H_{n+1}(\phi) + \frac{1+a^2}{a} H_n(\phi) + H_{n-1}(\phi) &= -\frac{\cos n\phi}{n} \\ (n=1, 2, 3, \dots) \\ H_0(\phi) &= \frac{a}{1-a^2} [\log(1+2a \cos \phi + a^2) - \log 2] \\ H_1(\phi) &= -\frac{1+a^2}{2(1-a^2)} [\log(1+2a \cos \phi + a^2) \\ &\quad - \frac{2a^2}{1+a^2} \log 2] \end{aligned} \right\} \quad \dots (46)$$

$$\left. \begin{aligned} Q_{n+1} + \frac{1+a^2}{a} Q_n + Q_{n-1} &= -\frac{(-a)^n}{n} \\ (n=1, 2, 3, \dots) \\ Q_0 &= \frac{2a}{1-a^2} \log(1-a^2) - \frac{a}{1-a^2} \log 2a \\ Q_1 &= -\frac{1+a^2}{1-a^2} \log(1-a^2) + \frac{a^2}{1-a^2} \log 2a \end{aligned} \right\} \quad \dots (47)$$

Then, $G_1(\phi)$, $G_2(\phi)$ and $G_3(\phi)$ are calculated and shown as follows:

$$G_1(\phi) = \frac{1}{2} - (1+a^2)R(\phi) + a^2 r_0 \\ G_2(\phi) = -a + \frac{1+a^4}{a} R(\phi) - a^3 r_0 \\ G_3(\phi) = \cos^2 \phi + \frac{1+a^2}{a} \cos \phi + \frac{3}{2} a^2 + 1 \\ - \frac{1+a^6}{a^2} R(\phi) + a^4 r_0$$

$$\text{where } R(\phi) \equiv \frac{1}{2(1-a^2)} [\log(1+2a \cos \phi + a^2) \\ - 2 \log(1-a^2)] \\ r_0 \equiv (1 - \log a) / (1 - a^2)$$

Furthermore, when we calculate $G_4(\phi)$, $G_5(\phi)$, ..., etc. in succession, we can find that the highest power of $\cos \phi$ contained in each of them becomes $\cos^3 \phi$, $\cos^4 \phi$, ..., etc. On the other hand, the right-hand side of Eq. (43) is written as

$$\frac{2a^2}{(1+a^2)^2} \left[\cos^2 \phi + \frac{1+a^2}{a} \cos \phi \right] + \frac{1}{2}$$

The highest power of $\cos \phi$ in this expression is $\cos^2 \phi$. Therefore, in order that Eq. (43) may be valid for an arbitrary value of ϕ , the coefficients of the series of more than fourth order, such as A_4 , A_5 , ..., are all required to be equal to zero, and hence the left-hand side of Eq. (43) should be expressed with the sum, $\sum_{n=1}^3 A_n G_n(\phi)$. So, Eq. (43) yields finally,

$$A_1 G_1(\phi) + A_2 G_2(\phi) + A_3 G_3(\phi) \\ = \frac{2a^2}{(1+a^2)^2} \left[\cos^2 \phi + \frac{1+a^2}{a} \cos \phi \right] + \frac{1}{2} \quad \dots (48)$$

The same term as bracketed in Eq. (48),

$$\cos^2 \phi + \frac{1+a^2}{a} \cos \phi.$$

is included in $G_3(\phi)$. Therefore, if a suitable value is given to A_3 , Eq. (43) can be valid for an arbitrary value of ϕ , concerning the term,

$$\cos^2\phi + \frac{1+a^2}{a}\cos\phi.$$

On the contrary, $R(\phi)$ which is included in $G_1(\phi)$, $G_2(\phi)$ and $G_3(\phi)$ in common does not appear in the right-hand side of Eq. (48). Hence, the coefficient of $R(\phi)$ in the left-hand side of Eq. (48) should be zero. And the constant terms on each side of the equation are also required to be the same. Under such conditions as mentioned above, the following simultaneous linear algebraic equations are given

$$\begin{cases} A_3 = \frac{2a^2}{(1+a^2)^2} \\ -\frac{1+a^6}{a^2}A_3 + \frac{1+a^4}{a}A_2 - (1+a^2)A_1 = 0 \\ \left(1 + \frac{3}{2}a^2 + a^4r_0\right)A_3 - (a+a^3r_0)A_2 + \left(\frac{1}{2} + a^2r_0\right)A_1 = 1/2 \end{cases}$$

From this set of equations, we obtain

$$\left. \begin{aligned} A_1 &= \frac{(1-a^2)(1+a^2)^3 - 4a^4 \log a}{(1-a^2)(1+a^2)^3 - 2a^2(1+a^2)^2 \log a} \\ A_2 &= \frac{3a(1-a^2)(1+a^2) - 4a^3 \log a}{(1-a^2)(1+a^2)^2 - 2a^2(1+a^2) \log a} \\ A_3 &= 2a^2/(1+a^2)^2 \\ A_4 &= A_5 = A_6 = \dots = 0 \end{aligned} \right\} \dots (49)$$

where $a = \lambda - \sqrt{\lambda^2 - 1}$.

Equation (49) shows that A_n 's are expressed as a function of a , i.e., λ only. The parameter λ represents the location of the point at which the velocity of free shear flow is zero. For instance, when λ tends to infinity with the constant value of Kc , which represents the velocity at $y=0$, this uniform shear flow tends to a uniform flow whose velocity is Kc . On the contrary, when λ tends to unity, the location of the zero-velocity point of free flow comes infinitely closer to the wing tip. Therefore, λ is considered to represent the inverse dimensionless

gradient of free stream as well.

In Fig. 4, we illustrate curves of $A_1(\lambda)$, $A_2(\lambda)$ and $A_3(\lambda)$. The case that $1/\lambda$ is equal to zero corresponds to that of uniform flow, in which case $A_2(\lambda)$ and $A_3(\lambda)$ are both equal to zero and $A_1(\lambda)$ has a value of unity.

As all the coefficients A_n 's of the series have been obtained, now the integral equation (40) is solved completely. Here, we shall show $\varphi(\theta)$ again.

$$\varphi(\theta) = -\frac{2\alpha Kc\lambda}{(\lambda + \cos\theta)\sin\theta} \sum_{n=1}^3 A_n(\lambda) \cos n\theta$$

$$\theta = \cos^{-1}(y/b) \dots (50)$$

In Fig. 5, we illustrate the curves $\varphi(y)/2\alpha Kc$ for several values of λ . Since $\varphi(y)/2$ is equal to v_+ and $-v_-$, $\varphi(y)/2\alpha Kc$ can be written as follows:

$$\frac{\varphi(y)}{2Kc\alpha} = \frac{v_+}{Kc\alpha} = -\frac{v_-}{Kc\alpha} \dots (51)$$

Accordingly, the curves in Fig. 5 can be regarded as that of the dimensionless velocity in the y -direction divided by $Kc\alpha$. $\varphi(y)$ becomes infinite at both tips of the wing except for $\lambda=1$. This is because the wing is supposed to be very thin in the present research, and so the velocity of the flow at the wing tip becomes infinite theoretically. This phenomenon is the same as that of the flow around a plate. The point at which $\varphi(y)$ becomes zero, that is, both v_+ and v_- become zero, corresponds to the "stagnation point" of "two-dimensional flow" in the π -plane. This "stagnation point" is located at the mid-point of the local wing breadth when the wing is in uniform flow (in the case of $1/\lambda=0$). In uniform shear flow, however, this point moves toward the side where the velocity of on-coming free flow is large and, in other words, the total head is large. The reason is that the fluid flows from a higher position of the total head to a lower one. In the case

Fig. 4 Curves of $A_n(\lambda)$

Fig. 5 Curves of $\varphi(y)$ for several λ

of $\lambda=1$, $\varphi(y)$ becomes zero at the wing tip of the left-hand side as well as the "stagnation point", because the velocity of the on-coming flow toward the wing tip becomes zero.

Making use of $\varphi(y)$ obtained thus, we can express the total lift force, spanwise lift distribution, etc. as follows:

(1) The pressure distribution on the wing

$$\Delta p(y; x') = 2\rho K^2 \frac{\partial}{\partial x'} \left[\alpha b c^2 \sum_{n=1}^3 \frac{A_n(\lambda)}{n} \sin n\phi \right]$$

$$\phi = \cos^{-1}(y/b) \quad \dots\dots\dots (52)$$

(2) The spanwise lift distribution

$$l_y(y) = 2\rho K^2 c_e^2 b_e \alpha_e \sum_{n=1}^3 \frac{A_n(\lambda_e)}{n} \sin n\phi_e$$

$$\phi_e = \cos^{-1}(y/b_e) \quad \dots\dots\dots (53)$$

(3) The chordwise lift distribution

$$l_x(x') = \pi \rho K^2 \frac{d}{dx'} [c^2 b^2 \alpha A_1(\lambda)] \quad \dots\dots\dots (54)$$

(4) The total lift

$$L = \pi \rho K^2 c_e^2 b_e \alpha_e A_1(\lambda_e) \quad \dots\dots\dots (55)$$

(5) The spanwise induced drag distribution

$$d_i(y) = \rho K^2 c_e^2 b_e \alpha_e^2 \sum_{n=1}^3 \frac{A_n(\lambda_e)}{n} \sin n\phi_e$$

$$\phi_e = \cos^{-1}(y/b_e) \quad \dots\dots\dots (56)$$

(6) The total induced drag

$$D_i = \frac{\pi}{2} \rho K^2 c_e^2 b_e^2 \alpha_e A_1(\lambda_e) \quad \dots\dots\dots (57)$$

As shown in these equations, $l_x(x')$, L and D_i are independent of both $A_2(\lambda)$ and $A_3(\lambda)$. When λ tends to infinity with Kc (or Kc_e) kept constant U_0 in these equations, each of them agrees with that given by the Jones' theory.

5. Discussions about the wing characteristics

In this chapter we shall consider the characteristics of a low-aspect-ratio wing in uniform shear flow, quoting several numerical examples.

As mentioned by now in the present research, the total lift force, the spanwise lift distribution and the induced drag are determined when the values of the "effective trailing edge" such as b_e , c_e , α_e , etc. are given. In the first place, dividing $\varphi(y)$ by $2\rho K^2 c_e^2 b_e \alpha_e$, we get the pattern of the dimensionless spanwise lift distribution, which is known to be a function of λ_e alone by Eq. (53). This means that the pattern of the spanwise lift distribution depends not on the wing planform, but on the location of the zero-velocity point of uniform shear flow. In Fig. 6 we illustrate the patterns of the spanwise lift distribution for several λ_e . The lift distribution pattern which is determined under the condition when λ_e tends to infinity agrees with that of the wing in uniform flow, and is elliptic. According as λ_e tends to unity, that is, the zero-

velocity point of uniform shear flow comes infinitely closer to the wing tip, the lift distribution acting on the starboard side of the wing becomes gradually larger and larger. This is because the on-coming shear flow velocity toward this portion becomes large in comparison with that toward the port side of the wing. Meanwhile, opposite phenomena occur on the port side of the wing.

Dividing the total lift force L by $(1/2)\rho(Kc_e)^2 S_w$ (S_w is wing area), we define the lift coefficient C_L .

$$C_L = \frac{\pi}{2} AR \alpha_e A_1(\lambda_e), \quad AR = \frac{(2b_e)^2}{S_w} \quad \dots\dots\dots (58)$$

where AR is the aspect ratio.

When λ_e tends to infinity, let us represent C_L by C_{L0} . Then we examine the ratio of C_L to C_{L0} . Taking it into consideration that $A_1(\lambda)$ becomes unity when $1/\lambda_e$ tends to zero, we get the following expression:

$$C_L/C_{L0} = A_1(\lambda_e) \quad \dots\dots\dots (59)$$

From this it follows that the ratio of the lift coefficient of the wing in uniform shear flow to that in uniform flow is equal to $A_1(\lambda_e)$. Figure 4 shows that C_L becomes about 20 percent smaller than C_{L0} . Similarly, dividing the induced drag D_i by $(1/2)\rho(Kc_e)^2 S_w$, we can define the drag coefficient C_D . Equation (37) gives the relation,

$$C_D = (\alpha_e/2) C_L \quad \dots\dots\dots (60)$$

This expression shows that the same [thing as for C_L can be said for the coefficient C_D .

The spanwise lift distributions, the lift coefficients and the drag coefficients can be, as mentioned above, determined when the condition of the

Fig. 6 The spanwise lift distributions for several λ_e

effective trailing edge is given. Nevertheless, the chordwise lift distribution $l_x(x')$ can not be determined even when the same condition is shown: $l_x(x')$ can, as shown in Eq. (54), be calculated only when the planform and the camber line i.e. $b(x')$, $c(x')$ and $\alpha(x')$ are given.

Now, let us examine the wing with isosceles triangle planform, as shown in Fig. 7. The wing, which has no camber, is placed in uniform shear flows for several λ_e . The center line of the wing is parallel to the x' -axis. Therefore, c is independent of x' . α is also independent of x' , since the wing has no camber. So, we can easily differentiate the right-hand side of Eq. (54) with respect to x' , and calculate the value of $l_x(x')$. Figure 8 shows the chordwise lift distributions of this wing. The lift distribution in the case of $\lambda_e = \infty$ is linear, and so it corresponds with that given by the Jones' theory. However, when we let λ_e gradually approach to unity, the influence of shear flow on the lift distribution becomes large. Accordingly, the lift distributions become different from that by the Jones' theory. However, near the leading edge, since b is much smaller than c (which is always equal to c_e in the present case), the local value of λ is very large regardless of λ_e . Therefore, the lift distribution near the leading edge is nearly similar to that in the case of $\lambda_e = \infty$.

By now we have obtained the chordwise lift distributions, giving the concrete planform to the wing. From now on, we shall try to get the wing

Fig. 7 A schematic wing with an isosceles triangle planform in uniform shear flow

Fig. 8 The chordwise lift distributions of a delta wing for several λ_e

planform when the chordwise lift distribution is given. In Fig. 9 we illustrate the two planforms: one has a lift distribution which is linear in the chordwise direction, and the other has a uniform lift distribution. The camber lines of those wings are both circular arcs of 4% camber ratio. In addition, these planforms are determined so that the "stagnation points" of the stream lines in each π -planes may be placed parallel to the on-coming free flow. The dotted lines in Fig. 9 are rows of these "stagnation points". When the uniform shear flow shown in Fig. 9 comes toward the wing, all of the fluid that flows into the lower surface of the wing on the right side of these lines, at first, goes to the right tip and next into the upper side. This is also true of the left side; the fluid at first, goes to the left tip and next into the upper side. At the position near the leading edge, $\lambda = c/b$ is very large. Accordingly, the flow field is little influenced by shear flow. Therefore, the "stagnation point" is located at the mid-span at every chordwise station. At the trailing edge, however, λ becomes equal to unity, and the "stagnation point" moves to the position $y = (5/6)b$.

Above-mentioned are numerical examples of the present theory concerning the low-aspect-ratio wing in uniform shear flow.

6. Conclusions

The theoretical analysis of the characteristics of the finite low-aspect-ratio wings in uniform shear flow has been presented on the assumption analogous to that in the Jones' theory for the low-aspect-ratio wings in uniform flow.

As the result, the hydrodynamic forces acting on the wing such as the total lift L , the total induced drag D_i , the spanwise distributions of the lift and of the induced drag are all determined

Fig. 9 The wing planforms for given chordwise lift distributions
(Camber line: arc of 4% camber ratio)

when the conditions of the "effective trailing edge" are given. Furthermore, the present theory has been able to clarify that the hydrodynamic forces acting on the wing in uniform shear flow are strongly influenced by the parameter λ or λ_e , which corresponds to the dimensionless inverse velocity gradient of free shear flow: when λ_e tends to unity, that is, when the zero-velocity point of the free shear flow approaches to a wing tip, the lift coefficient C_L and the induced drag coefficient C_D become respectively about 20% smaller than those of the wing in uniform flow.

The chordwise lift distribution $l_x(x')$ can be determined by giving both planform and camber line of the wing. This lift distribution is also strongly influenced by λ or λ_e , the inverse velocity gradient of free shear flow.

In the present research, we have treated the flow in the plane normal to the free stream as two-dimensional one. In uniform flow, the "stagnation point", at which the spanwise component of the velocity at the wing surface becomes equal to zero, is located at the mid-span at every chordwise station. On the other hand, in uniform shear flow,

this "stagnation point" moves toward the side where the free flow velocity is large, and when λ tends to unity, this point reaches the position $y=(5/6)b$.

In the present theory, when the parameter λ or λ_e tends to infinity, the solution agrees perfectly with the Jones' one.

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Discussion

M. NAMBA: (1) Near the zero-velocity point $y=-c$, of the uniform shear flow velocity $U(y)=K(y+c)$, it seems not correct to assume

$$U \gg u, v, w.$$

Therefore, it appears proper to apply the linearization for the equation of motion only when the wing is far from the zero-velocity point of the free flow.

(2) If w/U has continuous value across the region $y=-c$, the following relation is certified to be valid from Eq. (15)

$$[w]_{y=-c} = -\frac{1}{2\pi} \int_{-b}^b \frac{\varphi(\xi)}{1-c-\xi} d\xi = 0$$

However, as $\varphi(\theta)$ has such a form as introduced in Eq. (50), the above relation seems not valid generally. It is rather due to the wing characteristics that v/U and w/U become infinite at the wing tips. Meanwhile, the fact that these become infinite in the region $y=-c$ is based on the reason that U itself becomes zero in this region. Furthermore, since this singularity is of order of $(y+c)^{-1}$, it is not small. Therefore, the questioner thinks that the assumption $|v|/U, |w|/U \ll 1$ is not applicable even when the singularity is not on the wing. And, from the result obtained by the linearized theory, which is shown in Append.-Fig. 1, we can not exactly know whether the influence of the

neglected nonlinear terms is large or not.

Author's closure

(1) It is the flow just around the wing that gives great influence on the wing characteristics: if we can prove that the perturbation velocities just around the wing are small enough compared with the free shear flow velocity, $U(y)=K(y+c_e)$, especially when $1/\lambda_e$ is nearly equal to unity, we can, at the same time, prove the correctness of the present theory. For this purpose, v and w on the spanwise wing section and on its extension line are calculated, as shown in Append.-Figs. 1 and 2. According as $1/\lambda_e$ approaches to unity, the absolute value of $v/U(y)$ becomes larger in the region where the free flow velocity is small. Its magnitude becomes only a few times as large as that in uniform flow. For instance, at the point $y/b_e=-0.9$, the magnitude of $v/U(y)$ in the case of $1/\lambda_e=1$ becomes only about 3.5 times larger than that in the case of uniform flow. Consequently, the assumption of small perturbation $U(y) \gg v$, is considered reasonable. Meanwhile, at both tips, $y/b_e=-1$ and $y/b_e=1$, v/U and w/U both become infinite regardless of λ_e . Such singularity is unavoidable in the thin wing theory. On the extension line of the spanwise wing

section $|y/b_e| > 1$, $v/U(y)$ for all the values of λ_e is equal to zero.

On the other hand, w/U is independent of λ_e on the spanwise wing section. And in the region $|y/b_e| > 1$, the value of w/U is little influenced by λ_e . Therefore, the values of w/U for every λ_e are in the same order of magnitude.

As we examined by now, the ratio of the perturbation velocities to the free flow velocity $U(y)$ becomes, at the most, a few times as much as the angle of attack α . For such reason, the present author presumes that the present theory can be applied validly even if $1/\lambda_e$ is nearly unity.

In the flow field ($y = -c$, $z \neq 0$), v/U becomes infinite (on the contrary, w/U is continuous across this region, and is small enough). However, singularity in this region may be considered to be in the same order of magnitude as that at the wing tips. Moreover, it is not on the wing. Therefore, it is presumed that the wing characteristics are

Append.-Fig. 1 Calculated values of v near the wing surface

Append.-Fig. 2 Calculated values of w near the wing surface

little influenced by this singularity.

(2) About the validity of the relation $[w]_{y=-c, z=0} = 0$, the author will reply as follows: Equation (15) will be expressed as follows:

$$\frac{w(y, 0; x')}{U(y)} = -\frac{1}{2\pi K(y+c)} \int_{-b}^b \frac{\varphi(\xi)}{y-\xi} d\xi + \frac{1}{2\pi K(y+c)^2} \int_{-b}^b \varphi(\xi) \log \left| \frac{y-\xi}{c+\xi} \right| d\xi$$

Now y , which is nearly equal to $-c$, will be written as follows:

$$y = -c + \varepsilon \quad (c > b, \varepsilon: \text{small quantity})$$

Then, we have the following two relations:

$$\frac{1}{y-\xi} = -\frac{1}{c+\xi} - \frac{\varepsilon}{(c+\xi)^2} + O(\varepsilon^2)$$

$$\log \left| \frac{y-\xi}{c+\xi} \right| = -\frac{\varepsilon}{c+\xi} - \frac{1}{2} \frac{\varepsilon^2}{(c+\xi)^2} + O(\varepsilon^3)$$

When we put these into above, equation, the relation

$$\frac{w(-c, 0; x')}{U(y)} = \frac{1}{4\pi K} \int_{-b}^b \frac{\varphi(\xi)}{(c+\xi)^2} d\xi + O(\varepsilon) \quad (\varepsilon \rightarrow 0)$$

can be obtained. In this equation, w/U is proved to have a finite value and not to have singularity in the region $y = -c$. The above concerns the case $z = 0$. The same thing can be shown easily in the case $z \neq 0$, too.

On the other hand, v/U becomes infinite in the region $y = -c$. From a purely mathematical point of view, it may be concluded that there may be no meaning in such linearization of the equations of motion. [The fact that v has finite value in this region is associated with the fact that there is a crossing flow from the region $U(y) > 0$ into that $U(y) < 0$.] However, ratio of the perturbation velocities to the free flow velocity diverges on the wing even in the ordinary thin wing theory, and the wing characteristics obtained by such linearized theory are little different from those given by the exact theory, such as by conformal mapping. This is because that singularity gives little influence on the wing characteristics. Especially in the present research, singularity is not on the wing. Hence, even if v/U diverges locally, singularity gives little influence on the result.

Concerning the argument that the influence of the neglected nonlinear terms on the result can not be estimated by the results obtained by the linearized theory, the author replies as follows: as for the problems of a thin airfoil in incompressible and inviscid fluid flow, the influence of the neglected terms can be estimated for such a reason as mentioned above.